HAUSDORFF MEASURES AND TWO POINT SET EXTENSIONS

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Abstract. We investigate the question: under which conditions is a σ-compact partial two point contained in a two point set? We show that no reasonable measure or capacity (when applied to the set itself) can provide a sufficient condition for a compact partial two point set to be extendable to a two point set. On the other hand, we prove that under Martin's Axiom any σ-compact partial two point set such that its square has Hausdorff 1-measure zero is extendable.

1. Introduction

A planar set is called a *two point set* if every line intersects the set in exactly two points and a *partial two point set* if every line intersects the set in at most two points. We call a partial two point set *extendable* if it is a subset of some two point set. The existence of two point sets is due to Mazurkiewicz [7]. His proof shows that every partial two point set with cardinality less than $\mathfrak{c}$ is extendable. A circle is the standard example of a nonextendable partial two point set.

In [6, 9, Problem 1070] R. D. Mauldin asked whether every compact zero-dimensional partial two point set can be extended to a two point set. It was shown by two of the authors in [1] and independently by Mauldin in [7] that there exist partial two point Cantor sets that are not extendable to two point sets. In both papers the proof rests on the fact that the examples have positive linear Lebesgue measure. It is therefore natural to ask whether there exist nonextendable partial two point compacta with vanishing linear measure (or Hausdorff 1-measure zero). In addition, what about Hausdorff dimension zero or logarithmic capacity zero? In this paper the authors answer these questions by showing that no reasonable measure or capacity (when applied to the

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set itself) can provide a sufficient condition for a compact partial two point set to be extendable to a two point set. These results can be found in section 2. In section 3 we present sufficient conditions for a partial two point set to be extendable. For example, under Martin's Axiom any σ-compact partial two point set such that its square has Hausdorff 1-measure zero is extendable.

2. Nonextendable sets

We denote the space of nonempty compacta in a metric space X equipped with the usual Hausdorff metric by $\mathcal{K}(X)$. The space of projection directions in the plane is the circle $\Theta = \mathbf{R}/\pi\mathbf{Z}$. If $\theta \in \Theta$ then p_θ is the projection of the plane onto the line through the origin that is perpendicular to θ. Obviously, the function $p : \Theta \times \mathbf{R}^2 \to \mathbf{R}^2$ defined by $p(\theta, u) = p_\theta(u)$ is a continuous mapping and hence it generates a continuous map from $\Theta \times \mathcal{K}(\mathbf{R}^2)$ to $\mathcal{K}(\mathbf{R}^2)$. If u is a nonzero vector in the plane then $\varphi(u) \in \Theta$ stands for the direction parallel to u. If u and v are two distinct points in the plane then $L(u, v)$ stands for the line through u and v. If A is a planar set then

$$\mathfrak{L}(A) = \bigcup\{L(u, v) : u, v \in A \text{ and } u \neq v\}.$$

and

$$\mathfrak{B}(A) = \{p(\varphi(u - v), u) : u, v \in A \text{ and } u \neq v\}.$$

Observe that $p(\varphi(u - v), u)$ is the point of intersection of $L(u, v)$ with the line through 0 that is perpendicular to $L(u, v)$ and that we have in terms of the dot product in $\mathbf{R}^2$:

$$p(\varphi(u - v), u) = \frac{((v - u) \cdot v)u + ((u - v) \cdot u)v}{|u - v|^2}.$$

If $v \in \mathbf{R}^2 \setminus \{0\}$ then C_v stands for the circle that has the line segment from 0 to v as diameter.

Lemma 2.1. *If $A \subset \mathbf{R}^2$ then*

$$\{v \in \mathbf{R}^2 \setminus \{0\} : C_v \cap \mathfrak{B}(A) \setminus \{0\} \neq \emptyset\} \subset \mathfrak{L}(A).$$

Proof. Let $u \in C_v \cap \mathfrak{B}(A)$ with $u \neq 0$. Then there exists a line ℓ through u that is perpendicular to the line through 0 and u and that intersects A in two distinct points a and b, see Figure 1.

Figure 1

Since the angle at u is $90°$ we have by elementary planar geometry that $\ell = L(a,b)$ intersects C_v at v.

Let S^1 be the unit circle centered at the origin. If $A \subset S^1$ then we simply have $\mathfrak{B}(A) = \{\frac{1}{2}(u+v) : u, v \in A \text{ and } u \neq v\}$.

We will show in the next section (see Proposition 3.7) that the nonextendable elements of $\mathcal{K}(S^1)$ form a category I subset of $\mathcal{K}(S^1)$. So finding nonextendable elements in $\mathcal{K}(S^1)$ is relatively hard because the set to choose from is small and standard Baire category arguments will not work. However, our next result gives us a rich supply of nonextendable compact subsets of S^1. Since the concept (partial) two point set is invariant under affine transformations the results we obtain for S^1 apply to any ellipse.

Theorem 2.2. *Every dense G_δ-subset of $\mathcal{K}(S^1)$ contains two elements with a nonextendable union.*

Proof. Let $G_0 \supset G_1 \supset \cdots$ be a sequence of dense open subsets of $\mathcal{K}(S^1)$ and define $G = \bigcap_{n=0}^{\infty} G_n$. Let $D^1 = \{u \in \mathbf{R}^2 : |u| \leq 1\}$ and $D^1_0 = D^1 \setminus \{0\}$. Define the continuous maps $\alpha, \beta : D^1_0 \to S^1$ as in Figure 2.

Figure 2

To be precise α and β are given by:

$$\alpha(u) = (x + ty, y - tx) \quad \text{and} \quad \beta(u) = (x - ty, y + tx),$$

where $u = (x, y) \in D^1_0$ and $t = \sqrt{1 - |u|^2}/|u|$. Then for each $u \in D^1_0$ we have

$$p(\varphi(\alpha(u) - \beta(u)), \alpha(u)) = \frac{1}{2}(\alpha(u) + \beta(u)) = u.$$

The semicircles with diameter 1 in Figure 2 correspond to preimages of the form $\alpha^{-1}(w)$ and $\beta^{-1}(w')$ where $w, w' \in S^1$. The building blocks of our construction will be segments of such semicircles.

We shall construct an element A of $\mathcal{K}(S^1)$ such that every circle through the origin with diameter at least 2 intersects $\mathfrak{B}(A)$ in a point that is not the origin. Then by Lemma 2.1 we have $\{v \in \mathbf{R}^2 : |v| \geq 2\} \subset \mathfrak{L}(A)$. So if we add any point v with $|v| \geq 2$ to A then the resulting set is no longer a partial two point set and hence A is not extendable to a two point set.

Let d be the arc length metric on S^1. Note that $u \mapsto (\alpha(u), \beta(u))$ is an imbedding of D^1_0 in the product $S^1 \times S^1$. This observation allows us to define a metric ρ on D^1_0 by pulling back the max metric of $S^1 \times S^1$: for $u, v \in D^1_0$,

$$\rho(u, v) = \max\{d(\alpha(u), \alpha(v)), d(\beta(u), \beta(v))\}.$$

Let $\tilde{d}$ and $\tilde{\rho}$ be the corresponding Hausdorff metrics on $\mathcal{K}(S^1)$ respectively $\mathcal{K}(D^1_0)$. If X is a subset of S^1 and $\varepsilon > 0$ let $U_\varepsilon(X) = \{w \in S^1 : d(w,X) \le \varepsilon\}$. If $w \in S^1$ then $U_\varepsilon(w) = U_\varepsilon(\{w\})$. If $u, v \in S^1$ then $[u,v]$ stands for the closed segment of S^1 from u to v (counterclockwise orientation). If $X \in \mathcal{K}(S^1)$ then $\tilde{U}_\varepsilon(X) \subset \mathcal{K}(S^1)$ denotes the closed ε-ball around X with respect to the Hausdorff metric $\tilde{d}$.

Let $\mathcal{C}$ be the collection of all semicircles C with diameter at least 2 such that one of the endpoints is the origin. We shall construct inductively a sequence $B_0, B_1, \ldots$ in $\mathcal{K}(D^1_0)$ and a sequence $\varepsilon_0, \varepsilon_1, \ldots$ of positive numbers such that

(1) $\varepsilon_n \le \varepsilon_{n-1}/2$,
(2) $\tilde{\rho}(B_n, B_{n-1}) \le \varepsilon_{n-1}/2$,
(3) B_n intersects every element of $\mathcal{C}$,
(4) $\tilde{U}_{\varepsilon_n}(\alpha(B_n)) \subset G_n$ if n is even,
(5) $\tilde{U}_{\varepsilon_n}(\beta(B_n)) \subset G_n$ if n is odd,
(6) if n is even then B_n is a finite union of sets of the form $\alpha^{-1}(p) \cap \beta^{-1}([u,v])$, where $p, u, v \in S^1$,
(7) if n is odd then B_n is a finite union of sets of the form $\beta^{-1}(p) \cap \alpha^{-1}([u,v])$, where $p, u, v \in S^1$,

Let F be a δ-net in S^1, i.e. a finite subset of S^1 with $\tilde{d}(F, S^1) \le \delta$. Consider the set

$$P_F = \alpha^{-1}(F) \cap \{u \in D^1 : \tfrac{1}{4} \le |u| \le \tfrac{3}{4}\},$$

see Figure 3.

Figure 3

We choose δ so small that for any δ-net F and any semicircle $C \in \mathcal{C}$ we have that P_F and C intersect. This is possible since there exists a δ such that for every $C \in \mathcal{C}$, the projection $C' = \alpha(\{u \in C : \frac{1}{4} \le |u| \le \frac{3}{4}\})$ is an interval of length at least 2δ. Since the finite sets are dense in $\mathcal{K}(S^1)$ we may select a δ-net $F \in G_0$. We put $B_0 = P_F$ and we select an $\varepsilon_0 > 0$ such that $\tilde{U}_{\varepsilon_0}(F)$ is contained in G_0 and such that for any $X \in \mathcal{K}(D^1_0)$, $\tilde{\rho}(X, B_0) \le \varepsilon_0$ implies that X is contained in $D' = \{w \in D^1 : \frac{1}{8} \le |w| \le \frac{7}{8}\}$. Consequently, every B_n will be a subset of D'. The applicable induction hypotheses are obviously satisfied.

Let us assume that B_n and ε_n have been found. We now describe the procedure for constructing B_{n+1}, assuming that n is even. The procedure for n odd is simply the mirror image (interchange α and β). The set B_n is the union of the finite collection $\{E_1, \ldots, E_k\}$, where $E_j = \alpha^{-1}(p_j) \cap \beta^{-1}([u_j, v_j])$ for some $p_j, u_j, v_j \in S^1$. Consider E_j and pick a $\delta \le \varepsilon_n/4$. We select a $\gamma_j \le \delta$ such that every finite $F \subset S^1$ with $\tilde{d}(F, U_\delta([u_j, v_j])) \le \gamma_j$ has the property that

$$Q_j(F) = \alpha^{-1}(U_\delta(p_j)) \cap \beta^{-1}(F)$$

intersects every $C \in \mathcal{C}$ that intersects E_j (see Figure 4).

Figure 4

Let $\gamma = \min\{\gamma_1, \dots, \gamma_k\}$. Select a finite set $F \in G_{n+1}$ such that

$$\tilde{d}\Big(F, \bigcup_{j=1}^{k} U_\delta([u_j, v_j])\Big) < \gamma.$$

We define the compactum

$$B_{n+1} = \bigcup_{j=1}^{k} Q_j(F_j),$$

where $F_j = F \cap U_{\delta+\gamma}([u_j, v_j])$. Let $\varepsilon_{n+1} > 0$ be such that $\varepsilon_{n+1} < \varepsilon_n/2$ and $\tilde{U}_{\varepsilon_{n+1}}(F) \subset G_{n+1}$.

Since every $Q_j(F_j)$ is obviously a finite union of sets of the form $\beta^{-1}(w) \cap \alpha^{-1}(U_\delta(p))$ the set B_{n+1} satisfies condition (7). Condition (1) is trivially true. Note that $\beta(Q_j(F_j)) = F_j$ and that $F = \bigcup_{j=1}^{k} F_j$. Consequently, $\beta(B_{n+1}) = F$ and condition (5) is satisfied. In addition, $\tilde{d}(F_j, U_\delta([u_j, v_j])) \le \gamma \le \gamma_j$ so if $C \in \mathcal{C}$ intersects E_j then C intersects $Q_j(F_j)$ and B_{n+1}. By hypothesis, every $C \in \mathcal{C}$ intersects some E_j and hence (3) is valid for $n+1$. We have

$$\tilde{d}(\{p_j\}, U_\delta(p_j)) = \delta \quad \text{and} \quad \tilde{d}([u_j, v_j], F_j) \le \delta + \gamma \le 2\delta.$$

Since E_j and $Q_j(F_j)$ are essentially Cartesian products of these sets and $\tilde{\rho}$ corresponds to the product max metric it follows that $\tilde{\rho}(E_j, Q_j(F_j)) \le 2\delta \le \varepsilon_n/2$ (see Figure 4). Consequently, we have $\tilde{\rho}(B_n, B_{n+1}) \le \varepsilon_n/2$ and the induction is complete.

Induction hypotheses (1) and (2) show that $B_0, B_1, \dots$ is a Cauchy sequence with respect to $\tilde{\rho}$. In addition they show that $\tilde{\rho}(B_0, B_n) \le \varepsilon_0$ for each n and hence we have a Cauchy sequence in the compact space $\mathcal{K}(D')$. Put

$$B = \lim_{n\to\infty} B_n \in \mathcal{K}(D') \quad \text{and} \quad A = \alpha(B) \cup \beta(B) \in \mathcal{K}(S^1).$$

Obviously, we have $B \subset \mathfrak{B}(A)$. Since every $C \in \mathcal{C}$ intersects every B_n we have by compactness that B intersects every $C \in \mathcal{C}$. So $\mathfrak{B}(A)$ intersects every element of $\mathcal{C}$ in a point other than the origin. As argued above this result implies that A is not extendable.

Let n be even and $k \in \mathbf{N}$. Observe that

$$\tilde{d}(\alpha(B_n), \alpha(B_{n+k})) \leq \tilde{\rho}(B_n, B_{n+k}) \leq \sum_{i=1}^{k} \varepsilon_n 2^{-i} \leq \varepsilon_n.$$

So $\alpha(B_{n+k}) \in \tilde{U}_{\varepsilon_n}(\alpha(B_n))$ and $\alpha(B) = \lim_{k\to\infty} \alpha(B_{n+k}) \in \tilde{U}_{\varepsilon_n}(B_n)$. By condition (4) we have $\alpha(B) \in G_n$ and hence $\alpha(B) \in \bigcap_{n=0}^{\infty} G_{2n} = G$. The same argument for n odd yields $\beta(B) \in G$.

Theorem 2.2 is sharp in the sense that there exist dense G_δ-subsets of $\mathcal{K}(S^1)$ consisting entirely of extendable elements (see Proposition 3.7).

A realvalued function is called *upper semicontinuous* if the preimage of every interval of the form $(-\infty, t)$ is open. We call a function $\gamma : \mathcal{K}(X) \to [0, \infty]$ *null-subadditive* if $\gamma(A) = \gamma(B) = 0$ implies $\gamma(A \cup B) = 0$. Measures and capacities satisfy these conditions.

Theorem 2.3. *If $\gamma : \mathcal{K}(S^1) \to [0, \infty]$ is a null-subadditive and upper semicontinuous function that vanishes on finite sets then there exists a nonextendable $C \in \mathcal{K}(S^1)$ with $\gamma(C) = 0$.*

Proof. The set $\gamma^{-1}(0)$ is a dense G_δ in $\mathcal{K}(S^1)$ that is closed under unions.

Note that instead of upper semicontinuity it suffices to know that sets of the form $\gamma^{-1}([0, \varepsilon))$ are neighbourhoods of sets with γ equal to zero.

Corollary 2.4. *There exists a nonextendable element of $\mathcal{K}(S^1)$ with linear Lebesgue measure zero.*

It is well known that capacities satisfy the premise of Theorem 2.3. The most interesting capacity in the plane is the logarithmic or Newtonian capacity (see e.g. [4, 12]).

Corollary 2.5. *There exists a nonextendable element of $\mathcal{K}(S^1)$ with logarithmic capacity zero.*

Let us have a look at Hausdorff type measures. For every $h : [0, t_0] \to [0, \infty)$ we define the *h-measure* m_h of a subset X of $\mathbf{R}^n$ by

$$m_h(X) = \lim_{\varepsilon \searrow 0} \inf \Big\{ \sum_{i=1}^{\infty} h(\mathrm{diam}(A_i)) : \{A_i : i \in \mathbf{N}\} \text{ is a covering in } \mathbf{R}^n \text{ of } X \text{ with sets of diameter at most } \varepsilon \Big\}.$$

$\mathbf{R}^n$ is assumed to be equipped with the metric that is generated by the standard norm $\sqrt{\sum_{i=1}^n x_i^2}$. If $s > 0$ then the *Hausdorff s-measure* $\mathcal{H}^s$ equals m_h, where $h(t) = t^s$. Recall that the *Hausdorff dimension* of X is defined by

$$\dim_{\mathrm{H}}(X) = \inf\{s : \mathcal{H}^s(X) = 0\}.$$

Since a set with vanishing logarithmic capacity has Hausdorff dimension zero (see [4, Theorem 3.13]) we have:

Corollary 2.6. *There exists a nonextendable element of $\mathcal{K}(S^1)$ with Hausdorff dimension zero.*

Note that if $A \in \mathcal{K}(S^1)$ has $\dim_{\mathrm{H}}(A) = 0$ then $\dim_{\mathrm{H}}(A \times S^1) = 1$ and hence $\dim_{\mathrm{H}}(A \times A) \leq 1$. So we have the following result whose significance will become apparent in the next section.

Corollary 2.7. *There exists a nonextendable $A \in \mathcal{K}(S^1)$ with $\dim_{\mathrm{H}}(A \times A) \leq 1$.*

Corollary 2.6 also follows if we substitute $h(t) = 1/\log(1/t)$ into the following general statement about h-measures.

Corollary 2.8. *There exists a nonextendable $C \in \mathcal{K}(S^1)$ with $m_h(C) = 0$ if and only if $\liminf_{t \searrow 0} h(t) = 0$.*

Proof. If $\liminf_{t \searrow 0} h(t) > 0$ and $m_h(C) = 0$ then C is countable. Every partial two point set with cardinality $< \mathfrak{c}$ is extendable.

Assume that $\liminf_{t \searrow 0} h(t) = 0$. Let for each $n \in \mathbf{N}$ the set O_n be the interior in $\mathcal{K}(S^1)$ of the collection

$$\Big\{C \in \mathcal{K}(S^1) : \text{there is a covering } \{A_i : i \in \mathbf{N}\} \text{ in } \mathbf{R}^2 \text{ of } C \\ \text{with sets of diameter at most } t_0/n \\ \text{such that } \sum_{i=1}^{\infty} h(\operatorname{diam}(A_i)) \leq 1/n\Big\}.$$

Obviously, every element of $G = \bigcap_{n=1}^{\infty} O_n$ has m_h equal to 0. Let $F \in \mathcal{K}(S^1)$ be a finite set with m points and let $n \in \mathbf{N}$. Select an $\varepsilon > 0$ such that $\varepsilon \leq t_0/n$ and $h(\varepsilon) \leq (mn)^{-1}$. Let $C \in \mathcal{K}(S^1)$ be a set whose distance towards F in the Hausdorff metric is less than $\varepsilon/2$. Then C can be covered by intervals $A_1, \ldots, A_m$ each with diameter equal to ε. Consequently,

$$\sum_{i=1}^{m} h(\operatorname{diam}(A_i)) = mh(\varepsilon) \leq \frac{1}{n}$$

and we may conclude that $F \in O_n$. So G is a G_δ in $\mathcal{K}(S^1)$ which contains all finite sets and Theorem 2.2 applies. It is easily seen that if $A, B \in O_{2n}$ then $A \cup B \in O_n$. This implies that G is closed under unions.

Corollary 2.8 allows us to improve upon Corollary 2.7 which follows if we substitute $h(t) = t/\log(1/t)$ in

Proposition 2.9. *If every $A \in \mathcal{K}(S^1)$ with $m_h(A \times A) = 0$ is extendable then the following two equivalent statements are valid:*

(1) *$m_h(X) = 0$ implies $\mathcal{H}^1(X) = 0$ for every X,*
(2) *$\liminf_{t \searrow 0} h(t)/t > 0$.*

Proof. First we verify the equivalence of (1) and (2). Assume first that (2) is valid. Then there are $\varepsilon, \delta > 0$ such that $h(t) \geq \delta t$ for $0 < t < \varepsilon$. Consequently, $m_h(X) \geq \delta \mathcal{H}^1(X)$ for any X which implies statement (1). Consider now the case $\liminf_{t \searrow 0} h(t)/t = 0$. Select a sequence $t_1, t_2, \dots$ of positive numbers less than 1 and t_0 such that $\lim_{i\to\infty} t_i = 0$ and $\lim_{i\to\infty} h(t_i)/t_i = 0$. If I is the unit interval $[0,1]$ then $\mathcal{H}^1(I) = 1$. For every $i \in \mathbf{N}$ put $n_i = \lceil 1/t_i \rceil$ and write I as a union of a collection $\mathcal{B}_i$ consisting of n_i intervals, each with length equal to t_i. We have

$$\begin{aligned} m_h(I) &\leq \lim_{i\to\infty} \sum_{B \in \mathcal{B}_i} h(\operatorname{diam}(B)) = \lim_{i\to\infty} n_i h(t_i) \\ &\leq \lim_{i\to\infty} (1 + t_i) \frac{h(t_i)}{t_i} = 0. \end{aligned}$$

Assume now that $\liminf_{t \searrow 0} h(t)/t = 0$. We define $h'(t) = h(t)/t$ for $0 < t \leq t_0$ and $h'(0) = 0$ and we apply Corollary 2.8 to $m_{h'}$ yielding a nonextendable $A \in \mathcal{K}(S^1)$ with $m_{h'}(A) = 0$. We will prove that $m_h(A \times A) = 0$ which violates the premise of the proposition. Let $0 < \varepsilon < \min\{t_0, 1\}$ and select a collection $\{A_1, A_2, \dots\}$ of subsets of S^1 such that $A \subset \bigcup_{i=1}^\infty A_i$, $\operatorname{diam}(A_i) < \varepsilon$, and $\sum_{i=1}^\infty h'(\operatorname{diam}(A_i)) \leq \varepsilon$. Put $a_i = \operatorname{diam}(A_i)$ for each $i \in \mathbf{N}$. Since $\liminf_{t \searrow 0} h'(t) = 0$ we may assume that every a_i is positive. Define for every i the natural number $n_i = \lceil 2\pi\sqrt{2}/a_i \rceil$. Partition S^1 into n_i sets $D_i^1, \dots, D_i^{n_i}$ each with diameter no greater than $a_i/\sqrt{2}$. Write every A_i as a union of two sets B_i and C_i with diameter at most $a_i/\sqrt{2}$. Note that in $\mathbf{R}^4$ $\operatorname{diam}(B_i \times D_i^j) \leq a_i$ and find a $\tilde{B}_i^j$ such that $B_i \times D_i^j \subset \tilde{B}_i^j$ and $\operatorname{diam}(\tilde{B}_i^j) = a_i$. Analogously, let $C_i \times D_i^j \subset \tilde{C}_i^j$ and $\operatorname{diam}(\tilde{C}_i^j) = a_i$.

Note that

$$\mathcal{D} = \{\tilde{B}_i^j : i \in \mathbf{N}, 1 \leq j \leq n_i\} \cup \{\tilde{C}_i^j : i \in \mathbf{N}, 1 \leq j \leq n_i\}$$

is a countable covering of $A \times A$ with sets of diameter less than ε. Consider the sum

$$\begin{aligned}\sum_{D\in\mathcal{D}} h(\operatorname{diam}(D)) &= \sum_{i=1}^{\infty}\sum_{j=1}^{n_i}(h(\operatorname{diam}(\tilde{B}_i^j)) + h(\operatorname{diam}(\tilde{C}_i^j))\\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{n_i} 2a_i h'(a_i) = \sum_{i=1}^{\infty} 2n_i a_i h'(a_i)\\ &\le \sum_{i=1}^{\infty} 2(2\pi\sqrt{2}+a_i)h'(a_i) \le 2(2\pi\sqrt{2}+1)\sum_{i=1}^{\infty} h'(a_i)\\ &\le 2(2\pi\sqrt{2}+1)\varepsilon.\end{aligned}$$

Since ε can be chosen arbitrarily small, $m_h(A \times A) = 0$ and the proposition is proved.

The following proposition improves upon [1]. Let σ stand for the arc length measure on S^1.

Proposition 2.10. *If A is a measurable and extendable subset of S^1 then $\sigma(A) = 0$.*

Proof. Let A be a subset of $S^1 \cap B$ where $\sigma(A) > 0$ and B is a two point set. According to [11, 7.12] almost every $u \in A$ has the property

$$\lim_{\varepsilon \searrow 0} \frac{\sigma(A \cap U_\varepsilon(u))}{\sigma(U_\varepsilon(u))} = 1.$$

Let u be such a Lebesgue point and let ℓ be the tangent line to S^1 at u. Since B is a two point set we can find a $v \in B \cap \ell$ such that $v \neq u$. The two tangent lines to S^1 through v divide the circle into two open segments E and F as in Figure 5.

Figure 5

Let p be an element of E and let $f : E \to F$ be the radial projection centered at v. Note that f is a contraction so $\sigma(f(C)) \le \sigma(C)$ for any measurable $C \subset E$. (Since a line has the same angle of intersection with a circle at both points the contraction factor of f at p is simply the ratio $\frac{|f(p)-v|}{|p-v|}$.) Consequently, we have

$$\lim_{p\to u} \frac{\sigma([u,p])}{\sigma([f(p),u])} = \lim_{p\to u} \frac{\sigma([u,p])}{\int_u^p \frac{|f(q)-v|}{|q-v|}\, d\sigma(q)} = \lim_{p\to u} \frac{|p-v|}{|f(p)-v|} = 1$$

and

$$\lim_{p\to u}\frac{\sigma([f(p),u]\setminus f(A\cap E))}{\sigma([u,p])}\leq\lim_{p\to u}\frac{\sigma([u,p]\setminus A)}{\sigma([u,p])}=0.$$

Note that

$$\lim_{p\to u}\frac{\sigma([f(p),u]\setminus(A\cap f(A\cap E)))}{\sigma([f(p),u])}\leq\lim_{p\to u}\frac{\sigma([f(p),u]\setminus A)}{\sigma([f(p),u])}$$
$$+\lim_{p\to u}\frac{\sigma([f(p),u]\setminus f(A\cap E))}{\sigma([u,p])}\cdot\lim_{p\to u}\frac{\sigma([u,p])}{\sigma([f(p),u])}=0.$$

This result obviously implies that $\sigma(A\cap f(A\cap E))>0$ and we can choose a $p\in A\cap E$ such that $f(p)\in A$. Then p, $f(p)$, and v are three collinear elements of the two point set B and we have arrived at the contradiction that proves the proposition.

The following observation is based on a suggestion by W. Rudin.

Proposition 2.11. *If C is a partial two point set with $\dim(C)>0$ then every dense G_δ-subset of C is nonextendable.*

Proof. Let C be a partial two point set with $\dim(C)>0$. According to Kulesza [3, Theorem 1] C contains an arc so let $\alpha: I\to C$ be an imbedding. Kulesza also shows in the proof of [3, Lemma 3] that there is a line ℓ that meets C in a single point $u\in\alpha((0,1))$ such that the arc $\alpha(I)$ is contained in one of the half planes determined by ℓ. Let $r\in(0,1)$ be such that $u=\alpha(r)$. Assume that A is a dense G_δ-subset of C which is contained in some two point set B.

There exists a point $v\in B\cap\ell$ which is distinct from u. By connectedness we have that the line through v and $\alpha(0)$ intersects $\alpha((r,1])$ or that the line through v and $\alpha(1)$ intersects $\alpha((r,0])$. If necessary shorten the arc on one end so that we obtain that $\alpha(0)$, $\alpha(1)$, and v are collinear. By connectivity and the fact that $\alpha(I)$ is a partial two point set we find that for every $p\in\alpha([0,r])$ the line through p and v intersects the arc $\alpha([r,1])$ in precisely one point which we denote by $f(p)$. By compactness $f:\alpha([0,r])\to\alpha([r,1])$ is a homeomorphism. Since C is a partial two point set every line through v and an arbitrary point $p\in\alpha((0,r))$ intersects C only in the points p and $f(p)$. Consequently, $E=\alpha((0,r))$ and $F=\alpha((r,1))$ are open subsets of C and hence $E\cap A$ and $F\cap A$ are dense G_δ-subsets of E respectively F. So we have that $E\cap A\cap f^{-1}(F\cap A)$ is a dense G_δ in E and we may select a $p\in E\cap A$ such that $f(p)\in A$. Then p, $f(p)$, and v are three collinear elements of the two point set B, a contradiction.

The following generalization of Proposition 2.10 seems reasonable:

Conjecture. *Every σ-compact subset of a two point set has Hausdorff 1-measure zero.*

We will find in the next section that the existence of extendable elements of $\mathcal{K}(S^1)$ with positive Hausdorff dimension and capacity is consistent with ZFC.

3. Extendable sets

Several of the results in this section use Martin's Axiom. MA is weaker than the continuum hypothesis and it implies that no compact metric space can be written as the union of less than $\mathfrak{c} = 2^\omega$ nowhere dense subsets. For more information on MA see e.g. [10].

We call a partial two point set A a *pre-2-point set* if every line ℓ in the plane such that $\ell \setminus \mathfrak{L}(A)$ is nowhere dense in ℓ intersects A in two points. According to Kulesza [3] every two point set is zero-dimensional. Kulesza shows in effect that if A is a partial two point set with $\dim(A) > 0$ then there is a line $\ell \subset \mathfrak{L}(A)$ that intersects A in only one point (see the proof of Lemma 3 in [3]). Note that this result also proves that every pre-2-point set is zero-dimensional.

Theorem 3.1. (MA) *Every σ-compact pre-2-point set is extendable.*

Proof. Let A be a σ-compact pre-2-point set. Consider the σ-compactum $E = \{(u, v) \in A \times A : u \neq v\} \times \mathbf{R}$ and the map $g : E \to \mathbf{R}^2$ defined by $g(u, v, t) = u + t(v - u)$. Observe that $\mathfrak{L}(A) = g(E)$ and hence the set is σ-compact.

Let $\{\ell_\alpha : \alpha < \mathfrak{c}\}$ enumerate the lines in the plane. We shall construct by transfinite induction a nondecreasing sequence $(C_\alpha)_{\alpha \leq \mathfrak{c}}$ of subsets of $\mathbf{R}^2 \setminus A$ with induction hypotheses:

(1) $|C_\alpha| \leq |\alpha| + \omega$,
(2) $A \cup C_\alpha$ is a partial two point set.

Put $C_0 = \emptyset$ and if $\lambda \leq \mathfrak{c}$ is a limit ordinal then $C_\lambda = \bigcup_{\alpha<\lambda} C_\alpha$. Let α be a fixed ordinal $< \mathfrak{c}$ and consider C_α and ℓ_α. Assume for the moment that $(A \cup C_\alpha) \cap \ell_\alpha$ contains at most one point. Let I be a nondegenerate compact interval in ℓ_α such that $I \subset \mathrm{cl}(\ell_\alpha \setminus \mathfrak{L}(A))$. Note that since $I \cap \mathfrak{L}(A)$ is σ-compact it is a category I set in I.

Let u be a point in C_α. If $u \in \ell_\alpha$ then we put $B_u = \{u\} \cap I$; otherwise we define
$$B_u = \{v \in I : L(u, v) \cap A \neq \emptyset\}.$$
Let $u \notin \ell_\alpha$ and note that $D = \bigcup\{L(u, v) : v \in I\}$ is closed. Since A is a zero-dimensional σ-compactum, $D \cap A$ can be written as a countable union $\bigcup_{i=1}^\infty D_i$ of zero-dimensional compacta. Let $f : D \setminus \{u\} \to I$ be the

projection defined by $\{f(z)\} = L(u,z) \cap I$. Since $D_i \cup \{u\}$ is a subset of $A \cup C_\alpha$ it is also a partial two point set and $f|D_i$ is one-to-one and hence an imbedding. Consequently, $B_u = \bigcup_{i=1}^{\infty} f(D_i)$ is a countable union of zero-dimensional compacta. So B_u is of category I in I for each $u \in C_\alpha$.

Consider now two distinct points u and v in C_α. Since u and v do not both lie on ℓ_α the set $B_{uv} = L(u,v) \cap I$ contains at most one point. Note that

$$\mathcal{D} = \{I \cap \mathfrak{L}(A)\} \cup \{B_u : u \in C_\alpha\} \cup \{B_{uv} : u, v \in C_\alpha \text{ and } u \neq v\}$$

is a collection of category I subsets of I with $|\mathcal{D}| \leq |\alpha| + \omega < \mathfrak{c}$ and that $\bigcup \mathcal{D} = I \cap \mathfrak{L}(A \cup C_\alpha)$. Martin's Axiom implies that $I \setminus \bigcup \mathcal{D}$ is dense in I and we can select two distinct points a and b in that dense set.

Since $A \cup C_\alpha$ is a partial two point set we may define

$$C_{\alpha+1} = \begin{cases} C_\alpha \cup \{a,b\}, & \text{if } |(A \cup C_\alpha) \cap \ell_\alpha| = 0 \\ C_\alpha \cup \{a\}, & \text{if } |(A \cup C_\alpha) \cap \ell_\alpha| = 1 \\ C_\alpha, & \text{if } |(A \cup C_\alpha) \cap \ell_\alpha| = 2. \end{cases}$$

By the construction it is obvious that $|C_{\alpha+1}| \leq |\alpha + 1| + \omega$ and that $A \cup C_{\alpha+1}$ is a partial two point set which intersects ℓ_α in two points. Then $A \cup C_\mathfrak{c}$ is a two point set.

Theorem 3.2. (MA) *Any σ-compact partial two point set A with $\mathcal{H}^1(\mathfrak{B}(A)) = 0$ is extendable.*

Proof. First we show that A is zero-dimensional. Assume that $\dim(A) \geq 1$. Select a $u \in A$ such that $u \neq 0$. Then $A \setminus \{u\}$ is a σ-compactum with $\dim \geq 1$ and hence it contains a compactum B with $\dim(B) \geq 1$. Select a nontrivial continuum $C \subset B$. Define the continuous map $h : C \to \mathbf{R}^2$ by $h(v) = p(\varphi(v-u), v)$. So $h(C)$ is a continuum in $\mathfrak{B}(A)$ and hence $\operatorname{diam}(h(C)) \leq \mathcal{H}^1(h(C)) \leq \mathcal{H}^1(\mathfrak{B}(A)) = 0$. Let v and w be distinct elements of C and note that $h(v) = h(w) \in L(u,v) \cap L(u,w)$. Since A is a partial two point set u, v, and w are not collinear which implies that $L(u,v) \cap L(u,w) = \{u\}$. So $u = h(v) = h(w)$ which means since $u \neq 0$ that v and w lie on the line through u that is perpendicular to the line through 0 and u. So u, v, and w are collinear, a contradiction.

In order to show that A is a pre-2-point set let ℓ be a line that intersects A in less than two points. We shall prove that $\mathfrak{L}(A) \cap \ell$ is of category I in ℓ. By rotating the xy-axes we can arrange that ℓ is the line $x = a$. Define a function f from $D = \{(x,y) \in \mathbf{R}^2 : y \neq 0\}$ to ℓ by $f(x,y) = (a, (x^2 + y^2 - ax)/y)$. Note that if u is a vector from D then u and $f(u) - u$ are perpendicular. Since A is σ-compact $\mathfrak{B}(A)$ and $\mathfrak{B}(A) \cap D$

are σ-compact. Write $P = \mathfrak{B}(A) \cap D = \bigcup_{i=1}^{\infty} P_i$ where the P_i's are compact. Since f is analytic the maps $f|P_i$ are Lipschitz and we have $\mathcal{H}^1(f(P_i)) = 0$ because $\mathcal{H}^1(P_i) \leq \mathcal{H}^1(\mathfrak{B}(A)) = 0$. Consequently, $f(P)$ is a countable union of zero-dimensional compacta and hence it is a category I set in ℓ.

Consider the case $a = 0$, i.e. ℓ is the y-axis. Let $u \neq 0$ be an element of $\mathfrak{L}(A) \cap \ell$. There are v and w in A such that $u \in L(v, w)$. Since ℓ intersects A in at most one point we know that $L(u, v)$ is not vertical. Note that $z = p(\varphi(v - w), v)$ is an element of P such that $f(z) = u$. We may conclude that $\mathfrak{L}(A) \cap \ell$ is a subset of $\{0\} \cup f(P)$ and hence it is a category I set in ℓ.

Assume now that $a \neq 0$. Consider the zero-dimensional σ-compactum $A' = A \cap (D \cup \{0\})$. Define the set

$$R = \{(u, v) \in A' \times A' : u, v, 0 \text{ collinear and } u \neq v\}.$$

Since R is the intersection of an open and a closed subset of $A' \times A'$ it is also zero-dimensional and σ-compact. Since R is disjoint from the diagonal of $A' \times A'$ we can write R as a countable union of compacta $R_1, R_2, \ldots$, where $(u, v) \in R_i$ implies $(v, u) \notin R_i$. Define the continuous map $g : R \to \ell$ by $\{g(u, v)\} = L(u, v) \cap \ell$. Let (u, v) and (w, z) be elements of R_i such that $g(u, v) = g(w, z)$. Then $L(u, v)$ and $L(w, z)$ have the origin and $g(u, v)$ in common so they coincide. Since A is a partial two point set we have $\{u, v\} = \{w, z\}$ and hence $(u, v) = (w, z)$. Consequently, $g|R_i$ is one-to-one and an imbedding. This implies that $g(R)$ is a countable union of zero-dimensional compacta and hence a category I set in ℓ. Let v and w be elements of A such that $L(v, w)$ contains some $u \in \ell$. If $L(v, w)$ does not go through the origin then by the same argument as given above we have $u \in f(P)$. If $L(v, w)$ does contain the origin then $(v, w) \in R$ and $g(v, w) = u$. So we may conclude that $\mathfrak{L}(A) \cap \ell \subset f(P) \cup g(R)$ and hence it is a category I set in ℓ.

According to section 2 there exist nonextendable compact partial two point sets with Hausdorff 1-measure zero (or even with Hausdorff dimension zero). On the other hand we have:

Theorem 3.3. (MA) *Any σ-compact partial two point set A with $\mathcal{H}^1(A \times A) = 0$ is extendable.*

Proof. We assume that $A \times A$ is equipped with the standard metric it inherits from $\mathbf{R}^4$. Let $\Delta = \{(u, v) \in \mathbf{R}^2 \times \mathbf{R}^2 : u = v\}$ and define the map $f : \mathbf{R}^4 \setminus \Delta \to \mathbf{R}^2$ by $f(u, v) = p(\varphi(u - v), u)$ for $u, v \in \mathbf{R}^2$, $u \neq v$. Obviously, $B = A \times A \setminus \Delta$ can be written as a countable union of compacta $B_1, B_2, \ldots$. Since f is analytic $f|B_i$ is Lipschitz and we may conclude that $\mathcal{H}^1(f(B_i)) = 0$ because $\mathcal{H}^1(B_i) \leq \mathcal{H}^1(A \times A) = 0$ for each $i \in \mathbf{N}$. By σ-additivity of Hausdorff measures we have $\mathcal{H}^1(\mathfrak{B}(A)) = \mathcal{H}^1(\bigcup_{i=1}^{\infty} f(B_i)) = 0$ and Theorem 3.2 applies.

According to Corollary 2.7 there exist nonextendabele elements A of $\mathcal{K}(S^1)$ such that $\dim_{\mathrm{H}}(A \times A) \leq 1$. So Theorem 3.3 is sharp in the sense that we cannot replace the condition $\mathcal{H}^1(A \times A) = 0$ by $\forall s > 1, \mathcal{H}^s(A \times A) = 0$. In fact, the following result, which combines Theorem 3.3 and Proposition 2.9, shows that $\mathcal{H}^1$ is the optimum choice among the h-measures.

Theorem 3.4. (MA) *If h is an arbitrary function from $[0, t_0]$ into $[0, \infty)$ then the following statements are equivalent:*

1. *Every σ-compact partial two point set A with $m_h(A \times A) = 0$ is extendable,*
2. *$m_h(X) = 0$ implies $\mathcal{H}^1(X) = 0$ for every X,*
3. *Every $A \in \mathcal{K}(S^1)$ with $m_h(A \times A) = 0$ is extendable,*
4. *$\liminf_{t \searrow 0} h(t)/t > 0$.*

Note that $\mathcal{H}^1(A \times A)$ is an upper semicontinuous function of A that vanishes on finite sets. Consequently, the null-subadditivity condition in Theorem 2.3 is essential and we have

Corollary 3.5. (MA) *If X is any partial two point set then there exists a dense G_δ-subset of $\mathcal{K}(X)$ consisting entirely of extendable sets.*

So the nonextendable sets in $\mathcal{K}(X)$ form merely a category I set.

We now describe a standard procedure for constructing a Cantor set C. Let C_0 be a continuum. As step 1 we replace C_0 by n_1 disjoint subcontinua $C_1^1, \ldots, C_1^{n_1}$. In general, for $i \in \mathbf{N}$ we replace every C_{i-1}^j, for $1 \leq j \leq n_1 \cdots n_{i-1}$, by n_i disjoint subcontinua. This produces a set $C_i = C_i^1 \cup \cdots \cup C_i^k$, where $k = n_1 \cdots n_i$. Define

$$d_i = \max\{\mathrm{diam}(C_i^j) : 1 \leq j \leq n_1 \cdots n_i\}$$

and $r_i = d_i / d_{i-1}$. C is then the intersection $\bigcap_{i=0}^{\infty} C_i$.

Corollary 3.6. (MA) *Let C be a planar Cantor set as constructed above. If C is a partial two point set such that $\prod_{i=1}^{\infty} n_i^2 r_i = 0$ then C is extendable.*

Proof. Consider the covering

$$\mathcal{B}_i = \{C_i^j \times C_i^k : 1 \leq j, k \leq n_1 \cdots n_i\}$$

of $C \times C$. Let $d_0 = \operatorname{diam}(C_0)$ and let $\operatorname{mesh}(\mathcal{B}_i) = \max\{\operatorname{diam}(B_i) : B \in \mathcal{B}_i\}$. We have

$$\lim_{i\to\infty} \operatorname{mesh}(\mathcal{B}_i) \le \lim_{i\to\infty} \sqrt{2}\, d_0 r_1 \cdots r_i \le \sqrt{2}\, d_0 \prod_{i=1}^{\infty} n_i^2 r_i = 0.$$

In addition,

$$\lim_{i\to\infty} \sum_{B\in\mathcal{B}_i} \operatorname{diam}(B) \le \lim_{i\to\infty} |\mathcal{B}_i| \operatorname{mesh}(\mathcal{B}_i) \le \lim_{i\to\infty} (n_1 \cdots n_i)^2 \sqrt{2}\, d_0 r_i \cdots r_i$$
$$= \sqrt{2}\, d_0 \prod_{i=1}^{\infty} n_i^2 r_i = 0.$$

Consequently, $\mathcal{H}^1(C \times C) = 0$ and we may apply Theorem 3.3.

For instance, if we start with an interval in S^1 and we obtain C_{i+1} from C_i by deleting the middle three fifth of every interval of C_i then $n_i = 2$ and $r_i = 1/5$. Consequently, $\prod_{i=1}^{\infty} n_i^2 r_i = (4/5)^\infty = 0$ and the resulting Cantor set is extendable to a two point set. In addition, one may verify that the set has positive Hausdorff dimension and hence positive logarithmic capacity.

Corollary 3.6 gives us many Cantor sets that are extendable to two point sets provided that Martin's Axiom is valid, suggesting the following question: is the existence of a two point set that contains a Cantor set provable in ZFC? Noting that every dense G_δ in $\mathcal{K}(S^1)$ contains Cantor sets we have an affirmative answer in the following result.

Proposition 3.7. *There exists a dense G_δ-subset of $\mathcal{K}(S^1)$ consisting entirely of extendable sets.*

We need some definitions (cf. [5, §VII.1]) and two technical lemmas. If A is a subset of a field then the subfield generated by A is denoted by $\mathbf{Q}(A)$. A subset A of a field is called *algebraically independent* if every $x \in A$ is transcendental over the field $\mathbf{Q}(A \setminus \{x\})$. A *transcendence base* A for a field F is an algebraically independent subset such that F is algebraic in $\mathbf{Q}(A)$. The *transcendence degree* of a field F over a subset A equals $\min\{|B| : B \subset F$ and F is algebraic in $\mathbf{Q}(A \cup B)\}$.

Lemma 3.8. *If X and Y are spaces, F is a closed subset of X, O is an open subset of Y, and $f : F \to Y$ is continuous then the set $\mathcal{O} = \{K \in \mathcal{K}(X) : f(K \cap F) \subset O\}$ is open in $\mathcal{K}(X)$.*

Proof. The set $f^{-1}(O)$ is open in F and hence $f^{-1}(O) \cup (X \setminus F)$ is open in X. Note that $\mathcal{O} = \{K \in \mathcal{K}(X) : K \subset f^{-1}(O) \cup (X \setminus F)\}$ and hence $\mathcal{O}$ is obviously open.

Lemma 3.9. *If D is an algebraically independent compact subset of $\mathbf{R}$ such that $\mathbf{R}$ has infinite transcendence degree over D then*

$$\mathcal{G}_D = \{C \in \mathcal{K}(\mathbf{R} \setminus D) : D \cup C \text{ is algebraically independent}\}$$

is a dense G_δ-subset of $\mathcal{K}(\mathbf{R})$.

Proof. Let D be such a subset of $\mathbf{R}$ and let $p_{1n}, p_{2n}, \dots$ be an enumeration of all nonzero polynomials in n variables with integral coefficients. Let Δ_n be the closed subset of $\mathbf{R}^n$ that consists of all points with at least two identical coordinates. Define for $m, n \in \mathbf{N}$ the following set of compacta:

$$G_{mn} = \{C \in \mathcal{K}(\mathbf{R} \setminus D) : 0 \notin p_{mn}((D \cup C)^n \setminus \Delta_n)\}.$$

Note that $\mathcal{G}_D = \bigcap\{G_{mn} : m, n \in \mathbf{N}\}$. Write $\mathbf{R}^n \setminus \Delta_n$ as a countable union of compacta $K_1 \subset K_2 \subset \dots$. Fix $m, n \in \mathbf{N}$. Consider for each $i \in \mathbf{N}$ the set

$$U_i = \{C \in \mathcal{K}(\mathbf{R} \setminus D) : 0 \notin p_{mn}((D \cup C)^n \cap K_i)\}.$$

According to Lemma 3.8 every U_i is open in $\mathcal{K}(\mathbf{R})$. Since $G_{mn} = \bigcap_{i=1}^\infty U_i$ it is a G_δ-set and hence $\mathcal{G}_D$ is a G_δ-set as well.

Let $F = \{x_1, \dots, x_n\}$ be a finite set in $\mathbf{R}$ and let $\varepsilon > 0$. We define by induction sets $C_0, \dots, C_n$ with $|C_i| = i$. Put $C_0 = \emptyset$ and let $0 \le i < n$. Since $\mathbf{R}$ has infinite transcendence degree over D there exists an $a \in \mathbf{R}$ that is transcendental over $\mathbf{Q}(D \cup C_i)$. Consequently, every nonzero element of the dense set $\mathbf{Q}a$ is transcendental over $\mathbf{Q}(D \cup C_i)$ and we can select a $y_{i+1} \in \mathbf{Q}a \setminus \{0\}$ that is ε-close to x_{i+1}. Put $C_{i+1} = C_i \cup \{y_{i+1}\}$. Note that C_n is ε-close to F in the Hausdorff metric and that $C_n \in \mathcal{G}_D$. So the closure of $\mathcal{G}_D$ contains all finite sets and hence it is $\mathcal{K}(\mathbf{R})$.

Proof of Proposition 3.7. Let A be an algebraically independent Cantor set in $\mathbf{R}$. The existence of such sets is known and follows if we substitute $D = \emptyset$ in Lemma 3.9. Let D be a Cantor set that is a proper subset of A. Then the transcendence degree of $\mathbf{R}$ over $\mathbf{Q}(D)$ is $\mathfrak{c} = |A \setminus D|$, see [5,Theorem VII.1.1]. By Lemma 3.9 $\mathcal{G}_D$ is a dense G_δ in $\mathcal{K}(\mathbf{R})$. Define $\pi_1, \pi_2 : \mathbf{R}^2 \to \mathbf{R}$ by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. It is easily seen that $G = \{C \in \mathcal{K}(S^1) : \pi_1(C) \in \mathcal{G}_D\}$ is a dense G_δ in $\mathcal{K}(S^1)$.

We show that every $C \in G$ is extendable to a two point set. $\pi_1(C) \cup D$ is algebraically independent so we can extend this set to a transcendence base B for $\mathbf{R}$. Let $\{\ell_\alpha : \alpha < \mathfrak{c}\}$ enumerate the lines in the plane. We shall construct by transfinite induction a nondecreasing sequence $(E_\alpha)_{\alpha \le \mathfrak{c}}$ of subsets of $\mathbf{R}^2 \setminus C$ with induction hypotheses:

(1) $|E_\alpha| \le |\alpha| + \omega$,
(2) $C \cup E_\alpha$ is a partial two point set.

Put $E_0 = \emptyset$ and if $\lambda \leq \mathfrak{c}$ is a limit ordinal then $E_\lambda = \bigcup_{\alpha<\lambda} E_\alpha$. Let α be a fixed ordinal $< \mathfrak{c}$ and consider E_α and ℓ_α. If $(C \cup E_\alpha) \cap \ell_\alpha$ contains two points then $E_{\alpha+1} = E_\alpha$. Assume now that $|(C \cup E_\alpha) \cap \ell_\alpha| \leq 1$ and that ℓ_α is the graph of $ax + by = c$. Consider the set $Z = \mathfrak{L}(C \cup E_\alpha) \cap \ell_\alpha$. If $(x, y) \in Z$ then there exist two distinct points $(x_1, y_1), (x_2, y_2) \in C \cup E_\alpha$ such that (x, y) is the point of intersection of ℓ_α and the line through (x_1, y_1) and (x_2, y_2). Consequently, x and y are elements of the field $\mathbf{Q}(\{x_1, x_2, y_1, y_2, a, b, c\})$. Since $|E_\alpha| \leq |\alpha| + \omega$ we can find a $B' \subset B$ such that $|B'| \leq |\alpha| + \omega$ and $\pi_1(E_\alpha) \cup \pi_2(E_\alpha) \cup \{a, b, c\}$ is algebraic in $\mathbf{Q}(B')$. Since S^1 is an algebraic curve $\pi_2(C)$ is algebraic in $\mathbf{Q}(\pi_1(C))$. As the set of points that are algebraic in a given field is itself a field we may conclude that $\pi_1(Z) \cup \pi_2(Z)$ is algebraic in $\mathbf{Q}(\pi_1(C) \cup B')$. Since $|B'| < \mathfrak{c}$ we can select two distinct points u and v in ℓ_α such that at least one of the their coordinates is in $D \setminus B'$ and hence transcendental over $\mathbf{Q}(\pi_1(C) \cup B')$. Consequently, u and v are not in $\mathfrak{L}(C \cup E_\alpha)$. Putting $E_{\alpha+1}$ equal to $E_\alpha \cup \{u\}$ or $E_\alpha \cup \{u, v\}$ completes the induction. It is obvious that $C \cup E_\mathfrak{c}$ is a two point set.

Note that in this proof we may replace S^1 by any algebraic curve that is a partial two point set. In fact, it is shown in [2] that Corollary 3.5 is provable in ZFC.

In connection with Proposition 3.7 A. W. Miller asked the following

Question. *What is the Borel type of the set of extendable elements of* $\mathcal{K}(S^1)$*?*

If we combine Proposition 3.7 with Theorem 2.2 we find

Proposition 3.10. *There exist two extendable elements of* $\mathcal{K}(S^1)$ *whose union is not extendable to a two point set.*

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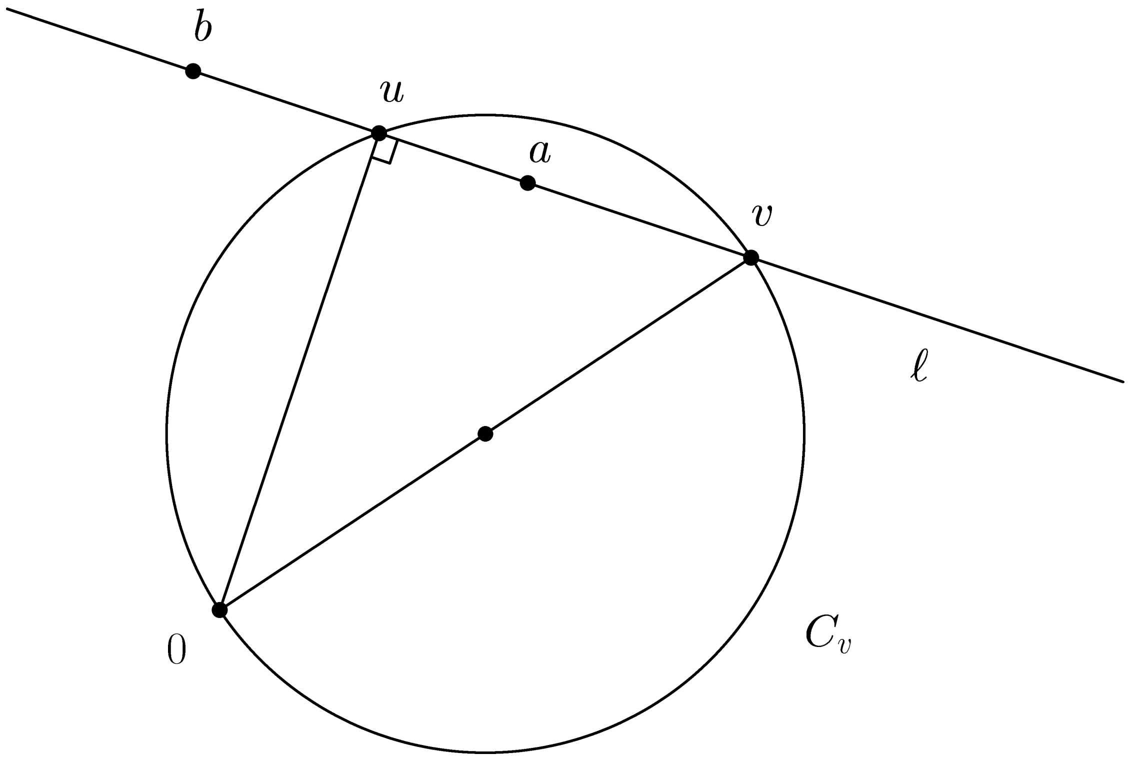

Figure 1

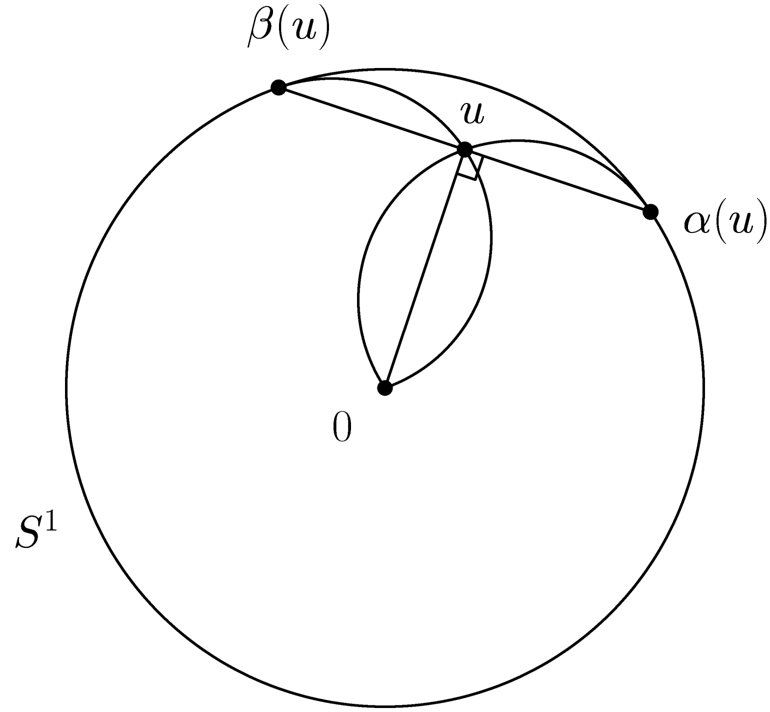

Figure 2

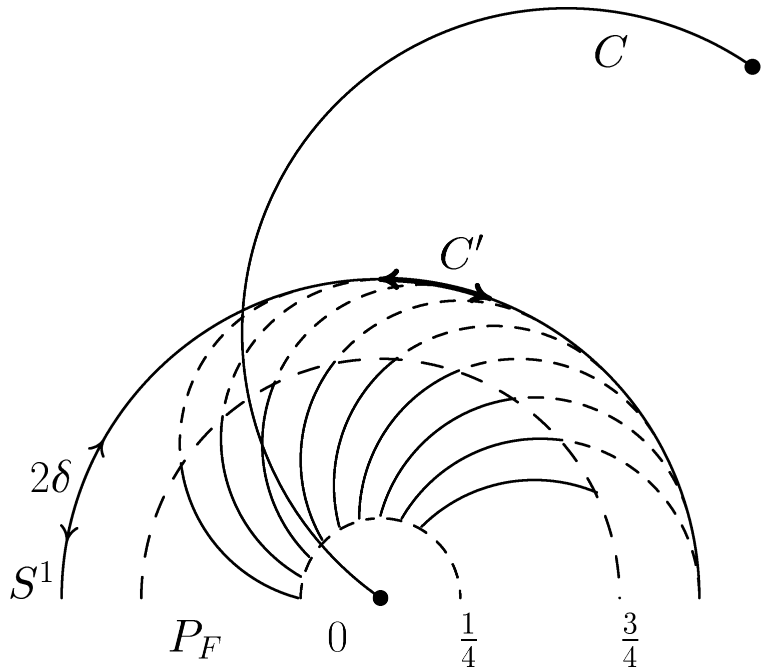

Figure 3

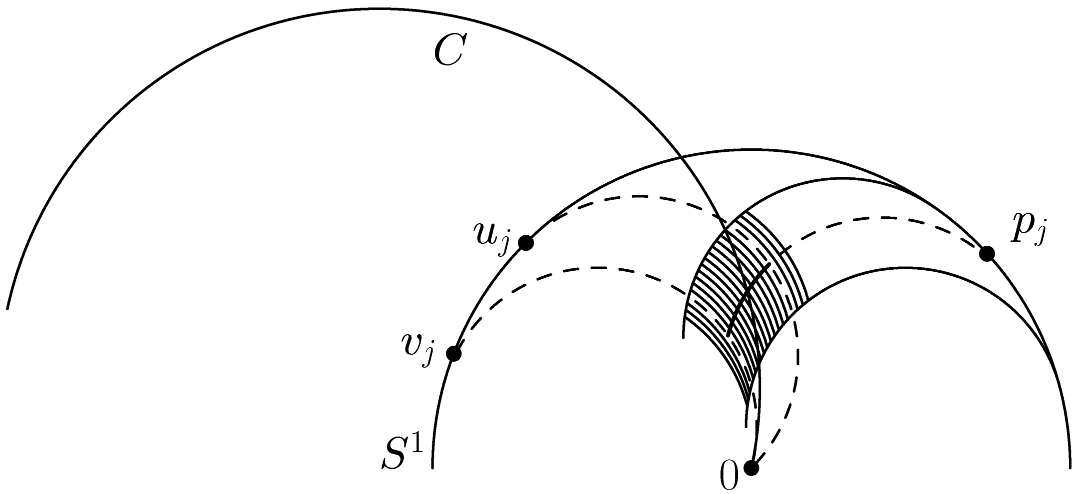

Figure 4

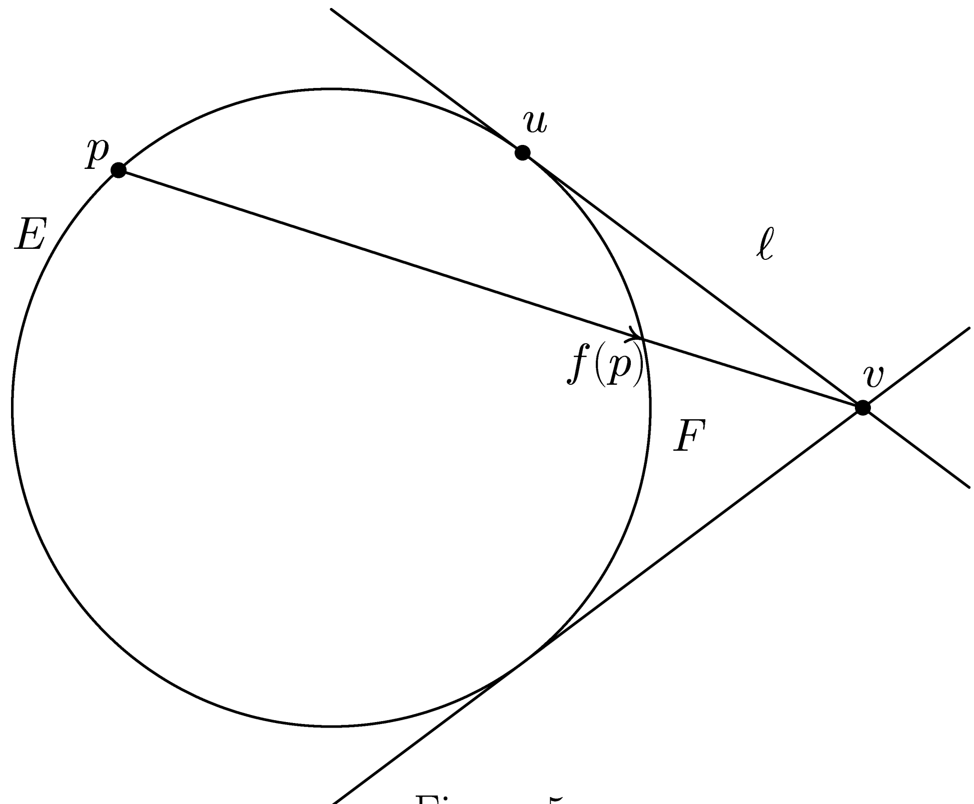

Figure 5