In Memory of Ken Kunen

Mirna Džamonja, Joan Hart, Andrea Medini, and Andrés Villaveces

Kenneth Kunen (1943–2020) made deep and wide-ranging contributions to mathematics: to topology, analysis, theoretical computer science, and algebra, but first and foremost, to set theory. Ken supervised over thirty doctoral students, several of whom have mathematical descendants of their own. The topologist Mary Ellen Rudin, in her note for the 2011 *Topology and its Applications* tribute to Ken, observed that "he didn't make waves, but he made both mathematics and mathematicians." Another close colleague, the set-theorist Arnie Miller, wrote that Ken was "not just a brilliant and productive mathematician but what we really admire most about him is his generosity with his mathematical ideas, conjectures, and problems." And he perfectly captured Ken's personality, by describing him as "always affable and always unflappable." Ken did what he enjoyed: even after retiring, he still thought about interesting problems and discussed them with colleagues, and continued to publish results.

We thank the AMS for the opportunity to edit the following compilation of personal recollections and technical accounts of Ken's work. We are also grateful to all who participated in this project for their enthusiastic responses and insightful contributions: Alan Dow, Michael Hrušák, Stephen Jackson, István Juhász, H. Jerome Keisler, Steffen Lempp, Donald Martin, Adrian Mathias,[1] Jan van Mill, Arnold W. Miller, Justin Moore, Dilip Raghavan, John Steel, Frank Tall, and Hugh Woodin. In particular, István was the first to propose this idea, and Steffen made it possible for us, as Ken's students and colleagues, to volunteer for this job. As our long (but still not exhaustive) list of contributors suggests, thanks to Ken and the scope of his work, keeping our tribute within the prescribed limits was yet another nontrivial problem.

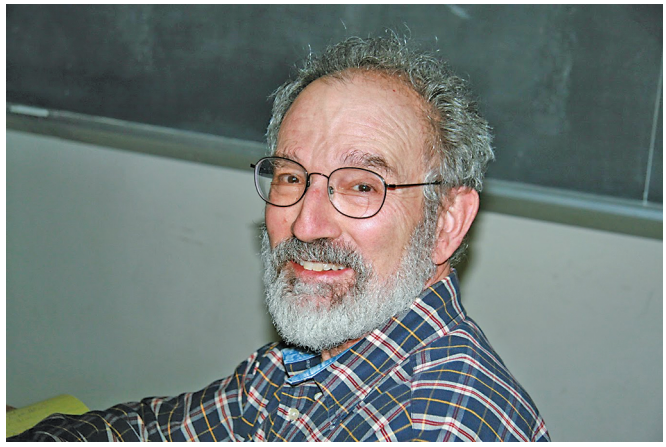

Figure 1. Ken at the Kunenfest in 2009.

István Juhász

I was deeply shocked by the death of Ken Kunen, because I was unaware that my old friend had any health problems. In fact, a relatively short time before learning this, for me, completely unexpected news, we corresponded via email concerning a math problem that I had turned to him about, as I had done so often. Also, he was younger than

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Communicated by Notices *Associate Editor Antonio Montalbán.*

DOI: https://doi.org/10.1090/noti2570

[1] *The contribution of Mathias is included in the section by Steel and Woodin. István Juhász is a research professor emeritus of mathematics at the Alfréd Rényi Institute of Mathematics. His email address is* juhasz.istvan@renyi.hu.

I, by exactly one month, and in my eyes he had "a professor's head on an athlete's body." His devotion to physical fitness was well-known. I remember how, sometime in the 70s, Paul Erdős greeted Ken not by asking about mathematics, as he usually did, but with "How is your cycling?"

Our acquaintance and friendship goes back a long time. I first heard his name mentioned in 1969, at a memorable ASL summer meeting in Manchester, England. That was just after he proved that there is no non-trivial elementary embedding of V, the set-theoretic universe, into itself. Though technically relatively simple, this result was completely unexpected, so it was perhaps the biggest sensation there. (I remember how excitedly Jan Mycielski tried to convince us at the previous ASL meeting in Italy that assuming the existence of such an elementary embedding is intuitively obvious.)

Another sensation, at least for me, at this meeting was the appearance of Martin's Axiom. After the meeting, Dana Scott, who had been Ken's PhD advisor and, like me, was visiting Amsterdam that year, proudly told me that the axiom could just as well be called Kunen's Axiom because Ken invented it independently of Tony Martin. This initiated my correspondence with Ken about topological applications of Martin's Axiom and his advice helped me a lot in writing the chapter on Martin's Axiom in my tract on cardinal functions in topology.

We first met in person at the Cambridge Set Theory Summer School in 1971, and that was when we worked on our first joint paper [JK73]. I'll omit the mathematical details of this as well as our later joint work because in my article for the 2011 special issue of *Topology and its Applications*, I wrote in detail about Ken's decisive and plentiful contributions to set-theoretic topology.

We really got close in the academic year 1974/75 that I spent at the UW–Madison. Although Mary Ellen Rudin arranged my invitation to Madison, Ken was also a gracious host. For instance, he took care of finding a place for us to stay in the same building where he then lived.

My professional benefits from this visit were enormous, mainly due to my connection with Ken. Our numerous exchanges about our common interests resulted, for example, in the triple paper [JKR76], one of the most cited works in our field. More importantly, these exchanges taught me an awful lot. In addition, I had the good fortune to attend a recursion theory course that Ken taught and that opened up a completely new chapter of logic for me.

This visit of mine to Madison and Ken was followed by uncountably many others. I even visited him in Austin in 1980, when he temporarily left Madison. My last visit to Ken in Madison took place in 2009, at his retirement meeting. After that we only kept in frequent touch by email.

In contrast, Ken visited Hungary just four times. The first time was in the summer of 1978 at a topology conference in Budapest. In his talk he presented his very deep result on the existence of weak P-points in ω^*, [Kun80a]. Just recently, this result played an essential role in a result of ours that is to appear in a volume of *Topology and its Applications* commemorating Ken. His second visit to us was for the August 1998 summer topology conference, where he presented results on Bohr topologies. His many papers on Bohr topologies include one with Walter Rudin. So, given the earlier triple paper [JKR76] that he, Mary Ellen, and I published, the question Ken and Walter answered makes Ken the only person to have published results jointly with each of the Rudins. Following TOPOSYM2001 in Prague, where he was an invited speaker, Ken visited Budapest again, when 9/11 had just occurred. His last visit was in 2003, to a meeting that was organized on the occasion of my 60th birthday.

I hoped to convince Ken to take advantage of the Hungarian Academy of Sciences distinguished visitor program, but it turned out that before I could lure Ken back to Budapest he ran out of time.

H. Jerome Keisler

I first met Ken in 1968, when he was in his last year as a graduate student at Stanford, and he gave a lecture at UCLA, where I was visiting for the logic year. I had just been promoted to Professor at Wisconsin. The logic group at Wisconsin then consisted of Stephen Kleene and Barkley Rosser, who were late in their illustrious careers, and myself. In 1968, Ken was universally recognized as a budding star, and was probably the most sought after new PhD in the world of logic. It was a major coup for Wisconsin to add Ken to our department. Although Ken worked mainly in set theory and I worked mainly in model theory, our research was closely related. Throughout our careers we had a common interest in properties of ultrafilters. For the next 34 years, I had the incredible luxury of having Ken in the office next door to mine. He was always available to exchange ideas and answer any questions I had in set theory. There was no one better to ask. Ken would get to the heart of a question and explain things in the clearest possible way. He was a superb teacher whose courses were highly popular. His graduate textbook on set theory is widely regarded as the best. During our careers, I worked closely with several of Ken's PhD students, and Ken worked closely with several of mine, where model theory and set theory overlapped.

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Ken had an immediate and profound influence on the logic program at Wisconsin. His arrival made Wisconsin one of the few programs that covered all of computability theory, model theory, and set theory. That gave graduate students the option to learn all of them and then choose a research area. Ken had a great influence on the research of others. Ken's work as a graduate student and his seminar in set theory in his first year at Wisconsin inspired Rosser to write his book "Simplified Independence Proofs." When Ken arrived, Mary Ellen Rudin joined the Wisconsin logic seminar, and from then on much of her research was on the borderline between topology and set theory. Early in his career, in 1971–72, after spending the academic year at Berkeley, Ken returned with three Berkeley graduate students, William Fleissner, Judith Roitman, and Aki Kanamori. Each of them added much to the Wisconsin logic group as visiting students and then went on to have distinguished careers. For the next forty years, Ken attracted many of the top people in the field as visitors to Wisconsin, and some visited several times.

During the 1970s, Ken, Jon Barwise, and I worked together on the logic program at Wisconsin and on many other things, such as the Handbook of Mathematical Logic, and the Kleene Symposium in honor of Kleene's 70th birthday. I was extremely fortunate to have had the opportunity to work with Ken for so many years.

Steffen Lempp

When I arrived in Madison in 1988, twenty years after Ken, I felt like a very junior colleague among senior giants (including my current coauthors Jerry Keisler and Arnie Miller as well as Terry Millar, who unfortunately passed away two years ago). Each of them had his own unique style. The three things I remember the most about Ken are his boundless energy, his uncanny ability to attract very good students with a fairly hands-off approach, and his truly amazing ability to generate interesting qualifying exam problems.

During my first 25 years at UW, we almost always had our semesterly logic picnic at Devil's Lake State Park, and we always scaled East Bluff from the south shore. It was hard to keep up with Ken running up the steep path; in fact, most of the time, Ken didn't even bother with the path but just climbed up the boulder field in between, with the rest of us panting behind him (or taking the path instead).

He was immensely popular with students, many of whom went on to highly distinguished careers of their own, and almost all of whom finished their degree and found their niche; in fact, Ken's last student graduated the summer Ken passed away! He was very conscientious about meeting with his students, but especially after his retirement, they had to come in early, since Ken arrived in the department at 7 AM and was gone by 10 AM!

Finally, each semester when we had to make up the qualifying exam, and even long after Ken retired, he contributed a lot of problems. My problem with this was that I had a hard time even solving the ones he labeled as for the "elementary" section! Fortunately, Ken gave the solutions for later publication along with the problems, so I could check if the problems were doable for our students without spending hours trying to solve them!

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Arnold W. Miller

I want to say something about Ken's teaching. Some time ago one of our graduate students made a very astute observation about Ken's style of lecture. He said "Professor Kunen never gives the proof of anything; he just makes remarks." I attended a great many of Ken's lectures and can explain what the student meant. Here is how a typical Kunen lecture went.

First he states the Theorem and its relevant definitions. Then he explains what the Theorem says, maybe illustrates a few consequences, special cases, Corollaries, and problems it doesn't settle. Then he begins remarking about how the proof might go. "You might think" he says "that we could prove it this way—but that wouldn't work." But then he says "or you might prove it this way—but that doesn't work. But maybe something similar does." After discussing several false starts and why they don't work, he discusses strategies and how to handle various problems and details. He says "let's consider the following strategy. Prove Lemma 1 ✍, 2 ✍, 3✍." He explains how Lemmas 1, 2, and 3 would give us the Theorem. Then he says "Now to prove 1 you might think we could do this—but that wouldn't work." And then he says "Well a possible proof of Lemma 1 would be a, b, then c." After a while he starts to discuss Lemma 2. "Now" he says "Lemma 2 is false, and here's why. . . but maybe we could prove Lemma 2'". . .

So by the end of the hour the whole class sees the particular path the proof must take and how to take care of the problems and details along the way. And then finally Ken says "OK ah ah that's enough discussion, let's give the proof of the Theorem." So he writes on the board:

"Proof—see above remarks—☕."

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Justin Tatch Moore

Unfortunately, I arrived too late to know Ken well. I remember meeting him on only two occasions: once as the external examiner of my 2000 PhD thesis at the University of Toronto and later when I visited Madison in 2006. Instead, I was part of a generation of set-theorists educated through his book [Kun80b].

In 1995 while an undergraduate at Miami University, I worked on a summer research project with Dennis Burke funded through the college. There was a budget for "materials" and Dennis picked three books for me: *Open Problems in Topology*, *The Handbook of Set-theoretic Topology*, and Ken's *Set Theory*. All are still on my bookshelf; the last is by far the most worn. I still recommend "*Kunen*" to any student who expresses an interest in studying set theory. From transfinite recursion to combinatorial set theory to constructibility to iterated forcing, it gives a complete foundation for further reading in the subject. The treatment is timeless. The exercises are legendary. The tone is both precise and conversational.

Above all, it is a study in discipline. At the time the book was written, set theory was rapidly developing in many different directions. It must have been extremely tempting to include more, but Ken somehow knew the correct boundaries and kept within them. As a result, the book has stood the test of time extremely well. It is still the place I send students to learn about forcing or how to construct a Souslin tree. Even Chapter V on defining definability—which seemed somewhat obscure back in the 1990s—was prescient in light of the role that HOD plays in modern inner model theory.

Of course Ken's research itself was transformative and inspirational: the nonexistence of a nontrivial elementary $j : V_{\lambda+2} \to V_{\lambda+2}$, the inner model for a measurable cardinal, his work on random reals and RVMs, saturated ideals on small cardinals, and his S and L spaces. Most of these have had a significant influence on my own work. Still, his book stands out as an extremely important contribution to set theory and surely a lasting part of his legacy.

Frank Tall

I've written my reminiscences of Ken in my contribution to the forthcoming memorial issue of *Topology and its Applications*. I'll just repeat one item here. Ken had a genius for providing clear answers to murky questions. You could go to his office with a vaguely formulated question and he would divine its essence and answer it. Furthermore, he knew "everything" known about set theory.

I only really learned set theory when I started teaching from Ken's book [Kun80b]. Many of my students who seriously worked through the exercises became accomplished set theorists. His innovation of teaching Martin's Axiom before forcing was revolutionary at the time. For a discussion of Ken's expository talents, see Kanamori's article in the upcoming special issue of *Annals of Pure and Applied Logic*.

My most-cited article with Ken, and Ken's third most cited joint paper is *Between Martin's axiom and Souslin's hypothesis* [KT79]. I had noticed that most consequences of Martin's Axiom fell into two natural categories: combinatorial propositions about the real line or sets of natural numbers and propositions implying Souslin's Hypothesis. I asked Ken if these two could be distinguished. He had already done so! Ken had a habit of jotting down 4-page handwritten notes containing proofs of interesting facts that were too small to be an actual paper, but that he could pull out of his filing cabinet when the occasion called for them. One of these was that property K forcing preserved Souslin trees. It immediately followed that Martin's Axiom restricted to property K partial orders plus the negation of the Continuum Hypothesis did not imply Souslin's Hypothesis, although it did imply all the usual combinatorial consequences. This paper spawned many others; its terminology has become so commonplace that its origin is often not cited. A survey on this topic by Bagaria, entitled *The relative strength of fragments of Martin's Axiom*, will appear in the issue of *Annals of Pure and Applied Logic* mentioned above.

My second-most cited article with Ken, [KST86], provided an unexpected strong connection between saturated ideals and a problem Russian topologists were interested in—the existence of Baire irresolvable spaces.

Ken had high standards concerning what was publishable. Several topologists, e.g. Wis Comfort, had asked me about whether our Baire irresolvable results applied at singular cardinals, so I thought we should publish the fact that they did as a short note. Ken didn't want to, because it was essentially the same proof as in the regular case, so we posted it on Topology Atlas with the title *On the consistency of the non-existence of Baire irresolvable spaces*.

I had two other papers with Ken: [BGKT78] and [KT00]. The first was the usual story—the other authors couldn't solve a central problem in their work, so asked Ken. The second was a true collaboration, answering the question of whether an uncountable elementary submodel containing the real line as a member actually includes all of it. Not surprisingly, the answer depends on your set theory.

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Figure 2. A group picture from the Kunenfest, including many of the contributors and editors of this article.

Ken and $\beta\mathbb{N}$

Alan Dow, Michael Hrušák, and Jan van Mill

Ken's enormous contribution to the study of the *space of ultrafilters* $\beta\mathbb{N}$, and dually the Boolean algebras $\mathcal{P}(\mathbb{N})$ and $\mathcal{P}(\mathbb{N})/\mathrm{fin}$, cannot be overstated. Dealing with the difficult task of deciding which of his many fundamental results we should highlight, we have decided to focus almost exclusively on his ZFC theorems leaving aside his many important consistency proofs such as (1) the non-existence of selective ultrafilters, (2) the early results on cardinal invariants of the continuum ($\mathfrak{t} < \mathfrak{c}$, $\mathfrak{u} < \mathfrak{c}$ and $\mathfrak{a} < \mathfrak{c}$), (3) his joint work with Bell that every point in $\beta\mathbb{N}$ has π-character $\aleph_1 < \mathfrak{c}$ and there is a point with π-character at least $cf(\mathfrak{c})$, (4) the non-existence of gaps other than Hausdorff and Rothberger together with MA + ¬CH while inventing the technique of "freezing" a gap, and (5) along with some more "topological sounding" statements, e.g., the fact that $\mathbb{N}^* = \beta\mathbb{N} \setminus \mathbb{N}$ cannot be covered by nowhere dense closed P-sets under CH, the consistency of the statement that $\mathbb{N}^*$ does not map onto all compact spaces of weight $\mathfrak{c}$, and the consistency of $\mathbb{N}^* \setminus \{p\}$ is C^*-embedded in $\mathbb{N}^*$ for every $p \in \mathbb{N}^*$, to mention but a few.

In the presence of CH, information about $\mathcal{P}(\mathbb{N})$ and $\mathcal{P}(\mathbb{N})/\mathrm{fin}$ can often be found by a transfinite procedure dealing essentially with countable structures. This is no longer true in the absence of CH if one's aim is to get results in ZFC, since regardless of additional set theoretic assumptions one can, for example, run into a Hausdorff gap. It was Kunen [Kun72] who found in 1972 a remedy for this seemingly insurmountable barrier by creating beforehand enough "space" to make sure that a recursive process does not prematurely terminate. This is what we now call a "*guided transfinite recursion.*" We shall briefly outline this brilliant innovative idea that turned out to be extremely powerful in solving open problems dealing with ultrafilters in $\mathcal{P}(\mathbb{N})$. One first identifies an "independent" matrix of subsets of $\mathbb{N}$ with $\mathfrak{c}$-many rows. The cofinite filter is independent modulo the matrix in the sense that every member of it intersects an arbitrary finite collection of sets, each chosen from a different row of the matrix, in an infinite set. Suppose that the tranfinite process allows for steps in which only a finite number of rows of the matrix is used. Then at every intermediate step of the process, only fewer

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than $\mathfrak{c}$-many rows of the matrix are used and there are still $\mathfrak{c}$-many left for future use, hence the transfinite process does not terminate until it reaches its successful end.

Kunen applied it in two papers, solving major open problems. In [Kun72], motivated by the work of Frolík and M. E. Rudin, he used it to prove the existence of incomparable ultrafilters in the *Rudin-Keisler order*. Kunen's result which was generalized by Shelah and Rudin, had a very high impact. In the same paper [Kun72], motivated by problems from Model Theory, Kunen used the new method to prove the existence of so-called *good* ultrafilters in ZFC, which improved a well-known result of Keisler who proved it earlier using CH. The Rudin-Keisler incomparability of ultrafilters also had definitive topological applications. In [Kun90], Kunen used it to show that no infinite compact F-space (and more generally, no product of compact F-spaces at least one of which is infinite) is (topologically) homogeneous, which greatly improved an earlier result by Frolík for extremally disconnected compacta.

Kunen's second main, and still more striking, application of his method [Kun80a] that there are weak P-points in $\mathbb{N}^*$ gave the ultimate proof of non-homogeneity of $\mathbb{N}^*$. W. Rudin showed that P-points exist in $\mathbb{N}^*$ under CH, thereby demonstrating that $\mathbb{N}^*$ is not homogeneous by producing two points with obviously distinct topological behavior (recall that a *P-point* is a point with the property that the intersection of any countable family of its neighborhoods is again a neighborhood). That some set theoretic hypothesis is essential in Rudin's result was shown by Shelah, while a ZFC proof of non-homogeneity of $\mathbb{N}^*$ is due to Frolík. However, Frolík's proof is based on a cardinality argument, and does not yield two points with obvious different topological behavior. A point is a *weak P-point* if it is not in the closure of any countable subset contained in its complement. Every P-point is a weak P-point (but not vice-versa). Kunen's result about weak P-points in $\mathbb{N}^*$ not only gives an "honest" proof of the nonhomogeneity of $\mathbb{N}^*$, but also shows that the Shelah P-point independence theorem is in a certain sense sharp. The matrix of sets needed for the proof is much more complicated than the one from [Kun72]. Indeed the matrix discussed above had only two elements in each row and was shown to exist by Hausdorff; however for the results on weak P-points each row had $\mathfrak{c}$-many elements and proving such a structure exists in ZFC is in itself a major result. The ideas in [Kun80a] had again enormous impact. Several other major open problems on $\mathbb{N}^*$ were solved by applying Kunen's method: Simon's result that there is a separable closed subspace of $\mathbb{N}^*$ that is not a retract of $\mathbb{N}^*$, van Mill's results on weak P-points in general Čech-Stone remainders and construction of many other special points of $\mathbb{N}^*$, and Dow's solution to van Douwen's problem that there is a nontrivial copy of $\mathbb{N}^*$ in $\mathbb{N}^*$ (the latter result was recently generalized by Dow and van Mill that there even exists a nowhere dense weak P-set copy of $\mathbb{N}^*$ in $\mathbb{N}^*$).

Figure 3. Ken at UC Berkeley in 1972.

Kunen's Work on Determinacy and Descriptive Set Theory

Stephen Jackson and Donald A. Martin

Kunen's work in the early 70s in descriptive set theory, and particularly in the development of determinacy theory, deserves to be considered a fundamental and groundbreaking achievement.

Background for Kunen's work in determinacy. The axiom of determinacy, AD, which states that every two-player game on ω (or equivalently on $2 = \{0,1\}$) has a winning strategy, was introduced by Mycielski and Steinhaus in 1962. They proposed using this axiom to develop a theory of the sets of reals, as this axiom avoids pathological sets constructed from AC. Although AD contradicts AC, it was understood that this axiom was meant to apply to a more restricted universe, such as $L(\mathbb{R})$, in which the sets of reals have a more explicitly definable structure. However, it wasn't until much later through the work of Martin, Steel, and Woodin in the late 80s that it was shown that large cardinals imply $\mathsf{AD}^{L(\mathbb{R})}$. The early researchers realized,

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though, that AD, and even weaker forms such as projective determinacy, PD, might be enough to develop a theory of the projective sets and beyond similar to the theory of the Borel and analytic sets developed by the classical descriptive set theorists of the early 20th century (see Kechris's textbook).

In the late 60s, just prior to Kunen's main work in this area, several important developments occurred. Martin and Moschovakis (1968) independently proved the first periodicity theorem for propagating the prewellordering property under $\forall^{\omega^\omega}$ assuming a determinacy hypothesis. This was followed by Moschovakis's second periodicity theorem for propagating scales and the scale property under the same determinacy hypothesis (see Moschovakis's textbook for more history). The periodicity theorems give a structural representation for the projective sets, but the theory that emerges is largely in terms of the so-called projective ordinals. By definition,

$$\delta^1_n = \sup\{|\preceq| : \preceq \in \mathbf{\Delta}^1_n \text{ is a prewellordering of } \omega^\omega\},$$

that is, it is the supremum of the lengths of the $\mathbf{\Delta}^1_n$ prewellorderings of ω^ω. As we will discuss shortly, the initiation of a program for computing the δ^1_n would be an important contribution of Kunen's.

A second development in the late 60s which helps to set the stage for Kunen's work was Martin's proof of $\mathbf{\Pi}^1_1$ determinacy from a measurable cardinal. Aside from forging a connection between determinacy hypotheses and large cardinal axioms, implicit in this proof was the notion of a *homogeneous tree*. This notion also implicitly appears in the 1969 paper of Martin and Solovay [MS69]. What we now call the Martin-Solovay construction shows how to construct a scale/Suslin representation for $\omega^\omega \setminus A$ from a scale on A given that the tree T giving the Suslin representation for A is weakly homogeneous. The modern notion of a homogeneous tree and weakly homogeneous tree was formulated independently by Martin and Kechris in 1981, but arose implicitly earlier as just mentioned.

Kunen's work. With this as background, in 1971 Kunen began producing a series of remarkable results which would shape the future of descriptive set theory and determinacy theory. During this time, Kunen (and others) communicated their results through handwritten notes which were circulated to the other researchers in the area. While Kunen's work in determinacy became well-known, he did not publish this work.

In the same time period, Kunen and Martin independently proved what we now call the Kunen-Martin theorem. This theorem states that a κ-Suslin well-founded relation $\prec$ has length $|\prec| < \kappa^+$. This result has immediate ramifications for computing the δ^1_n. It shows that $\delta^1_{2n+2} = (\delta^1_{2n+1})^+$ and that $\delta^1_{2n+1} = \lambda^+_{2n+1}$, where λ_{2n+1} is the cardinal where $\mathbf{\Sigma}^1_{2n+1}$ sets and $\mathbf{\Pi}^1_{2n}$ sets admit scales. Kunen then developed a plan for computing the projective ordinals. Roughly speaking, if we assume δ^1_{2n-1} is known, and this is where the $\mathbf{\Sigma}^1_{2n}$ sets admit weakly homogeneous scales, and if we can transfer the scales on the $\mathbf{\Sigma}^1_{2n}$ sets to the $\mathbf{\Pi}^1_{2n}$ sets (something akin to the Martin-Solovay construction) then we would compute λ_{2n+1} and thus δ^1_{2n+1}, and thus complete the induction.

A central part of Kunen's plan for carrying out the above program is to establish *partition properties* of the projective ordinals. If one assumes the strong partition property at δ^1_{2n-1}, and the tree T_{2n-1} of a scale on a complete $\mathbf{\Pi}^1_{2n-1}$ set is homogeneous then the Martin-Solovay construction will propagate the tree T_{2n-1} to a homogeneous Suslin representation for a complete $\mathbf{\Pi}^1_{2n}$ set which will be on the ordinal $\sup_\mu j_\mu(\delta^1_{2n-1})$ where μ ranges over the measures in the homogeneous tree for T_{2n-1}, and j_μ is the ultrapower embedding. Thus we would compute $\lambda_{2n+1} = \sup_\mu j_\mu(\delta^1_{2n-1})$ and thereby compute δ^1_{2n+1}.

The above program of Kunen's depends on two key points. First, we must be able to prove the partition properties of the δ^1_{2n+1}, and second we must be able to analyze the homogeneity measures μ well enough to be able to compute the ultrapowers $j_\mu(\delta^1_{2n-1})$. Concerning the partition properties, Martin showed that $\delta^1_1 = \omega_1$ has the strong partition property, and established a general framework for proving partition results from determinacy. In a remarkable result which contained several important ideas, Kunen showed that δ^1_3 had the weak partition property $\delta^1_3 \to (\delta^1_3)^\lambda$ for all $\lambda < \delta^1_3$. Kunen's proof went by showing that there is a $\mathbf{\Delta}^1_3$ coding of the subsets of $\lambda_3 = \omega_\omega$, and then quoting the Martin framework (see Jackson's chapter in the *Handbook of Set Theory* for an exact statement). In particular, Kunen came up with a short but very clever argument that to analyze the subsets of κ, it suffices to analyze the measures on κ. Using the theory of indiscernibles, Kunen analyzed the measures on the ω_n, which then allowed him to show the weak partition relation on δ^1_3 (see [Sol78] for a presentation of Kunen's argument). A similar elegant result proved by Kunen is that under AD the set of measures on any ordinal $\alpha < \Theta$ is wellordered.

An important technical ingredient in the analysis of measures on the ω_n under AD is the notion of the *Kunen tree*. This concept, introduced by Kunen, plays a central role in almost all arguments involving the ω_n under AD. The Kunen tree T is a tree $T \subseteq (\omega \times \omega_1)^{<\omega}$ on $\omega \times \omega_1$ such that for all $f : \omega_1 \to \omega_1$ there is an $x \in \omega^\omega$ with the section $T_x = \{s \in \omega_1^{<\omega} : (x \restriction |s|, s) \in T\}$ of T wellfounded, such that for all $\alpha < \omega$ in a c.u.b. set we have

$$f(\alpha) < |T_x \restriction \alpha|$$

(the rank of the T_x restricted to ordinals less that α). An

immediate consequence of the Kunen tree, for example, is a bound for the ultrapower $j_{W_n^1}(\omega_1) \leq \omega_{n+1}$ where W_n^1 denotes the n-fold product of the normal measure on ω_1. From the perspective of Kunen's program, this can be viewed as computing an upper bound for δ_3^1 as we have $\delta_3^1 = (\lambda_3)^+ \leq \sup_n j_{W_n^1}(\omega_1) \leq (\omega_\omega)^+ = \omega_{\omega+1}$.

The program of computing the projective ordinals via Kunen's program stalled after the results mentioned above. It turns out that Kunen's overall plan was still sound, but several important ingredients were missing for extending the program to higher levels. One of the missing ingredients was a generalization of the Kunen tree to higher cofinalities above ω. For functions on the higher δ_{2n+1}^1, one needs the *Martin tree*, an appropriate generalization of the Kunen tree introduced by Martin in the early 80s. A second ingredient necessary for extending the program is the notion of a *description*. These are hereditarily finite objects which "describe" how to generate functions via certain iterated ultrapowers (in Jackson's chapter in the *Handbook of Set Theory* the earlier theory is redone from this perspective). Although the description analysis does not rely on the theory of indiscernibles, it is important to note that Kunen's idea of analyzing the sets of ordinals by analyzing the measures is still a key component in the analysis. Using the Martin tree and the description analysis, Jackson computed δ_5^1 and proved the strong partition relation on δ_3^1 in the early to mid 80s [Jac99], and then extended this analysis to complete Kunen's program of computing all of the δ_n^1 and establishing the strong partition relation at all of the odd projective ordinals. Extending the results of the projective analysis throughout the cardinal structure of a determinacy model below Θ remains an elusive goal.

Kunen on the Product Measure Extension Axiom and on S- and L-spaces

Peter Nyikos

As a good friend and colleague of Mary Ellen Rudin at the University of Wisconsin, Kunen could hardly avoid becoming interested in some of the problems in set-theoretic topology about which Mary Ellen wrote in her tremendously influential booklet *Lectures on Set Theoretic Topology*. But his most valuable contribution may have been a result that he never bothered to publish.

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This was his generalization of a 1971 theorem of Solovay [Sol71] which, by sacrificing translation-invariance, extended Lebesgue measure to a countably additive measure defined on all subsets of $[0, 1]$. For this, Solovay had to assume the consistency of there being a measurable cardinal. This is an especially bold large cardinal axiom: measurable cardinals are so large that there were serious suspicions back then that their existence is inconsistent. Their consistency has stood the test of time long enough for these suspicions to have quieted down, but there is still good reason to avoid relying on their consistency when possible.

Kunen needed an even larger cardinal, known as "strongly compact," to extend Solovay's theorem to a whole class of countably additive measures. He did this by way of the Product Measure Extension Axiom (PMEA), which can be succinctly described as extending, for all cardinals κ, the Haar measure on the product of κ-many copies of the 2-element group to a measure on all subsets of these groups.

This axiom turned out to be the key to the first of two breakthroughs that settled the Normal Moore Space Problem, a problem of great interest to set-theoretic topologists since 1937, when F. Burton Jones published a proof, assuming $2^{\aleph_0} < 2^{\aleph_1}$, that every normal Moore space with a countable dense subset is metrizable. In her booklet, Rudin devotes a whole chapter "strictly to the normal Moore space conjecture." The PMEA was exactly what I needed to get a complete generalization of Jones's topological theorem under a different axiom: every normal Moore space is metrizable.

Kunen's proof was finally published in 1984 by Bill Fleissner, in his chapter in the *Handbook of Set-theoretic Topology*, where he also presented the general result in the opposite direction. This was his construction of a non-metrizable normal Moore space using an axiom that is so weak, that to *negate* it entails the consistency of there being a proper class of measurable cardinals!

Kunen made several significant contributions to another topic to which Mary Ellen devoted a chapter (and hefty parts of two others) in her booklet: the theme of S- and L-spaces. An *S-space* is a regular hereditarily separable[2] space that is not hereditarily Lindelöf; to define *L-space*, switch "separable" and "Lindelöf."

Early on, the conventional wisdom was that the existence of an S-space is equivalent to that of an L-space. This was refuted by J. T. Moore's sensational construction of an L-space from ZFC combined with an earlier result by S. Todorčević showing the consistency of there being no S-spaces.

[2]*The word "hereditarily" refers to all subspaces having the stated property. A separable space is one with a countable dense subset, and a Lindelöf space is one for which every open cover has a countable subcover.*

One of Kunen's best papers on this topic [Kun77] contributed to the conventional wisdom long before Moore's breakthrough. It has to do with spaces whose finite powers are all S-spaces or all L-spaces, known as *strong S-spaces*, resp. *strong L-spaces*. Zenor had shown that if there is a strong S-space, there is a strong L-space, and vice versa. Under CH, there are many constructions of strong S-spaces (including some by Kunen) and, by Zenor's theorem, strong L-spaces. Kunen's big breakthrough in [Kun77] was to show that MA + ¬CH implies that there are no strong S-spaces, and the proof itself dualized easily to imply that there are no strong L-spaces, without invoking Zenor's theorem.

The "Kunen line" [JKR76] is probably the best known of Kunen's contributions in this area. It is a refinement of the topology of the real line under CH to a locally countable and locally compact, perfectly normal S-space. Kunen also constructed various examples of compact L-spaces under various axioms. Some, like his Corson compact L-space using CH, are relevant to functional analysis.

A very different construction was in a joint article with Eric van Douwen. They showed that the following are equivalent: (1) a combinatorial principle that they designated ↓; (2) $\mathcal{P}(\omega)$ with the Vietoris topology has an L-subspace; (3) $\mathcal{P}(\omega)$ with the Vietoris topology has an S-subspace. The equivalence of (2) and (3) is obtained by taking complements of the respective subsets of ω, as are the proofs that each statement is equivalent to ↓, again in line with the illusion of S- and L- duality. They also showed that these statements follow from CH, via a stronger principle designated ↑ [mnemonic: What goes up must come down.] and are negated by MA + ¬CH.

Kunen and Forcing

Dilip Raghavan

Kunen made seminal contributions to all of the principal threads in the study of forcing, forcing axioms, and their applications. Here are some of the greatest hits.

Iterated forcing was invented by Solovay and Tannenbaum to prove the consistency of Suslin's Hypothesis. Several people realized that the details of Solovay and Tannenbaum's argument could be carried out with all c.c.c. partial orders, leading to the formulation of Martin's Axiom (MA), the first example of a forcing axiom. Kunen, who had independently formulated MA, began exploring its consequences in his thesis of 1968. In Section 14 of his thesis, Kunen showed that MA implies that every set of reals of cardinality less than $2^{\aleph_0}$ is of strong measure zero. He went on to observe that if $\kappa \leq 2^{\aleph_0}$ is real-valued measurable, then because of $\mathbf{\Sigma}^1_2$ indescribability, there is a set of reals of cardinality less than κ which is not Lebesgue measurable, thereby proving that MA is incompatible with the assertion that the continuum is real-valued measurable. Section 14 also contains the first construction of a generalized Luzin set under MA. In Section 12 of his thesis, Kunen showed that MA implies that every subset of $\mathbb{R}\times\mathbb{R}$ belongs to the σ-algebra generated by arbitrary rectangles, while showing that this is not the case in the Cohen model. Section 13 contains a proof that the cardinal characteristic $\mathfrak{b}$ equals $2^{\aleph_0}$ under MA.

Kunen was the first to investigate the gap structure of $\mathcal{P}(\omega)$/fin under forcing axioms. Introducing the pivotal technique of freezing a gap in his handwritten notes from August 1975, Kunen was able to show that $\mathsf{MA}_{\aleph_1}$ alone is not sufficient to determine the gap structure of $\mathcal{P}(\omega)$/fin. Although they were never published, the mimeographed notes of 1975 were widely circulated and attracted a great deal of attention, featuring in Woodin's solution to Kaplansky's conjecture. The idea of freezing gaps went on to play a crucial role in the eventual complete characterization of the gap structure of $\mathcal{P}(\omega)$/fin under PFA, which in turn, was vital to the proof that $2^{\aleph_0} = \aleph_2$ under PFA. The reader may consult Todorčević's 1989 monograph for more details of this. Kunen went on to publish most of the key ideas from his 1975 notes in [Kun88], where using the technique of freezing gaps once again, he was able to show that the cardinal characteristic $\mathfrak{m}$, which marks the place where MA first fails, could consistently be equal to $\aleph_{\omega_1}$. In [Kun88], Kunen asked if it is possible for $\mathfrak{m}$ to be a singular cardinal of cofinality other than ω_1. Kunen's question is yet to be fully resolved.

A major application of MA to general topology is presented in [Kun77] where Kunen proved that strong S and L spaces do not exist under MA. More recently, Kunen and his collaborators have investigated the effect of forcing axioms on continua and differentiable functions in $\mathbb{R}^n$. In 2011, Hart and Kunen proved that PFA implies that every uncountable subset of $\mathbb{R}^n$ meets some C^1 arc in an uncountable set and that this is not provable from $\mathsf{MA}_{\aleph_1}$. In [Kun12], Kunen showed that PFA implies that if E is any subset of $\mathbb{R}$ of size at most $\aleph_1$, then E^2 can be covered by countably many graphs of C^1 functions and their inverses.

Kunen pioneered techniques for obtaining consistency results from large cardinals through iterated forcing. What is now known as the Kunen-Paris forcing was introduced by Kunen and Paris [KP70] in 1970 to show that it is consistent for a measurable cardinal κ to have 2^{2^κ} normal measures on it. At the end of their paper Kunen and Paris asked

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whether it is possible for the number of normal measures on a measurable cardinal κ to be strictly between 1 and 2^{2^κ}, a question which was fully resolved only in 2009 by Friedman and Magidor. Kunen's 1972 result, published in 1978 in [Kun78], showing that it is consistent relative to a huge cardinal to have an $\aleph_2$-saturated ideal on ω_1, has come to be regarded as a landmark in set theory. Kunen's method has been adapted and improved numerous times to show the consistency of various saturation type properties from large cardinals. While it is now known how an $\aleph_2$-saturated ideal on ω_1 can be obtained from more optimal large cardinal hypotheses, Kunen's original technique and its variations remain an indispensable item in every set theorist's toolkit. We refer the reader to Section 7 of Kanamori's article in the 2011 special issue of *Topology and its Applications* for a detailed account of the historical significance of Kunen's [Kun78].

Kunen systematically studied the combinatorial properties of the Cohen and Random models. In his thesis, Kunen pointed out that there are no towers of length ω_2 in $\mathcal{P}(\omega)/\mathrm{fin}$ in the Cohen model. The fact that Ramsey ultrafilters could not be constructed in ZFC was established by Kunen's observation in [Kun72] that they do not exist in the Random model. In the expository article [Kun84], Kunen introduced the crucial notion of an invariant c.c.c. ideal. This notion allowed him to abstract away the shared features of the Cohen and Random models, and to point out the reasons for the differences between them. Towards the end of his article Kunen asked for a classification of all invariant c.c.c. ideals. Kunen's notion of an invariant c.c.c. ideal and his call to classify them turned out to be very influential, inspiring a sequence of important works by Kechris and Solecki, Farah and Zapletal, and Rosłanowski and Shelah which provided partial answers to Kunen's problem. Jointly with Juhász in 2001, Kunen sought axioms that capture the combinatorics of $\mathcal{P}(\omega)$ and ω^ω in the Cohen model. The idea behind this interesting line of research is to identify a handful of principles that hold in the Cohen model from which most of the known properties of the Cohen model could be axiomatically derived.

Kunen was one of the first to obtain consistency results about combinatorial cardinal characteristics of the continuum by forcing. Kunen observed that $\mathfrak{a} = \aleph_1$ in the Cohen model, a result he published in his textbook [Kun80b]. He was the first to prove the consistency of $\mathfrak{u} < 2^{\aleph_0}$. There were, in fact, two models of this. Bell and Kunen [BK81] produced a model with $\mathfrak{u} = \aleph_1$ and $2^{\aleph_0} = \aleph_{\omega_1}$ (more is true in their model – every ultrafilter on ω has π-character $\aleph_1$). Exercise (A10) in Chapter VIII of [Kun80b] asks the student to build Kunen's simpler model where $\mathfrak{u} = \aleph_1$ and $2^{\aleph_0}$ can be arbitrary. Kunen asked whether it is possible to have a uniform ultrafilter on ω_1 that is generated by fewer than $2^{\aleph_1}$ sets, that is, whether $\mathfrak{u}(\aleph_1) < 2^{\aleph_1}$ is consistent. His question remains wide open.

I will end with some personal recollections. Kunen was an outstanding advisor, a perfect match for my personality. He was able to inspire his students by the sheer example of his deeply original work. I had complete intellectual freedom in choosing what I wanted to work on, receiving guidance only if I was hopelessly stuck and decided to ask him. And in that case he always had some valuable remarks. I was able to acquire what is arguably the single most important skill in research: discerning the right problems to tackle. He was the ideal advisor for me. Kunen was also well-known for providing generous financial support to his students through his research grants. I was the beneficiary of this generosity twice.

I saw Kunen for the last time in May 2018 in Madison, WI. At that time, I was working on some problems about the order dimension of uncountable partial orders. I explained to him the results I had obtained with my collaborators and told him some of the problems I couldn't solve at the time. He thought briefly about what I had said, told me enthusiastically that the results were really interesting, and remarked casually that the model obtained by adding $\aleph_3$ Cohen reals is likely to be the right place to look for a counterexample to one of the open problems. He was absolutely right. Lemma 5.1 of [KR21] proves Kunen's remark from our final meeting that day.

Kunen and Inner Model Theory

John Steel and Hugh Woodin

Kunen began his graduate studies at Stanford in 1965, and finished in 1968 with a PhD thesis that is truly remarkable for its depth and breadth, and for the importance of its ideas in later work. The first half of the thesis deals with $L[U]$, the canonical minimal inner model with one measurable cardinal. An augmented version of it was published in [Kun70], a classic paper that has become standard material in graduate set theory texts.

Kunen's thesis adviser Dana Scott had proved in 1961 that the existence of measurable cardinals implies that there are nonconstructible sets, that is, the full universe V of sets is larger than Gödel's canonical inner model L. Rowbottom and Gaifman had shown in 1964 that if there are

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measurable cardinals, then in fact there are only countably many constructible real numbers, and Silver had identified a canonical "least" nonconstructible real, now known as $0^\sharp$. $L[0^\sharp]$ is thus a canonical inner model slightly larger than L, but if there are measurable cardinals, then it too has only countably many reals. Solovay had observed that if U is an ultrafilter witnessing the measurability of a cardinal κ, then in $L[U]$, κ is measurable via $U \cap L[U]$, and the GCH holds above κ. So $L[U]$ is an inner model with a measurable cardinal, but we cannot say yet that it is canonical, because it seems to depend on the arbitrary parameter U. Silver showed in [Sil71] that the full GCH holds in $L[U]$, and its set of reals is independent of U.

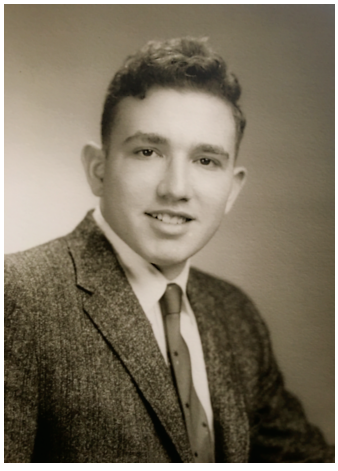

Figure 4. Ken in 1960.

With this as a foundation, Kunen established the basic canonicity theorems for $L[U]$. Silver's work on $L[U]$ had used Rowbottom's method of indiscernibles. Kunen's key first idea was to build on Gaifman's method of iterated ultrapowers instead. With that as his starting point, he showed that the model $L[U]$ depends only on the measurable cardinal κ, and not on U, and that the first order theory of $L[U]$ is independent of κ as well. Behind these results was a *Comparison Lemma*, which in Kunen's work took the form: if $\lambda > \kappa$ is any regular cardinal and if W is the filter generated by the closed unbounded subsets of λ, then in $L[W]$, $W \cap L[W]$ is an ultrafilter on λ, and the inner model $L[W]$ is an iterated ultrapower of $L[U]$.

With $L[U]$ established as a canonical object, it is natural to ask whether there are other ways to construct it, and whether there are canonical inner models for stronger large cardinal hypotheses. Kunen's [Kun70] took some important first steps in these directions. He showed that if there is a strongly compact cardinal, then there is a canonical inner model $L[\vec{U}]$ with a proper class of measurable cardinals. By later work of Magidor, the existence of strongly compacts does not imply the existence of two measurable cardinals, so this construction of $L[\vec{U}]$ cannot be the simple one that Solovay identified. Building on work of Solovay ([Sol71]), Kunen showed that if there is a κ^+-saturated (uniform and κ-complete) ideal on κ, then the canonical inner model with a measurable cardinal exists. (This leads to an equiconsistency.) The hypothesis here does not imply the existence of a measurable cardinal. Finally, Kunen showed that if the GCH fails at a measurable cardinal, then there are indiscernibles for $L[U]$. Again, the indiscernibles cannot come from a second measurable cardinal. Each of these theorems stands near the beginning of some substantial line of development in set theory.

Shortly after leaving Stanford for Madison, Kunen proved his well-known, very useful theorem that if there is a nontrivial elementary embedding j from L to itself, then $0^\sharp$ exists. (The converse is easy.) The key concept of an M-ultrafilter is studied in his thesis, and the equivalence of $0^\sharp$ with an iterable L-ultrafilter is at least implicit there. The heart of the new work is that the L-ultrafilter derived from j is iterable. Here Kunen's technique of considering the hull of $\{\alpha \mid j(\alpha) = \alpha\}$, much used by him and by others later, plays the key role.

Kunen left inner model theory not long after moving to Madison, but it is truly remarkable how far he went, near the beginning, and in such a short time. Since then, the theory has been extended so as to produce canonical inner models for much stronger large cardinal hypotheses, under a wide variety of assumptions. Kunen's Comparison Lemma has been greatly extended, both as to the models being compared, and as to the iteration methods used to compare them. Nevertheless, we still have very little information concerning canonical inner models at the level of strongly compact cardinals, and there are significant open problems well below that level.

Adrian Mathias was a Research Associate at Stanford in 1967–68 and a Visiting Lecturer at Madison in 1968–69. He writes of those times:

> *He had two excellent qualities not, alas, shared by every distinguished academic: if you asked to try a proof on him, he would listen carefully and give valuable feedback; and he scrupulously gave credit to others for their work. I remember receiving a letter once from him answering a question I'd posed, in which one paragraph began "An argument due to you shows that…" One knows mathematicians who hate to admit that someone else has had a good idea; but Ken was free of that fault.*
>
> *We had speculative conversations, and listened to each other's proposed proofs, but there was no intense collaboration such as I have had with some people; nor was there that intense competitiveness that has made me wary of some others. We were simply colleagues who helped each other. I learnt a lot from him: I remember his coming to my Stanford office to test a proof on me: the result was that if U is a normal ultrafilter on κ and you iterate $L[U]$ up to a larger regular cardinal λ, what you get is $L[F]$ where F is the club filter on λ; from which many things follow.*

In Madison, as spring approached, Kleene organised a picnic for the logicians, and at it Ken came up to me and said "I can prove that $0^\sharp$ exists." I asked him if he was feeling quite well, and then he admitted that his proof would require an assumption.

The assumption was that there is a nontrivial elementary embedding from L to itself.

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Credits

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