Characterizing Subgroups of Compact Abelian Groups

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Abstract

We prove that every countable subgroup of a compact metrizable abelian group has a characterizing set. As an application, we answer several questions on maximally almost periodic (MAP) groups and give a characterization of the class of (necessarily MAP) abelian topological groups whose Bohr topology has countable pseudocharacter.

1 Introduction

We shall write our abelian groups additively, and we view the circle $\mathbb{T}$ as $\mathbb{R}/\mathbb{Z}$; then, for $z \in \mathbb{T}$, $\|z\|$ is the distance to the nearest integer, so $0 \leq \|z\| \leq 1/2$. All topological groups are assumed to be Hausdorff unless otherwise noted.

For a topological abelian group X, $\widehat{X}$ denotes the group of all continuous characters on X (i.e., the continuous homomorphisms from X to $\mathbb{T}$), and $\widehat{X}$ is given the compact open topology. When X is locally compact, the *Pontryagin Duality Theorem* lets us identify X with $\widehat{\widehat{X}}$; see [19, 22, 27]. For most of the results in this paper, X will be compact, so that $\widehat{X}$ will be discrete.

One can use countable sequences or countable sets of characters to define subgroups of X as follows:

Definition 1.1 *Let X be a topological abelian group:*

a. [16] *For a sequence $\underline{u} = \langle u_n : n \in \omega \rangle$ of elements of $\widehat{X}$, let $s_{\underline{u}}(X)$ be the set of all $x \in X$ such that $u_n(x) \to 0$. If $H = s_{\underline{u}}(X)$, we say that $\underline{u}$* characterizes *H.*

b. [20, 21] *For a countably infinite subset B of $\widehat{X}$, let $\mathcal{C}_B(X)$ be the set of all $x \in X$ such that $\langle \varphi(x) : \varphi \in B \rangle$ converges to 0 in $\mathbb{T}$. If $H = \mathcal{C}_B(X)$, we say that B* characterizes *H.*

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Note that $s_{\underline{u}}(X)$ and $\mathcal{C}_B(X)$ are both subgroups of X. We study here which subgroups of X can be characterized as an $s_{\underline{u}}(X)$ or a $\mathcal{C}_B(X)$. Observe that $s_{\underline{u}}(X)$ and $\mathcal{C}_B(X)$ are minor variants of the same notion:

Lemma 1.2 *Let X be any compact topological abelian group.*

1. *$C_B(X) = s_{\underline{u}}(X)$ for any $\underline{u}$ which is a 1-1 enumeration of B.*
2. *If H is closed and $|X : H|$ is finite, then $H = s_{\underline{u}}(X)$ for some u.*
3. *Every $\mathcal{C}_B(X)$ is a Haar null set.*
4. *If $|X : H|$ is infinite and $H = s_{\underline{u}}(X)$, then $H = \mathcal{C}_B(X)$ for some B.*

Proof. (1) is clear from the definitions. For (2), $H^{\perp} = \{\varphi \in \widehat{X} : \varphi(H) = \{0\}\}$ is finite (of size $|X : H|$), and $H = s_{\underline{u}}(X)$ for any $\underline{u} : \omega \to H^{\perp}$ which lists each character in $H^{\perp}$ infinitely often. (3) follows easily from the orthogonality of the characters; see, e.g., [4, 13, 21]. (4) will be proved in Section 2. $\square$

Note also that every $C_B(X)$ and $s_{\underline{u}}(X)$ is a Borel set (in fact, an $F_{\sigma\delta}$), and every Borel subgroup of finite index is closed and open and of positive Haar measure. By Lemma 1.2, characterization as an $s_{\underline{u}}(X)$ is equivalent to characterization as a $\mathcal{C}_B(X)$ except in the trivial case of subgroups of finite index.

The subgroups $s_{\underline{u}}(X)$ have been studied by many authors, especially in the case $X = \mathbb{T}$, where $\widehat{X}$ is identified with $\mathbb{Z}$ (see [1, 6, 7, 10, 11, 15, 18, 24, 25]). When $x \in \mathbb{T}$ is a non-torsion element, the behavior of the sequences of the form $\langle u_n(x) : n \in \omega\rangle$ is related to Diophantine approximation and dynamical systems (more specifically, the Sturmian flow [26]), as well as to the study of precompact group topologies with converging sequences [4, 5, 16].

Some instances of characterization in $\mathbb{T}$ were described in Armacost's book [1], although this book predates the "characterization" terminology. Specifically, for each prime p, the Prüfer subgroup $\mathbb{Z}_{p^\infty}$ of $\mathbb{T}$ is characterized by $\{p^n : n \in \omega\}$. Also, let $D = \{n! : n \in \omega\}$. Then the set of rational points of $\mathbb{T}$ (i.e., $\mathbb{Q}/\mathbb{Z}$) is properly contained in $\mathcal{C}_D$ (in fact, this $\mathcal{C}_D$ is uncountable). The elements of $\mathcal{C}_D$ are called the *topologically torsion elements* of $\mathbb{T}$ in [1]; they are described further in [14, 15, 17].

It is easy to see that $\mathbb{Q}/\mathbb{Z}$ can be characterized by some set of characters:

Proposition 1.3 *If $B = \{k \cdot n! : 0 < k \leq n \in \omega\}$, then $\mathbb{Q}/\mathbb{Z} = \mathcal{C}_B(\mathbb{T})$.*

It is not always so easy to write down a characterizing set explicitly, but we shall show:

Theorem 1.4 *Every countably infinite subgroup of a compact metrizable abelian group X has a characterizing set.*

This theorem resolves Problem 5.3 from [16]. The proposition and theorem will be proved in Section 2. A number of important special cases of the theorem are already in the literature. In particular, when $X = \mathbb{T}$, it was proved by Bíró, Deshouillers, and Sós [10]. Earlier results by Larcher [25] and Kraaikamp and Liardet [24] relate the characterizability of infinite cyclic subgroups $\langle\alpha\rangle$ of $\mathbb{T}$ to the continued fraction representation of α; see also [6]. It was proved in [16] that every cyclic subgroup of the group $\widehat{\mathbb{Z}_{p^\infty}}$ of p-adic integers has a characterizing set. Furthermore, Bíró [9] has proved Theorem 1.4 in the case that the subgroup is dense in X, finitely generated, and torsion-free. Also, Beiglböck, Steineder, and Winkler [8] have independently proved Theorem 1.4 by a different method.

Note that in Theorem 1.4, X must be metrizable (equivalently, second countable), since $\mathcal{C}_B(X)$ always contains $\bigcap_{\varphi\in B}\ker(\varphi)$, and this set will be uncountable if X is not metrizable. We do not know a simple general criterion for deciding whether $H \leq X$ is characterizable. For closed H, this is easy; H is of the form $s_{\underline{u}}(X)$ iff H is a G_δ (see Proposition 1.8 below), and then, applying Lemma 1.2, H is of the form $\mathcal{C}_B(X)$ iff H is a G_δ of infinite index in X. In Section 2, we shall prove the following theorem, which shows that not all F_σ subgroups can be characterized, even in metrizable compact groups:

Theorem 1.5 *Suppose that X is a compact abelian group and $H = \bigcup_n F_n$, where each $F_n \leq X$ is closed and each $F_n \lneqq F_{n+1}$. Then the following are equivalent:*

- *a.* $H = \mathcal{C}_B(X)$ *for some countably infinite* $B \subseteq \widehat{X}$.
- *b.* $H = s_{\underline{u}}(X)$ *for some* $\underline{u} : \omega \to \widehat{X}$.
- *c. For some* m*:* X/F_m *is metrizable and* $|F_{n+1} : F_n|$ *is finite for all* $n \geq m$.

Next, we give an application of Theorem 1.4 to non-compact groups. If (G, τ) is a topological abelian group, we let (G, τ^+) be its *Bohr modification*. So, τ^+ is the coarsest topology which makes all (continuous) characters of (G, τ) continuous. Clearly, τ^+ is coarser than τ. When τ is clear from context, we refer to (G, τ^+) as G^+. It is easy to find Hausdorff G for which G^+ is indiscrete, but we are primarily interested in the case where G^+ is also Hausdorff; such G are called *MAP* (*maximally almost periodic*).

A topological space X has *countable pseudocharacter* iff every singleton $\{x\}$ is a G_δ set. This implies that X is a T_1 space, which, in the case of topological groups, is equivalent to being Hausdorff.

Corollary 1.6 *For a MAP topological abelian group G the following are equivalent:*

- *a.* Every *countable* $H \leq G$ *is of the form* $s_{\underline{u}}(G)$ *for some* $\underline{u} : \omega \to \widehat{G}$.
- *b.* Some *countable* $H \leq G$ *is of the form* $s_{\underline{u}}(G)$ *for some* $\underline{u} : \omega \to \widehat{G}$.
- *c.* G^+ *has countable pseudocharacter.*

Proof. $(a) \to (b)$ is obvious. For $(b) \to (c)$, note that $K = \bigcap_n \ker(u_n)$ is a G_δ set in G^+, and $K \leq H$, so K is countable. Since G^+ is Hausdorff, it follows that $\{0\}$ is a G_δ set in G^+.

For $(c) \to (a)$: Since $\{0\}$ is a G_δ set in G^+, there is a countable $\Phi \subseteq \widehat{G}$ such that $\{0\} = \bigcap\{\ker(\varphi) : \varphi \in \Phi\}$. Let $\Delta : G \to \mathbb{T}^\Phi$ be the evaluation map: $\Delta(x)(\varphi) = \varphi(x)$. Then Δ is 1-1 and continuous. Applying Theorem 1.4 and Lemma 1.2.1, let $\underline{v} = \langle v_n : n \in \omega\rangle$ be a sequence of characters of $\mathbb{T}^\Phi$ such that $\Delta(H) = s_{\underline{v}}(\mathbb{T}^\Phi)$. If $u_n = v_n \circ \Delta$, then $H = s_{\underline{u}}(G)$. $\square$

Note that if we did not assume that G is MAP in Corollary 1.6, either of condition (a) (just using $\{0\} = s_{\underline{u}}(G)$) or condition (c) would imply that G is MAP anyway.

Since having a characterizing sequence (or set) is a rather restrictive property, one can relax it in the following way, following [16]:

Definition 1.7 *For a subgroup H of a topological abelian group X,*

$$\mathfrak{g}\mathrm{cl}(H) = \bigcap \left\{ s_{\underline{u}}(X) : \quad u : \omega \to \widehat{X} \ \& \ H \subseteq s_{\underline{u}}(X) \right\} \quad .$$

H is $\mathfrak{g}$-closed iff $\mathfrak{g}\mathrm{cl}(H) = H$.

Of course, H is $\mathfrak{g}$-closed whenever $H = s_{\underline{u}}(X)$ for some $\underline{u}$, but a $\mathfrak{g}$-closed subgroup need not be of the form $s_{\underline{u}}(X)$. For example,

Proposition 1.8 *Suppose that H is a closed subgroup of the compact X. Then:*

1. *H is $\mathfrak{g}$-closed.*
2. *$H = s_{\underline{u}}(X)$ for some $\underline{u}$ iff H is a G_δ set.*

Proof. For (1), fix $x \notin H$, and then fix $\varphi \in \widehat{X}$ with $\varphi(x) \neq 0$ but $\varphi(H) = \{0\}$. If $u_n = \varphi$ for all n, then $x \notin s_{\underline{u}}(X)$. Thus, $x \notin \mathfrak{g}\mathrm{cl}(H)$. For (2) $\leftarrow$, if H is a closed G_δ, then $H^\perp = \{\varphi \in \widehat{X} : \varphi(H) = \{0\}\}$ is countable, and then $H = s_{\underline{u}}(X)$ for any $\underline{u} : \omega \to H^\perp$ which lists each character in $H^\perp$ infinitely often. For (2) $\to$, let $K = \bigcap_n \ker(u_n)$, and let $\pi : X \to X/K$ be the natural quotient map. Then X/K is second countable, and $\pi(H)$ is closed and hence a G_δ in X/K, so $H = \pi^{-1}\pi(H)$ is a G_δ in X. $\square$

Likewise, Theorem 1.4 holds only for compact metrizable X, but metrizability is not needed for:

Corollary 1.9 *Every countably infinite subgroup of a compact abelian group X is $\mathfrak{g}$-closed.*

Proof. Fix a countable H and $x \notin H$. Then, let Γ be a countable subgroup of $\widehat{X}$ which separates the elements of $H \cup \{x\}$, let $K = \bigcap\{\ker(\psi) : \psi \in \Gamma\}$, and let $\pi : X \to X/K$ be the natural quotient map. Then π is 1-1 on $H \cup \{x\}$, and one can apply Theorem 1.4 in X/K to fix $B \subseteq \widehat{X/K}$ with $\pi(H) = \mathcal{C}_B(X/K)$. If $\{v_n : n \in \omega\}$ is a 1-1 listing of B and $u_n = v_n \circ \pi$, then $H \leq s_{\underline{u}}(X)$ and $x \notin s_{\underline{u}}(X)$. $\square$

This generalizes Theorem 1.2 of [16], which proved the result for cyclic subgroups, and it answers Problem 5.1 and Question 5.2 of [16]. Note that Corollary 1.9 actually holds for all MAP X, since one can apply the corollary in the Bohr compactification, $\mathsf{b}(X)$. We do not know if every F_σ subgroup of a compact X is $\mathfrak{g}$-closed, although, in contrast to Theorem 1.5,

Proposition 1.10 *Suppose that X is a compact abelian group and $H = \bigcup_n F_n$, where each $F_n \leq X$ is closed and each $F_n \leq F_{n+1}$. Then H is $\mathfrak{g}$-closed.*

Proof. Fix $x \in X \backslash H$. Since F_n is closed, we can choose $u_n \in \widehat{X}$ such that $F_n \leq \ker(u_n)$ and $\|u_n(x)\| \geq 1/4$. Then $H \leq s_{\underline{u}}(X)$ and $x \notin s_{\underline{u}}(X)$. $\square$

$F_{\sigma\delta}$ subgroups certainly need not be $\mathfrak{g}$-closed. In fact, whenever X is infinite, there is an $F_{\sigma\delta}$ subgroup H such that H is a Haar null set and $\mathfrak{g}\mathrm{cl}(H) = X$; see [21], which extends earlier arguments in [4, 20].

The next corollary says that every totally bounded countable metrizable group topology can be "encoded" by means of a *single* τ-convergent zero-sequence:

Corollary 1.11 *Let (G, τ) be a countable totally bounded metrizable abelian group. Then there exists a sequence $\underline{u} = \langle u_n : n \in \omega\rangle$ with $u_n \to 0$ in (G, τ) such that τ is the finest totally bounded group topology on G in which $u_n \to 0$.*

Proof. Let G_d denote the group G with the discrete topology, and let $X = \widehat{(G_d)}$. Let $H \leq X$ be the set of $\varphi : G \to \mathbb{T}$ which are τ-continuous. Then H is countable and X is metrizable. Identifying G_d with $\widehat{X}$, apply Theorem 1.4 to fix $u_n \in G$ with $H = s_{\underline{u}}(X)$. Now, suppose that τ' is a strictly finer totally bounded group topology on G. Then there is a τ'-continuous character x of G which is not τ-continuous, so that $x \in X$ but $x \notin H$. Then $x(u_n) = u_n(x) \not\to 0$ in $\mathbb{T}$, so $u_n \not\to 0$ in (G, τ'). $\square$

It is easy to see that countability is necessary; that is, if (G, τ) is a totally bounded metrizable abelian group such that τ is the finest totally bounded group topology on G in which some sequence $u_n \to 0$, then G is countable.

Corollary 1.11 should be compared to the fact that the finest *Hausdorff* group topology that makes a given sequence converge to 0 is never Fréchet-Urysohn [28]. In the case $G = \mathbb{Z}$, the encoding sequence $\underline{u}$ can often be displayed explicitly. For example, if τ is the p-adic topology, one can take $u_n = p^n$. If τ is the profinite topology, then one can let $\underline{u}$ enumerate $\{k \cdot n! : 0 \leq k \leq n \in \omega\}$ by Proposition 1.3.

2 The Proofs

We begin by relating characterizable subgroups of N to characterizable subgroups of X, where N is a subgroup of X:

Lemma 2.1 *Suppose that $H \le N \le X$, where X is a compact abelian group, N is closed in X, X/N is metrizable, and $H = \mathcal{C}_B(N)$ for some countably infinite $B \subseteq \widehat{N}$. Then $H = \mathcal{C}_D(X)$ for some countably infinite $D \subseteq \widehat{X}$.*

Proof. Let $N^\perp = \{\varphi \in \widehat{X} : \varphi(N) = \{0\}\}$, which is countable.

If $|N^\perp| = \aleph_0$: For $\varphi \in B$, choose $\widetilde{\varphi} \in \widehat{X}$ such that $\widetilde{\varphi}\restriction N = \varphi$. Let $D = \{\widetilde{\varphi} : \varphi \in B\} \cup N^\perp$. Then $\mathcal{C}_D^X \cap N = \mathcal{C}_B^N$, and if $y \in X \backslash N$, then $y \notin \mathcal{C}_{N^\perp}^X$, so $y \notin \mathcal{C}_D^X$.

If $|N^\perp| = k < \aleph_0$: Then $|X : N| = k$. Let $D = \{\psi \in \widehat{X} : \psi\restriction N \in B\}$. For $\varphi \in B$, $|\{\psi \in D : \psi\restriction N = \varphi\}| = k$, so $\mathcal{C}_D^X \cap N = \mathcal{C}_B^N$. Now, fix $y \in X\backslash N$, and fix any $\alpha \in N^\perp$ such that $\alpha(y) \ne 0$. Note that $\psi \in D$ iff $\alpha + \psi \in D$, so $\langle \psi(y) : \psi \in D\rangle$ cannot converge to 0. $\square$

Proof of Lemma 1.2.4. We have X compact, $H \le X$, $|X : H|$ infinite, and $H = s_{\underline{u}}(X)$. We shall produce a countably infinite $D \subseteq \widehat{X}$ such that $H = \mathcal{C}_D(X)$. Let $N = \overline{H}$ and let $K = \bigcap\{\ker(u_n) : n \in \omega\}$. Then $K \le H \le N \le X$, and X/K is metrizable. If H is closed in X, then X/H is metrizable and infinite, so $H = \mathcal{C}_{H^\perp}(X)$.

Now, assume that H is not closed, so $H \ne N$. Let $v_n = u_n\restriction N$, so $H = s_{\underline{v}}(N)$. Let $B = \{v_n : n \in \omega\} \subseteq \widehat{N}$. Then B is infinite (since otherwise H would be closed), and $\{n : u_n = \psi\}$ is finite for each non-zero $\psi \in \widehat{N}$ (since $N = \overline{H}$), so $H = \mathcal{C}_B(N)$. We are now done by Lemma 2.1. $\square$

Proof of Proposition 1.3. It is clear that $\mathbb{Q}/\mathbb{Z} \subseteq \mathcal{C}_B(\mathbb{T})$. Now, any $x \in [0,1]$ can be written in a factorial expansion as $x = \sum_{n=1}^\infty c_n/(n+1)!$, where the integers c_n satisfy $0 \le c_n \le n$. Note that $\sum_{n=k}^\infty n/(n+1)! = 1/k!$. Thus, if $x \notin \mathbb{Q}$, then $c_n \notin \{0, n\}$ for infinitely many n. For any n, we have $n!\,x \equiv y_n \pmod 1$, where $y_n = n! \sum_{m=n}^\infty c_m/(m+1)!$. When $c_n \notin \{0, n\}$,

$$\frac{1}{n+1} = n!\left[\frac{1}{(n+1)!}\right] \le y_n \le n!\left[\frac{n-1}{(n+1)!} + \frac{1}{(n+1)!}\right] = \frac{n}{n+1} \quad ,$$

so we can find a k with $0 < k \le n$ such that $\|kn!x\| = \|kn!y_n\| \ge 1/4$. When $x \notin \mathbb{Q}$, there are infinitely many such k, n, so $x \notin \mathcal{C}_B(\mathbb{T})$. $\square$

We now prove three lemmas for Theorem 1.4. As is common in metric spaces, $B(x;\varepsilon)$ denotes an open ball, and $N(F;\varepsilon)$ denotes the open set $\bigcup\{B(x;\varepsilon) : x \in F\}$.

Lemma 2.2 *Let $(X;d)$ be a metric space, F any countably infinite subset of X, and F_n finite subsets such that $F_n \nearrow F$. Then there are positive $\varepsilon_n \searrow 0$ such that for all $x \notin F$, there are infinitely many n with $x \notin N(F_n;\varepsilon_n)$.*

Proof. Choose $\varepsilon_n \searrow 0$ so that $2\varepsilon_n < d(u,v)$ whenever u,v are distinct elements of F_{n+1}. Now, fix m and assume that $x \in N(F_n;\varepsilon_n)$ for all $n \geq m$; we show that $x \in F$. For each $n \geq m$, choose $u_n \in F_n$ such that $d(x,u_n) < \varepsilon_n$. Then $d(u_n,u_{n+1}) \leq d(x,u_n) + d(x,u_{n+1}) < \varepsilon_n + \varepsilon_{n+1} \leq 2\varepsilon_n$, so $u_n = u_{n+1}$. Thus, the u_n are all equal to some $u \in F$, and $x = u$. $\square$

If G is a discrete group, then G is dense in its Bohr compactification; if this fact is stated in terms of the compact group $X = \widehat{G}$, we get the following lemma, which generalizes an old result of Kronecker; see [12, p. 1188], [22, Cor. 26.16], or [23].

Lemma 2.3 *Let X be a compact abelian group, $x_1,\ldots,x_t$ a finite list of elements of X, $\psi : X \to \mathbb{T}$ a possibly discontinuous homomorphism, and $\delta > 0$. Then there is a $\varphi \in \widehat{X}$ such that $\|\psi(x_i) - \varphi(x_i)\| < \delta$ for $i = 1,\ldots,t$.*

Using this, one can give a direct proof of Corollary 1.9, without using Theorem 1.4. We have $H < X$, where H is countable and X is compact. Fix $x \notin H$ and list H as $\{e_n : n \in \omega\}$. For each n, $x \notin \langle e_0,e_1,\ldots,e_n\rangle$ implies that there is a homomorphism $\psi_n : X \to \mathbb{T}$ such that $\|\psi_n(x)\| > 1/4$ and $\ker(\psi_n)$ contains $e_0,e_1,\ldots,e_n$. By Lemma 2.3, there is a $u_n \in \widehat{X}$ such that $\|u_n(x)\| > 1/4$ and $\|u_n(e_j)\| < 1/n$ for $j = 0,1,\ldots,n$. Then $H \leq s_{\underline{u}}(X)$ and $x \notin s_{\underline{u}}(X)$.

Definition 2.4 *Let X be any topological abelian group and $E \subseteq X$. The m*-quasi-convex hull $\mathrm{qc}_m(E)$ *of E is the set*

$$\Big\{x \in X : \quad \forall \varphi \in \widehat{X} \ \big[\forall e \in E\,[\|\varphi(e)\| \leq 2^{-m-2}] \implies \|\varphi(x)\| \leq 1/4\big]\Big\}$$

$\mathrm{qc}(E) = \mathrm{qc}_0(E)$.

The *quasi-convex hull* $\mathrm{qc}(E)$ is discussed in [2, 3]. Note that $\mathrm{qc}_m(E)$ gets bigger as m gets bigger, and $E \subseteq \mathrm{qc}_0(E)$.

Lemma 2.5 *Let $m \in \omega$ and let E be a finite subset of a compact abelian group X. Then:*

a. $\mathrm{qc}_m(E) \subseteq \langle E\rangle$.

b. $\mathrm{qc}_m(E)$ *is finite.*

Proof. For (a), fix $x \notin \langle E\rangle$. There exists a possibly discontinuous homomorphism ψ such that $\psi(E) = \{0\}$ and $\|\psi(x)\| > 1/4$. Apply Lemma 2.3 to find a $\varphi \in \widehat{X}$ with $|\varphi(e)| < 2^{-m-2}$ for each $e \in E$ and $\|\varphi(x)\| > 1/4$, so that $x \notin \mathrm{qc}_m(E)$.

For (b), let $H = \langle E\rangle$. Then H is an internal direct product of cyclic groups; say $H \cong \langle a_1\rangle \times \cdots \times \langle a_n\rangle$, where $a_1, \ldots, a_n \in H$. For each j, the order $o(a_j)$ is either a positive integer or ∞. Fix a positive integer $M \geq \max\{o(a_j) : o(a_j) \neq \infty\}$ such that also each $e \in E$ is of the form $\sum_{j=1}^n \mu_j a_j$, where each $|\mu_j| \leq M$. Let F_0 be the set of all elements $\sum_{j=1}^n \nu_j a_j$ such that each $|\nu_j| \leq 2^{m+1}M$. Then F_0 is finite, and we show $\mathrm{qc}_m(E) \subseteq F_0$.

Fix $x = \sum_{j=1}^n \nu_j a_j \notin F_0$, with each $|\nu_j| < o(a_j)$. Then, fix ℓ such that $|\nu_\ell| > 2^{m+1}M$. Then $o(a_\ell) > |\nu_\ell| > M$, so $o(a_\ell) = \infty$. We can then define $\psi \in \mathrm{Hom}(H, \mathbb{T})$ so that $\psi(\sum_{j=1}^n \mu_j a_j) = \mu_\ell/(2\nu_\ell) \pmod 1$ for all $\sum_{j=1}^n \mu_j a_j \in H$. Then $\psi(x) = 1/2$, but for each $e = \sum_{j=1}^n \mu_j a_j \in E$, we have $\|\psi(e)\| < M \cdot 1/(2^{m+2}M) = 2^{-m-2}$. By Lemma 2.3, there is a $\varphi \in \widehat{X}$ such that $\|\varphi(x)\| > 1/4$, but $\|\varphi(e)\| < 2^{-m-2}$ for each $e \in E$. Thus, $x \notin \mathrm{qc}_m(E)$. $\square$

This lemma actually holds for all MAP X, since one can apply the lemma in $\mathsf{b}(X)$. For $m = 0$, this lemma was proved in [2] (see Lemma 7.10 and Theorem 7.11).

The inclusion $\mathrm{qc}_m(E) \subseteq \langle E\rangle$ can fail when E is infinite. For example, $\mathrm{qc}_m(E) = \overline{E}$ for every $m \in \omega$ whenever $E = \langle E\rangle \leq X$.

Proof of Theorem 1.4. Let X be a compact metric abelian group with $G = \widehat{X}$. Let E a countably infinite subgroup of X. In view of Lemma 2.1, we may assume that E is dense in X. Let $E_n \nearrow E$, with each E_n finite. Let $F_n = \mathrm{qc}_n(E_n)$. Then $E_n \subseteq F_n \subseteq E$, $F_n \nearrow E$, and F_n is finite by Lemma 2.5. Let d be a metric for X. Applying Lemma 2.2, choose positive $\varepsilon_n \searrow 0$ such that for all $x \notin E$, there are infinitely many n with $x \notin N(F_n; \varepsilon_n)$.

Let $A_n = \{\varphi \in \widehat{X} : \forall e \in E_n\, [\|\varphi(e)\| \leq 2^{-n-2}]\}$. Applying the definition of $\mathrm{qc}_n(E_n)$,

$$\forall x \notin F_n\, \exists \varphi \in A_n\ [\|\varphi(x)\| > 1/4] \quad .$$

Hence, $\bigcup_{\varphi \in A_n} \{x \in X : \|\varphi(x)\| > 1/4\} \supseteq X \setminus N(F_n; \varepsilon_n)$. Since $X \setminus N(F_n; \varepsilon_n)$ is compact, we can choose a finite $B_n \subseteq A_n$ such that

$$\forall x \notin N(F_n, \varepsilon_n)\, \exists \varphi \in B_n\, [\|\varphi(x)\| > 1/4] \quad .$$

Set $B = \bigcup_{n \in \omega} B_n$ and note that B must be infinite, since (using $\overline{E} = X$)

$$\forall \varphi \neq 0\, [|\{n : \varphi \in A_n\}| < \aleph_0] \ . \tag{$*$}$$

Clearly, $E \subseteq \mathcal{C}_B$. To prove that $E = \mathcal{C}_B$ fix $x \notin E$. Then $x \notin N(F_n; \varepsilon_n)$ for infinitely many n. For these n, choose $\varphi_n \in B_n$ so that $\|\varphi_n(x)\| > 1/4$. By $(*)$, $\{\varphi_n : x \notin N(F_n; \varepsilon_n)\}$ is infinite, so $x \notin \mathcal{C}_B$. $\square$

Next, we prove three lemmas for Theorem 1.5.

Lemma 2.6 *Suppose that X is a compact abelian group, F a closed subgroup, and B a countably infinite subset of $\widehat{X}$. Then $F \le \mathcal{C}_B$ iff $F \le \ker(\varphi)$ for all but finitely many $\varphi \in B$.*

Proof. For the non-trivial direction, assume that $F \le \mathcal{C}_B$ but $F \not\le \ker(\varphi)$ for infinitely many $\varphi \in B$. Since $\mathcal{C}_B$ gets bigger as B gets smaller, we may assume that $\varphi{\restriction}F \ne 0$ for all $\varphi \in B$, and that one of the following two cases holds:

Case I: The $\varphi{\restriction}F = \psi \in \widehat{F}$ for all $\varphi \in B$: But then any $x \in F \backslash \ker(\psi)$ will be in $F \backslash \mathcal{C}_B$, contradicting $F \le \mathcal{C}_B$.

Case II: The $\varphi{\restriction}F$ for $\varphi \in B$ are all different: Let $D = \{\varphi{\restriction}F : \varphi \in B\}$. Then $\mathcal{C}_B \cap F = \mathcal{C}_D$, which is a null set in F by Lemma 1.2, contradicting $F \le \mathcal{C}_B$. $\square$

Lemma 2.7 *Suppose that X is a compact metric abelian group, with an invariant metric d. Let $H \le X$, where $H = \bigcup_n F_n$, each F_n is closed, and each $F_n < F_{n+1} < X$. Assume that $y_n \in F_{n+1} \backslash F_n$ are chosen so that each $d(y_{n+1}, 0) \le (d(y_n, F_n))/3$. Define $x = \sum_{k=0}^{\infty} y_k$. Then $x \notin H$.*

Proof. By induction on k, we have $d(y_{n+k}, 0) \le (d(y_n, F_n)) \cdot 3^{-k}$ for all $n \ge 0$ and $k > 0$. This shows that the sum defining x really converges, and also lets us define $x_n = \sum_{k=n}^{\infty} y_k$; so $x = x_0$. Now, fix n, and we show that $x \notin F_n$. Since $(x - x_n) \in F_n$, it is sufficient to show that $x_n \notin F_n$. Now $y_n \notin F_n$ and $x_n = y_n + x_{n+1}$. Also, $d(x_{n+1}, 0) \le \sum_{k=1}^{\infty} d(y_{n+k}, 0) \le d(y_n, F_n) \cdot \sum_{k=1}^{\infty} 3^{-k} = d(y_n, F_n)/2$. Thus, $d(x_n, F_n) \ge d(y_n, F_n) - d(x_{n+1}, 0) \ge d(y_n, F_n)/2 > 0$. $\square$

Finally, we need the (trivial) converse to Lemma 2.1:

Lemma 2.8 *Suppose that $H \le N \le X$, where X is a compact abelian group, N is closed in X, H is not closed in X, and $H = \mathcal{C}_D(X)$ for some countably infinite $D \subseteq \widehat{X}$. Then $H = \mathcal{C}_B(N)$ for some countably infinite $B \subseteq \widehat{N}$.*

Proof. By Lemma 1.2.1, $H = s_{\underline{u}}(X)$, where each $u_n \in \widehat{X}$. But then also $H = s_{\underline{v}}(N)$, where each $v_n = u_n{\restriction}N \in \widehat{N}$. We now get B by Lemma 1.2.4. $\square$

Proof of Theorem 1.5. $(a) \leftrightarrow (b)$ follows from Lemma 1.2, since $|X : H|$ is infinite. For $(c) \rightarrow (a)$, we may, replacing X by X/F_m assume that $F_m = \{0\}$. But then H is countable, so the result follows by Theorem 1.4.

For $(a) \rightarrow (c)$: Assume (a), and let $K = \bigcap\{\ker(\varphi) : \varphi \in B\}$. Then $K \le H$ and X/K is metrizable. Observe that $K \le F_m$ for some m; otherwise, we have $K \cap F_n \nearrow K$, where each $K \cap F_n \lneqq K$. If some $|K : K \cap F_n|$ is finite, this is clearly a contradiction, while if all $|K : K \cap F_n|$ are infinite, this is a contradiction by the Baire Category Theorem.

We are now done if we can show that $|F_{n+1} : F_n|$ is finite for all but finitely many n; so, assume that this is false and we shall derive a contradiction. Re-indexing, we may assume that $|F_{n+1} : F_n|$ infinite for each n, and we may assume that $F_0 = \{0\}$. We may also assume that $X = \overline{H}$; if not, replace X by $N = \overline{H}$ and apply Lemma 2.8. Then $\widehat{X} = F_0^\perp > F_1^\perp > F_2^\perp \cdots$, and $\bigcap_n F_n^\perp = \{0\}$.

We may assume that X is metrizable; otherwise replace X by X/K. We may assume that $0 \notin B$, so that $B = \bigcup_n B_n$, where $B_n = B \cap F_n^\perp \backslash F_{n+1}^\perp$. By Lemma 2.6, each B_n is finite.

We shall now produce an $x \in \mathcal{C}_B(X)\backslash H$, contradicting $H = \mathcal{C}_B(X)$.

Let d be an invariant metric for X. Inductively choose $y_n \in F_{n+1}\backslash F_n$ so that $\|\varphi(y_n)\| \leq 2^{-n}$ for all $\varphi \in B_0 \cup \cdots \cup B_n$ and each $d(y_{n+1}, 0) \leq (d(y_n, F_n))/3$; this is possible because each F_n is nowhere dense in F_{n+1}. Let $x = \sum_{k=0}^\infty y_k$. Then $x \notin H$ by Lemma 2.7.

If $\varphi \in B_n$, then $\|\varphi(x)\| \leq \sum_{k=0}^\infty \|\varphi(y_k)\| = \sum_{k=n}^\infty \|\varphi(y_k)\| \leq \sum_{k=n}^\infty 2^{-k} = 2^{1-n}$. Thus $\langle \varphi(x) : \varphi \in B \rangle$ converges to 0 in $\mathbb{T}$. □

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