Arcs in the Plane*

Joan E. Hart[†] and Kenneth Kunen[‡§]

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Abstract

Assuming PFA, every uncountable subset E of the plane meets some C^1 arc in an uncountable set. This is not provable from $\mathrm{MA}(\aleph_1)$, although in the case that E is analytic, this is a ZFC result. The result is false in ZFC for C^2 arcs, and the counter-example is a perfect set.

1 Introduction

As usual, an *arc* in $\mathbb{R}^n$ is a set homeomorphic to a closed bounded subinterval of $\mathbb{R}$. A (simple) *path* is a homeomorphism g mapping a compact interval onto A. For $k \geq 1$, a path is C^k iff it is a C^k function, and an arc A is C^k iff A is the image of some C^k path g, with $g'(t) \neq 0$ for all t; equivalently, A has a C^k arc length parameterization. Also, A is C^∞ iff it is C^k for all k. We consider the following:

Question. For $n \geq 2$, if $E \subseteq \mathbb{R}^n$ is uncountable, must there be a "nice" arc A such that $E \cap A$ is uncountable?

Obviously, the answer will depend on the definition of "nice". We should expect ZFC results for closed E (equivalently, for analytic E), and independence results for arbitrary E. In general, under CH things are as bad as possible, and under PFA, things are as good as possible. In most cases, the results are the same for all $n \geq 2$, and trivial for $n = 1$.

For arbitrary arcs, the results are quite old. In ZFC, every closed uncountable set meets some arc in an uncountable set. For $n \geq 2$, arcs are nowhere dense in $\mathbb{R}^n$; so under CH there is a Luzin set that meets every arc in a countable set. At

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[†]University of Wisconsin, Oshkosh, WI 54901, U.S.A., hartj@uwosh.edu

[‡]University of Wisconsin, Madison, WI 53706, U.S.A., kunen@math.wisc.edu

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the other extreme, under MA($\aleph_1$), every uncountable $E \subseteq \mathbb{R}^n$ meets some arc in an uncountable set.

If "nice" means "straight line", then there is a trivial counter-example: a perfect set E which meets every line in at most two points.

Paper [3] introduces results where "nice" means "almost straight":

Definition 1.1 *Let $\rho : \mathbb{R}^n \backslash \{0\} \twoheadrightarrow S^{n-1}$ be the perpendicular retraction given by $\rho(x) = x/\|x\|$. Then $A \subseteq \mathbb{R}^n$ is* ε–directed *iff for some $v \in S^{n-1}$, $\|\rho(x-y)-v\| \leq \varepsilon$ or $\|\rho(x-y)+v\| \leq \varepsilon$ whenever x, y are distinct points of A.*

The retraction $\rho(x-y)$ may be viewed as the *direction* from y to x. Every $A \subseteq \mathbb{R}^n$ is trivially $\sqrt{2}$–directed, and A is 0–directed iff A is contained in a straight line. If "nice" means "ε–directed", a counter-example to the Question is consistent with MA($\aleph_1$). By [3], the existence of a *weakly* Luzin set is consistent with MA($\aleph_1$), and whenever $\varepsilon < \sqrt{2}$, a weakly Luzin set (see [3] Definition 2.4) meets every ε–directed set in a countable set. However, under SOCA, which follows from PFA, whenever $\varepsilon > 0$, every uncountable set meets some ε–directed arc in an uncountable set (see Lemma 4.1). Every C^1 arc is a finite union of ε–directed arcs, and hence we get the stronger:

Theorem 1.2 *PFA implies that every uncountable subset of $\mathbb{R}^n$ meets some C^1 arc in an uncountable set.*

MA($\aleph_1$) is not sufficient for this theorem, because, as in the ε-directed case ($\varepsilon < \sqrt{2}$), a weakly Luzin set provides a counter-example. Theorem 1.2 and the following ZFC theorem for closed sets are proved in Section 4.

Theorem 1.3 *If $P \subseteq \mathbb{R}^n$ is closed and uncountable, then there is a C^1 arc A with a Cantor set $Q \subseteq P \cap A$. Hence, for every $\varepsilon > 0$, P meets some ε–directed arc in an uncountable set.*

If the Question asks for a C^2 arc, then a ZFC counter-example exists in the plane, and hence in any $\mathbb{R}^n$ ($n \geq 2$). The counter-example, given in Theorem 1.5, is a *non-squiggly* subset of the plane. A simple example of a non-squiggly set is a C^1 arc whose tangent vector either always rotates clockwise or always rotates counter-clockwise. In particular, such an arc may be the graph of a convex function $f \in C^1([0,1], \mathbb{R})$; a real differentiable function is *convex* iff its derivative is a monotonically increasing function. But non-squiggly makes sense for non-smooth arcs, and in fact for arbitrary subsets of the plane:

Definition 1.4 *$A \subseteq \mathbb{R}^2$ is* non-squiggly *iff there is a δ, with $0 < \delta \leq \infty$, such that whenever $\{x, y, z, t\} \in [A]^4$ and $diam(\{x, y, z, t\}) \leq \delta$, point t is not interior to triangle xyz.*

Theorem 1.5 *There is a perfect non-squiggly set* $P \subseteq \mathbb{R}^2$ *which lies in a* C^1 *arc* A *and which meets each* C^2 *arc in a finite set. Moreover, the* C^1 *arc* A *may be taken to be the graph of a convex function.*

As "nice" notions, non-squiggly is orthogonal to smooth:

Theorem 1.6 *There is a perfect set* $P \subseteq \mathbb{R}^2$ *which lies in a* C^∞ *arc and which meets every non-squiggly set in a countable set.*

Note that by Ramsey's Theorem, every infinite set in $\mathbb{R}^2$ has an infinite non-squiggly subset.

In Definition 1.4, allowing $\delta < \infty$ makes non-squiggly a local notion; so, piecewise linear arcs and some spirals (such as $r = \theta\ ;\ 0 \le \theta < \infty$) are non-squiggly. However, the results of this paper would be unchanged if we simply required $\delta = \infty$. For $0 < \delta \le \infty$, if $E \subseteq \mathbb{R}^2$ meets a non-squiggly set A in an uncountable set, then E has uncountable intersection with a subset of A whose diameter is at most δ.

The proof of Theorem 1.5 uses the assumption that each C^2 arc is parameterized by some g whose derivative is nowhere 0. Dropping this requirement on g' yields a weaker notion of C^∞, and a different result. Call a C^k arc *strongly* C^k, and say that an arc is *weakly* C^k iff it is the image of a C^k path. Then, an arc is weakly C^∞ iff it is weakly C^k for all k.

Theorem 1.7 *If* $E \subseteq \mathbb{R}^n$ *is bounded and infinite, then it meets some weakly* C^∞ *arc in an infinite set.*

Theorems 1.5 and 1.6 are proved in Section 5; Theorem 1.7 and some related facts are proved in Section 6.

2 Remarks on Hermite Splines

We construct the arc of Theorem 1.3 by first producing a "nice" Cantor set $Q \subseteq P$. Then we apply results, described in this section, that make it possible to draw a smooth curve through a closed set. These results are a natural extension of results of Hermite for drawing a curve through a finite set. Our proof of Theorem 1.3 reduces the problem to the case where $Q \subset \mathbb{R}^2$ is the graph of a function with domain $D \subset \mathbb{R}$; then we extend this function to all of $\mathbb{R}$ to produce the desired arc.

First consider the case $|D| = 2$, or interpolation on an interval $[a_1, a_2]$; we find $f \in C^1(\mathbb{R})$ with predetermined values b_1, b_2 and slopes s_1, s_2 at a_1, a_2, *and* we bound f, f' on $[a_1, a_2]$ in terms of the *three* slopes: $s := (b_2 - b_1)/(a_2 - a_1)$, and s_1, s_2. Following Hermite, f will be the natural cubic interpolation function. Our bounds show that if s, s_1, s_2 are all close to each other, then f is close to the linear interpolation function L.

Lemma 2.1 *Given* s_1, s_2, b_1, b_2 *and* $a_1 < a_2$, *let* $s = (b_2 - b_1)/(a_2 - a_1)$, *and let* $L(x) = b_1 + s(x - a_1)$. *Let* $M = \max(|s_1 - s|, |s_2 - s|)$. *Then there is a cubic* f *with each* $f(a_i) = b_i$ *and each* $f'(a_i) = s_i$, *such that*

1. $|(f(x_2) - f(x_1))/(x_2 - x_1) - s| \le 3M$ *whenever* $a_1 \le x_1 < x_2 \le a_2$.

Moreover, for all $x \in [a_1, a_2]$:

2. $|f'(x) - s| \le 3M$.
3. $|f(x) - L(x)| \le 2M(a_2 - a_1)$.

Proof. (1) follows from (2) and the Mean Value Theorem. Now, let

$$f(x) = L(x) + \beta_2(x - a_1)^2(x - a_2) + \beta_1(x - a_1)(x - a_2)^2$$
$$f'(x) = s + \beta_2(x - a_1)^2 + \beta_1(x - a_2)^2 + 2(\beta_2 + \beta_1)(x - a_1)(x - a_2) \quad .$$

Then $f(a_i) = b_i$ is obvious, and setting $\beta_i = (s_i - s)/(a_2 - a_1)^2$ we get $f'(a_i) = s_i$. To see (2) and (3), note that $|\beta_i| \le M/(a_2 - a_1)^2$, and $(x - a_1)(a_2 - x) \le (a_2 - a_1)^2/4$ (the maximum of $(x - a_1)(a_2 - x)$ occurs at the midpoint $x = \frac{a_1 + a_2}{2}$). ☕

Next, we consider extending, to all of $\mathbb{R}$, a C^1 function defined on a closed $D \subset \mathbb{R}$. First note that there are two possible meanings for "$f \in C^1(D)$":

Definition 2.2 *Assume that* $f, h \in C(D, \mathbb{R})$, *where* D *is a closed subset of* $\mathbb{R}$. *Then* $f' = h$ in the strong sense *iff*

$$\forall x \in D\ \forall \varepsilon > 0\ \exists \delta > 0\ \forall x_1, x_2 \in D$$
$$\left[x_1 \ne x_2 \ \&\ |x_1 - x|, |x_2 - x| < \delta \ \longrightarrow\ \left| \frac{f(x_2) - f(x_1)}{x_2 - x_1} - h(x) \right| < \varepsilon \right] \quad .$$

The usual or weak sense would only require this with x_1 replaced by the point x. When D is an interval, the two senses are equivalent by the continuity of h and the Mean Value Theorem. Note that $f' = h$ in the strong sense iff there is a $g \in C(D \times D, \mathbb{R})$ such that $g(x, x) = h(x)$ for each x and $g(x_1, x_2) = g(x_2, x_1) = (f(x_2) - f(x_1))/(x_2 - x_1)$ whenever $x_1 \ne x_2$.

If D is finite, then $f' = h$ in the strong sense for *any* $f, h : D \to \mathbb{R}$, and the cubic Hermite spline is an $\tilde{f} \in C^1(\mathbb{R}, \mathbb{R})$ with $\tilde{f} \restriction D = f$ and $\tilde{f}' \restriction D = h$. The following lemma generalizes this to an arbitrary closed D:

Lemma 2.3 *Assume that* $f, h \in C(D, \mathbb{R})$, *where* D *is a closed subset of* $\mathbb{R}$, *and* $f' = h$ *in the strong sense. Then there are* $\tilde{f}, \tilde{h} \in C(\mathbb{R}, \mathbb{R})$ *such that* $\tilde{f}' = \tilde{h}$, $\tilde{f} \supseteq f$, *and* $\tilde{h} \supseteq h$.

Proof. Let $\mathcal{J}$ be the collection of pairwise disjoint open intervals covering $\mathbb{R}\backslash D$. For each interval $J \in \mathcal{J}$, we shall define $\tilde{f}, \tilde{h}$ on J.

If J is the unbounded interval (a_1, ∞), with $a_1 \in D$, define $\tilde{f}$ and $\tilde{h}$ by the linear $\tilde{f}(x) = f(a_1) + (x - a_1)h(a_1)$ and $\tilde{h}(x) = h(a_1)$, for $x \in J$. Then $\tilde{f}, \tilde{h}$ are continuous on $\overline{J}$ and $\tilde{f}' = \tilde{h}$ on J. At a_1, the derivative of $\tilde{f}$ from the right is $h(a_1)$; the derivative of $\tilde{f}$ from the left, as well as the continuity of $\tilde{f}, \tilde{h}$ from the left, depend on how we extend f to the bounded intervals.

The unbounded interval $(-\infty, a_2)$ is handled likewise.

Say $J = (a_1, a_2)$, with $a_1, a_2 \in D$. On J, let $\tilde{f}$ be the cubic obtained from Lemma 2.1, with $b_i = f(a_i)$ and $s_i = h(a_i)$. Then $\tilde{h}$ is the quadratic $\tilde{f}'$ on J.

To finish, we verify that $\tilde{f}, \tilde{h}$ are continuous and $\tilde{f}' = \tilde{h}$ on $\mathbb{R}$. Fix $z \in D$. Since differentiability implies continuity, it suffices to show that $\tilde{h}$ is continuous at z, and that $h(z) = \tilde{f}'(z) = \lim_{x \to z}(\tilde{f}(x) - \tilde{f}(z))/(x - z)$. We verify the continuity of $\tilde{h}$ from the left at z, and the difference quotient's limit for x approaching z from the left; a similar argument handles these from the right. Let $\sigma = h(z) = \tilde{h}(z)$. Fix $\varepsilon > 0$. Apply continuity of f, h on D, and the fact that $f' = h$ in the strong sense, to fix $\delta > 0$ such that whenever $z - \delta < a_1 < a_2 < z$ with $a_1, a_2 \in D$, the quantities $|s - \sigma|$, $|s_i - \sigma|$, $|b_i - f(z)|$, $|(f(a_2) - f(z))/(a_2 - z) - \sigma|$ are all less than ε, where $s_i = h(a_i)$ and $b_i = f(a_i)$, for $i = 1, 2$, and $s = (b_2 - b_1)/(a_2 - a_1)$. Let $M = \max(|s_1 - s|, |s_2 - s|)$, as in Lemma 2.1; so $M \leq 2\varepsilon$.

Assume that z is a limit from the left of points of D and of points of $\mathbb{R}\backslash D$; otherwise checking continuity and the derivative from the left is trivial. Thus, δ may be taken small enough so that $(z - \delta, z)$ misses any unbounded interval in $\mathcal{J}$. For $a_1, a_2 \in D$ with $(a_1, a_2) \in \mathcal{J}$ and $x \in \mathbb{R}$ with $z - \delta < a_1 \leq x < a_2 < z$, the bounds from Lemma 2.1 imply that $|\tilde{h}(x) - \sigma| \leq |\tilde{h}(x) - s| + |s - \sigma| \leq 3M + \varepsilon \leq 7\varepsilon$. So $\tilde{h}$ is continuous. To see that $h(z) = \tilde{f}'(z)$, observe that by elementary geometry, the slope $(\tilde{f}(x) - \tilde{f}(z))/(x - z)$ is between the slopes $(\tilde{f}(x) - \tilde{f}(a_2))/(x - a_2)$ and $(\tilde{f}(a_2) - \tilde{f}(z))/(a_2 - z)$. Applying Lemma 2.1 again, $|(\tilde{f}(x) - \tilde{f}(a_2))/(x - a_2) - \sigma| \leq 3M + \varepsilon \leq 7\varepsilon$, so we are done.

3 Some Flavors of OCA

The proofs of Theorems 1.2 and 1.3 will require the results of this section.

Definition 3.1 *For any set E, let $E^\dagger = (E \times E) \setminus \{(x, x) : x \in E\}$. If $W \subseteq E^\dagger$ with $W = W^{-1}$, then $T \subseteq E$ is W–*free *iff $T^\dagger \cap W = \emptyset$, and T is W–*connected *iff $T^\dagger \subseteq W$.*

Then SOCA is the assertion that whenever E is an uncountable separable metric space and $W = W^{-1} \subseteq E^\dagger$ is open, there is either an uncountable W–free set or an uncountable W–connected set.

SOCA follows from PFA, but not from MA($\aleph_1$). It clearly contradicts CH. However, it is well-known [2] that SOCA is a ZFC theorem when E is Polish:

Lemma 3.2 *Assume that E is an uncountable Polish space, $W \subseteq E^\dagger$ is open, and $W = W^{-1}$. Then there is a Cantor set $Q \subseteq E$ which is either W–free or W–connected.*

Proof. Shrinking E, we may assume that E is a Cantor set; in particular, non-empty open sets are uncountable. Assume that no Cantor subset is W–free. Since W is open, the closure of a W–free set is W–free; thus every W–free set has countable closure, and is hence nowhere dense.

Now, inductively construct a tree, $\{P_s : s \in 2^{<\omega}\}$. Each P_s is a non-empty clopen subset of E, with $\mathrm{diam}(P_s) \leq 2^{-\mathrm{lh}(s)}$. $P_{s \frown 0}$ and $P_{s \frown 1}$ are disjoint subsets of P_s such that $(P_{s \frown 0} \times P_{s \frown 1}) \subseteq W$. Let $Q = \bigcup \{\bigcap_n P_{f \restriction n} : f \in 2^\omega\}$; then Q is W–connected. ☕

An "open covering" version of SOCA follows by induction on ℓ:

Lemma 3.3 *Let E be an uncountable separable metric space, with $E^\dagger = \bigcup_{i<\ell} W_i$, where $\ell \in \omega$ and each $W_i = W_i^{-1}$ is open in $E^\dagger$. Assuming* SOCA*, there is an uncountable $T \subseteq E$ such that T is W_i–connected for some i. In the case that E is Polish, this is a* ZFC *result and T can be made perfect.*

There is also a version of this lemma obtained by replacing the covering by a continuous function:

Lemma 3.4 *Assume that E is an uncountable Polish space, F is a compact metric space, $g \in C(E^\dagger, F)$, and $g(x, y) = g(y, x)$ whenever $x \neq y$. Then there is a Cantor set $Q \subseteq E$ such that $g \restriction Q^\dagger$ extends continuously to some $\hat{g} \in C(Q \times Q, F)$.*

Proof. Construct a tree, $\{P_s : s \in 2^{<\omega}\}$. Each P_s is a Cantor subset of E, with $\mathrm{diam}(P_s) \leq 2^{-\mathrm{lh}(s)}$. $P_{s \frown 0}$ and $P_{s \frown 1}$ are disjoint subsets of P_s. Also, apply Lemma 3.3 to get $\mathrm{diam}(g(P_s^\dagger)) \leq 2^{-\mathrm{lh}(s)}$. Let $Q = \bigcup \{\bigcap_n P_{f \restriction n} : f \in 2^\omega\}$. ☕

Now, to prove Theorem 1.2, we need, under PFA, a version of Lemma 3.4 where E is just an uncountable subset of a Polish space. We begin with the following, from Abraham, Rubin, and Shelah [1]:

Theorem 3.5 *Assume* PFA*. Then* $\mathrm{OCA}_{[\mathrm{ARS}]}$ *holds. That is, let E be a separable metric space of size $\aleph_1$. Assume that $E^\dagger = \bigcup_{i<\ell} W_i$, where $\ell \in \omega$ and each $W_i = W_i^{-1}$ is open in $E^\dagger$. Then E can be partitioned into sets $\{A_j : j \in \omega\}$ such that for each j, A_j is W_i–connected for some i.*

The terminology $\mathrm{OCA}_{[\mathrm{ARS}]}$ was used by Moore [4] to distinguish it from other flavors of the Open Coloring Axiom in the literature. Actually, [1] does not mention PFA, but rather its Theorem 3.1 shows, by iterated ccc forcing, that $\mathrm{OCA}_{[\mathrm{ARS}]}$ is consistent with $\mathrm{MA}(\aleph_1)$; but the same proof shows that it is true under PFA. In our proof of Theorem 1.2, we only need $\mathrm{MA}(\aleph_1)$ plus $\mathrm{OCA}_{[\mathrm{ARS}]}$, so in fact every model of $2^{\aleph_0} = \aleph_1 \wedge 2^{\aleph_1} = \aleph_2$ has a ccc extension satisfying the result of Theorem 1.2.

To use $\mathrm{OCA}_{[\mathrm{ARS}]}$ for our version of Lemma 3.4, we need the A_j of Theorem 3.5 to be clopen. This is not always possible, but can be achieved if we shrink E:

Lemma 3.6 *Assume* $\mathrm{MA}(\aleph_1)$. *Assume that X is a Polish space and $E \in [X]^{\aleph_1}$. For each $n \in \omega$, let $\{A^n_j : j \in \omega\}$ partition E into $\aleph_0$ sets. Then there is a Cantor set $Q \subseteq X$ and, for each n, a partition of Q into disjoint relatively clopen sets $\{K^n_j : j \in \omega\}$ such that $|Q \cap E| = \aleph_1$ and each $K^n_j \cap E = A^n_j \cap Q$.*

Proof. Note that for each n, compactness of Q implies that all but finitely many of the K^n_j will be empty.

For $s \in \omega^{<\omega}$, let $A_s = \bigcap\{A^n_{s(n)} : n < \mathrm{lh}(s)\}$, with $A_\emptyset = E$. Shrinking E, X, we may assume that whenever $U \subseteq X$ is open and non-empty, $|E \cap U| = \aleph_1$ and each $|A_s \cap U|$ is either 0 or $\aleph_1$.

Let $\mathcal{B}$ be a countable open base for X, with $X \in \mathcal{B}$. Call $\mathcal{T}$ a *nice tree* iff:

1. $\mathcal{T}$ is a non-empty subset of $\mathcal{B}\backslash\{\emptyset\}$ which is a tree under the order $\subset$, with root node X.
2. $\mathcal{T}$ has height $\mathrm{ht}(\mathcal{T})$, where $1 \le \mathrm{ht}(\mathcal{T}) \le \omega$.
3. If $U \in \mathcal{T}$ is at level ℓ with $\ell + 1 < \mathrm{ht}(\mathcal{T})$, then U has finitely many but at least two children in $\mathcal{T}$, and the closures of the children are pairwise disjoint and contained in U.
4. If $U \in \mathcal{T}$ is at level $\ell > 0$, then $\mathrm{diam}(U) \le 1/\ell$.

This labels the levels $0, 1, 2, \ldots$, with $\mathrm{ht}(\mathcal{T})$ the first empty level. Let $L_\ell(\mathcal{T})$ be the set of nodes at level ℓ. By (1)–(3), each $L_\ell(\mathcal{T})$ is a finite pairwise disjoint collection.

When $\mathrm{ht}(\mathcal{T}) = \omega$, let $Q_{\mathcal{T}} = \bigcap_{\ell\in\omega} \bigcup L_\ell(\mathcal{T}) = \bigcap_{\ell\in\omega} \mathrm{cl}(\bigcup L_\ell(\mathcal{T}))$. Then $Q_{\mathcal{T}}$ is a Cantor set, so it is natural to force with finite trees approximating $\mathcal{T}$. Since many Cantor sets are disjoint from E, each forcing condition p will have, as a side condition, a finite $I_p \subseteq E$ which is forced to be a subset of Q.

Define $p \in \mathbb{P}$ iff p is a triple $(\mathcal{T}, I, \varphi) = (\mathcal{T}_p, I_p, \varphi_p)$, such that:

a. $\mathcal{T}$ is a nice tree of some finite height $h = h_p \ge 1$.
b. I is finite and $I \subseteq E \cap \bigcup L_{h-1}(\mathcal{T})$.
c. $\varphi : \mathcal{T} \to \omega^{<\omega}$ with $\varphi(U) \in \omega^\ell$ for $U \in L_\ell(\mathcal{T})$.
d. $\varphi(V) \supseteq \varphi(U)$ whenever $V \subseteq U$.

e. If $s = \varphi(U)$ then $A_s \cap U \neq \emptyset$ and $I_p \subseteq A_s$.

Define $q \leq p$ iff $\mathcal{T}_q$ is an end extension of $\mathcal{T}_p$ and $I_q \supseteq I_p$ and $\varphi_q \supseteq \varphi_p$. Then $\mathbb{1} = (\{X\}, \emptyset, \{(X, \emptyset)\})$. $\mathbb{P}$ is ccc (and σ–centered) because p, q are compatible whenever $\mathcal{T}_p = \mathcal{T}_q$ and $\varphi_p = \varphi_q$. If G is a filter meeting the dense sets $\{p : h_p > n\}$ for each n, then G defines a tree $\mathcal{T} = \mathcal{T}_G = \bigcup\{\mathcal{T}_p : p \in G\}$ of height ω, and $Q = Q_{\mathcal{T}}$ is a Cantor set. We also have $\varphi_G = \bigcup\{\varphi_p : p \in G\}$, so $\varphi_G : \mathcal{T}_G \to \omega^{<\omega}$; also, let $I_G = \bigcup\{I_p : p \in G\}$.

Note that for each $x \in E$, $\{p : x \in I_p \vee x \notin \bigcup L_{h_p - 1}(\mathcal{T}_p)\}$ is dense in $\mathbb{P}$. If G meets all these dense sets, then $Q \cap E = I_G$. We may then let $K^n_j = Q \cap \bigcup\{U \in L_{n+1}(\mathcal{T}_G) : \varphi(U)(n) = j\}$.

Finally, if we list E as $\{e_\beta : \beta < \omega_1\}$, note that each set $\{p : \exists \beta > \alpha\, [e_\beta \in I_p]\}$ is dense, so that we may force $Q \cap E$ to be uncountable.

Lemma 3.7 *Assume* PFA. *Assume that X is a Polish space, F is a compact metric space, $E \in [X]^{\aleph_1}$, $g \in C(X^\dagger, F)$, and $g(x, y) = g(y, x)$ whenever $x \neq y$. Then there is a Cantor set $Q \subseteq X$ such that $|Q \cap E| = \aleph_1$ and $g \restriction Q^\dagger$ extends continuously to some $\hat{g} \in C(Q \times Q, F)$.*

Proof. For each n, we may use compactness of F to cover $X^\dagger$ by finitely many open sets, $W^n_i = (W^n_i)^{-1}$ for $i < \ell_n$, such that each $\mathrm{diam}(g(W^n_i)) \leq 2^{-n}$. It follows by Theorem 3.5 that for each n, we may partition E into sets $\{A^n_j : j \in \omega\}$ such that each A^n_j is W^n_i–connected for some i, so that $\mathrm{diam}(g((A^n_j)^\dagger)) \leq 2^{-n}$.

By Lemma 3.6, we have a Cantor set $Q \subseteq X$ and, for each n, a partition of Q into disjoint relatively clopen sets $\{K^n_j : j \in \omega\}$ such that $|Q \cap E| = \aleph_1$ and each $K^n_j \cap E = A^n_j \cap Q$. Shrinking Q, we may assume $Q \cap E$ is dense in Q, so that each $A^n_j \cap Q$ is dense in K^n_j and $\mathrm{diam}(g((K^n_j)^\dagger)) \leq 2^{-n}$.

Now, fix $x \in Q$. For each n, x lies in exactly one of the K^n_j, and we may let $H^n = \mathrm{cl}(g((K^n_j)^\dagger))$ for that j. Then $\bigcap_n H^n$ is a singleton, and we may define $\hat{g}$ on the diagonal by $\{\hat{g}(x, x)\} = \bigcap_n H^n$. It is easily seen that this $\hat{g}$ is continuous on $Q \times Q$.

4 Proofs of Positive Results

Lemma 4.1 *Fix an uncountable $E \subseteq \mathbb{R}^n$ and an $\varepsilon > 0$. Assuming* SOCA, *there is an uncountable $T \subseteq E$ such that T is ε–directed. In the case that E is Polish, this is a* ZFC *result and T can be made perfect.*

Proof. Let $\{V_i : i < \ell\}$ be an open cover of S^{n-1} by sets of diameter less than ε, and apply Lemma 3.3 with $W_i = \{(x,y) \in E^\dagger : \rho(x-y) \in V_i\}$.

Proof of Theorem 1.3. Applying Lemma 4.1 and shrinking P, we may assume that P is a Cantor set and that P is $2\sin(22.5°)$–directed; so, the direction between any two points of P is within 45° of some fixed direction. Rotating coordinates, we may assume that this fixed direction is along the x-axis, where we label our n axes as $x, y^1, \ldots, y^{n-1}$. Now, P is (the graph of) a function which expresses $(y^1, \ldots, y^{n-1})$ as a function of x, and $D := \mathrm{dom}(P)$ is a Cantor set. Write $P(x)$ as $(P^1(x), \ldots, P^{n-1}(x))$.

The xy^i-planar slopes of P are all in $[-1,1]$. That is, for $x_1, x_2 \in D$ with $x_1 \neq x_2$, let $g^i(x_1,x_2) = (P^i(x_2) - P^i(x_1))/(x_2 - x_1)$; then $|g^i(x_1,x_2)| \leq 1$ for all x_1, x_2. Each $g^i \in C(D^\dagger, [0,1])$ and $g^i(x_1,x_2) = g^i(x_2,x_1)$ whenever $x_1 \neq x_2$. Applying Lemma 3.4 with $F = [0,1]^{n-1}$ and shrinking D if necessary, we may assume that each g^i extends continuously to some $\hat{g}^i \in C(D \times D, [0,1])$. Let $h^i(x) = \hat{g}^i(x,x)$. Then h^i is the derivative of P^i in the strong sense. Now, we may apply Lemma 2.3 on each coordinate separately to obtain a C^1 arc $A \supseteq P$; A is the graph of a C^1 function $x \mapsto (A^1(x), \ldots, A^{n-1}(x))$ defined on an interval containing D.

Proof of Theorem 1.2. Given Lemma 3.7, the proof is almost identical to the proof of Theorem 1.3.

When $E \subseteq \mathbb{R}^n$ has size exactly $\aleph_1$, and the Question of Section 1 has a positive answer, it is natural to ask whether E can be covered by $\aleph_0$ "nice" arcs. For example, under $\mathrm{MA}(\aleph_1)$, E is covered by $\aleph_0$ Cantor sets, and hence by $\aleph_0$ arcs. One can also improve Theorem 1.2:

Theorem 4.2 *PFA implies every $E \subseteq \mathbb{R}^n$ of size $\aleph_1$ can be covered by $\aleph_0$ C^1 arcs.*

The proof mimics the proof of Theorem 1.2, but uses improved versions of Lemmas 4.1, 3.6 and 3.7. The new and improved Lemma 4.1 gets E covered by $\aleph_0$ ε–directed sets, using Theorem 3.5 rather than SOCA.

The covering versions of Lemmas 3.6 and 3.7 get Cantor sets $Q_\ell \subseteq X$ for $\ell \in \omega$ satisfying the conditions of the lemmas and so that $E \subseteq \bigcup_\ell Q_\ell$. To get the Q_ℓ for $\ell \in \omega$, force with the finite support product of ω copies of the poset $\mathbb{P}$ described in the proof of Lemma 3.6. Then, use the Q_ℓ to prove the covering version of Lemma 3.7. Even though the proof of Lemma 3.7 shrinks Q, it does so by deleting at most countably many points from E, so these points may be covered by $\aleph_0$ straight lines. Thus, E will be covered by $\bigcup_\ell Q_\ell$ together with a countable union of lines.

5 Proofs of Negative Results

Lemma 5.1 *Let $D \subset \mathbb{R}$ be closed. Then there is an $h \in C^\infty(\mathbb{R})$ such that $h(x) \geq 0$ for all x and $D = \{x \in \mathbb{R} : h(x) = 0\}$.*

Proof. Let $U = \mathbb{R}\backslash D$; we shall call our function h_U. If $U = (a, b)$, then such h_U are in standard texts; for example, let $h_{(a,b)}(x)$ be $\exp(-1 \div (x-a)(b-x))$ for $x \in (a, b)$ and 0 otherwise. Now, say $U = \bigcup_{n\in\omega} J_n$, where each J_n is a bounded open interval. Let $h_U = \sum_{n\in\omega} c_n h_{J_n}$, where each $c_n > 0$ and the c_n are small enough so that for each $\ell \in \omega$, the ℓ^{th} derivative $h_U^{(\ell)}$ is the uniform limit of the sum $\sum_{n\in\omega} c_n h_{J_n}^{(\ell)}$.

Proof of Theorem 1.6. Let $D \subset \mathbb{R}$ be a Cantor set. Integrating the function of Lemma 5.1, fix $f \in C^\infty(\mathbb{R})$ such that $f'(x) \geq 0$ for all x and $D = \{x \in \mathbb{R} : f'(x) = 0\}$. Then f is strictly increasing.

Let P be the graph of $f \restriction D$. Fix an uncountable $A \subseteq P$, and assume that A is non-squiggly; we shall derive a contradiction. Fix $\delta > 0$ as in Definition 1.4; then, shrinking A, we may assume that $\text{diam}(A) \leq \delta$ so that whenever $\{x, y, z, t\} \in [A]^4$, point t is not interior to triangle xyz.

Let S be an infinite subset of $\text{dom}(A)$ such that every point of S is a limit, from the left and right, of other points of S.

Now, fix $a, b, c \in S$ with $a < b < c$; then $f(a) < f(b) < f(c)$. Let L be the straight line passing through $(a, f(a))$ and $(c, f(c))$. Moving b slightly if necessary, we may assume (since $f'(b) = 0$) that L does not pass through $(b, f(b))$. Then either $L(b) > f(b)$ or $L(b) < f(b)$.

Suppose that $L(b) > f(b)$. Consider triangle $(a, f(a)), (b, f(b)), (c, f(c))$. One leg of this triangle is the graph of $L\restriction[a, c]$, which passes above the point $(b, f(b))$. Since all three legs have positive slope and $f'(b) = 0$, the points $(b - \varepsilon, f(b - \varepsilon))$ are interior to the triangle when $\varepsilon > 0$ is small enough. Choosing such an ε with $b - \varepsilon \in S$ yields a contradiction.

$L(b) < f(b)$ is likewise contradictory, using points $(b + \varepsilon, f(b + \varepsilon))$.

Observe that the arc in Theorem 1.6 cannot be real-analytic, since if $f : [0, 1] \to \mathbb{R}$ is real-analytic, then $[0, 1]$ can be decomposed into finitely many intervals on which either $f'' \geq 0$ or $f'' \leq 0$. On each of these intervals, the graph of f is non-squiggly.

Proof of Theorem 1.5. As in the proof of Theorem 1.6, let $D \subset \mathbb{R}$ be a Cantor set, and fix $f \in C^\infty(\mathbb{R})$ such that f is strictly increasing, $f'(y) \geq 0$ for all y, and $D = \{y \in \mathbb{R} : f'(y) = 0\}$. Also, to simplify notation, assume that $f(\mathbb{R}) = \mathbb{R}$, so that $\varphi := f^{-1} \in C(\mathbb{R})$ and is also a strictly increasing function. Let $K = f(D)$; so K is also a Cantor set. Then φ is C^∞ on $\mathbb{R}\backslash K$, and $\varphi'(x) = +\infty$ for $x \in K$. Integrating, fix $\psi \in C^1(\mathbb{R})$ such that $\psi' = \varphi$; so ψ is a convex function.

Note that whenever $x \in K$ and $M > 0$, there is an $\varepsilon > 0$ such that $\varphi'(u) \geq M$ whenever $|u - x| < \varepsilon$. When $x - \varepsilon < a \leq v \leq b < x + \varepsilon$, we can integrate this to get $\varphi(a) + M(v - a) \leq \varphi(v) \leq \varphi(b) - M(b - v)$. Integrating again yields

$$(b-a)\varphi(a) + (b-a)^2 M/2 \leq \psi(b) - \psi(a) \leq (b-a)\varphi(b) - (b-a)^2 M/2 \quad .$$

This implies that, for $x \in K$,

$$\lim_{t \to 0} \frac{(\psi(x+t) - \psi(x))/t - \varphi(x)}{t} = +\infty \quad ; \qquad (*)$$

the argument can be broken into two cases: $t \searrow 0$ (consider $a = x < x + t = b$) and $t \nearrow 0$ (consider $a = x + t < x = b$).

Now let $P = \psi \restriction K$; so P is a Cantor set in $\mathbb{R}^2$. Suppose that P meets the C^2 arc A in an infinite set. Since the intersection is compact, it contains a limit point (x_0, y_0). At (x_0, y_0), the tangent to the arc A is parallel to the tangent of the C^1 arc $y = \psi(x)$; in particular, this tangent is not vertical. Thus, replacing A by a segment thereof, we may assume that A is the arc $y = \xi(x)$, where ξ is a C^2 function defined in some neighborhood of x_0. Now $y_0 = \xi(x_0) = \psi(x_0)$ and $\xi'(x_0) = \psi'(x_0) = \varphi(x_0)$. Also, since (x_0, y_0) is a limit point of the intersection, there are non-zero t_k, for $k \in \omega$, converging to 0, such that each $\psi(x_0 + t_k) = \xi(x_0 + t_k)$. Applying Taylor's Theorem to ξ,

$$\psi(x_0 + t_k) = \psi(x_0) + \varphi(x_0) t_k + \frac{1}{2} \xi''(z_k) t_k^2 \ \text{ for some } z_k \text{ between } x_0 \text{ and } x_0 + t_k \quad .$$

Since $\xi''(z_k) \to \xi''(x_0)$, we have

$$\big[(\psi(x_0 + t_k) - \psi(x_0))/t_k - \varphi(x_0)\big]/t_k \to \xi''(x_0)/2 \quad ,$$

contradicting $(*)$. ☕

If ψ were C^2, the limit in $(*)$ would be $\psi''(x)/2 \neq \infty$ (by Taylor's Theorem). Moreover, the Cantor set $P = \psi \restriction K$ meets *any* C^2 arc in a finite set. This illustrates a difference between C^1 and C^2: rotation can cure an infinite derivative, but not an infinite second derivative. Even though $\varphi'(x) = \infty$ for $x \in K$, rotating the graph of $\varphi \restriction K$ gives us the graph of $f \restriction D$, which lies on a C^∞ arc.

6 Remarks on Arcs

Although the notion of *strongly* C^k is the one capturing the geometric notion of "smooth", every polygonal path is weakly C^∞. Moreover, the standard formulas for evaluating line integrals (e.g., $\int_A \vec{\Phi}(\vec{x}) \cdot d\vec{x} = \int_a^b \vec{\Phi}(\vec{g}(t)) \cdot \vec{g}\,'(t)\,dt$) only require the path $\vec{g}(t)$ to be *weakly* C^1; the arc A may have corners, with the velocity vector $\vec{g}'(t)$ becoming zero at a corner.

Theorems 1.2, 1.3, and 1.6 produce *strongly* C^k arcs. In contrast, Theorem 1.5 produces a perfect set which *meets* all strongly C^2 arcs in a finite set. Theorem 1.7 shows that the *weakly* version of this theorem is false.

To prove Theorem 1.7, we begin with an interpolation result.

Definition 6.1 *An* interpolation function *is a* $\psi \in C([0,1],[0,1])$ *such that* $\psi(0) = 0$ *and* $\psi(1) = 1$.

Definition 6.2 *Assume that* D *is a closed subset of* $[0,1]$ *with* $0, 1 \in D$. *Fix* $g \in C(D, \mathbb{R}^n)$, *and let* ψ *be an interpolation function. Then the* ψ interpolation for g *is the function* $\tilde{g} \in C([0,1], \mathbb{R}^n)$ *extending* g *such that whenever* (a,b) *is a maximal interval in* $[0,1] \backslash D$ *and* $u \in (a,b)$,

$$\tilde{g}(u) = g(a) + (g(b) - g(a))\psi((u-a)/(b-a)) \quad .$$

It is easily seen that $\tilde{g}$ is indeed continuous on $[0,1]$.

Definition 6.3 *Assume that* D *is a closed subset of* $[0,1]$ *with* $0, 1 \in D$. *Then* $g \in C(D, \mathbb{R}^n)$ *is* flat *iff for all* $\alpha \in \omega$, *there is a bound* M_α *such that for all* $u, t \in D$ $\|g(u) - g(t)\| \le M_\alpha |u-t|^\alpha$.

That is, g is flat iff for all $\alpha \in \mathbb{N} = \omega \setminus \{0\}$, g is uniformly Lipschitz of order α on D. If D is finite, then every $g : D \to \mathbb{R}^n$ is flat. If D contains an interval, then a flat g is constant on that interval, because it is Lipschitz of order 2 there; for $t < t+h$ in the interval: $\|g(t+h) - g(t)\| \le k \cdot M_2 \cdot h^2/k^2$ for all $k \ge 1$.

Lemma 6.4 *Assume that* D *is a closed subset of* $[0,1]$ *with* $0, 1 \in D$. *Assume that* $g \in C(D, \mathbb{R}^n)$ *is flat. Let* ψ *be an interpolation function such that* $\psi \in C^\infty([0,1],[0,1])$ *and* $\psi^{(k)}(0) = \psi^{(k)}(1) = 0$ *for all* $k \in \mathbb{N}$. *Let* $\tilde{g}$ *be the* ψ *interpolation for* g. *Then* $\tilde{g} \in C^\infty([0,1], \mathbb{R}^n)$ *and* $\tilde{g}^{(k)}(t) = 0$ *for all* $t \in D$ *and all* $k \in \mathbb{N}$.

Proof. It is sufficient to produce bounds B_k giving the following Lipschitz condition for all $t \in D$ and $u \notin D$:

1. $\|\tilde{g}(u) - \tilde{g}(t)\| \le B_0 |u-t|^2$.
2. $\|\tilde{g}^{(k)}(u)\| \le B_k |u-t|^2$ for $k \in \mathbb{N}$.

Note that (1)(2) fail for $u, t \notin D$, since the derivatives there need not be 0. On the other hand, (1) holds for $u, t \in D$, because g is flat.

Observe that (1) and 2-Lipschitz on D prove $\tilde{g}'(t) = 0$ for $t \in D$, so that (2) makes $\tilde{g} \in C^1([0,1], \mathbb{R}^n)$. For $k \geq 2$, induct on k to see that $\tilde{g} \in C^{(k)}([0,1], \mathbb{R}^n)$: (2) for $k-1$ and the fact that $\tilde{g}^{(k-1)}$ is 2-Lipschitz on D prove $\tilde{g}^{(k)}(t) = 0$ for $t \in D$, so (2) for k makes $g^{(k)}$ continuous.

To prove (1)(2), assume, without loss of generality, $t < u$. To handle (1)(2) together, let $Q_0(u,t) = \|\tilde{g}(u) - \tilde{g}(t)\|$, and for $k > 0$, $Q_k(u,t) = \|\tilde{g}^{(k)}(u)\|$. Consider the two cases:

Case I. $(t,u) \cap D = \emptyset$: Say $t = a < u < b$, where $a, b \in D$ and (a,b) is a maximal interval in $[0,1] \backslash D$. So

$$Q_k(u,t) = \|g(b) - g(a)\| \cdot \left| \psi^{(k)} \left(\frac{u-a}{b-a} \right) \right| \cdot \frac{1}{(b-a)^k} \quad .$$

Let S_k be the largest value taken by the function $|\psi^{(k)}|$. Consider:

Subcase I.1. $(b-a)^2 \leq (u-a)$: Here,

$$Q_k(u,t) \leq \|g(b) - g(a)\| \cdot S_k \cdot \frac{1}{(b-a)^k} \cdot \frac{(u-a)^2}{(u-a)^2} \leq M_{k+4} S_k (u-a)^2 \quad .$$

Subcase I.2. $(b-a)^2 \geq (u-a)$: In this case, use Taylor's Theorem and the assumption $\psi^{(n)}(0) = 0$, for all $n \in \mathbb{N}$, to bound $|\psi^{(k)}(z)|$ by $\frac{S_{2k+4}}{(k+4)!} z^4$. Then,

$$Q_k(u,t) \leq M_0 \cdot \left| \psi^{(k)} \left(\frac{u-a}{b-a} \right) \right| \cdot \frac{(b-a)^{k+4}}{(u-a)^{k+4}} \cdot \frac{(u-a)^{k+4}}{(b-a)^{2k+4}} \leq M_0 \cdot \frac{S_{2k+4}}{(k+4)!} \cdot (u-a)^2 \quad .$$

Case II. $(t,u) \cap D \neq \emptyset$: Let $a = \sup(D \cap [t,u])$, so $t < a < u$ and Case I applies to a, u. For (1), use the fact that g is flat, together with

$$\|\tilde{g}(u) - \tilde{g}(t)\| \leq \|\tilde{g}(u) - \tilde{g}(a)\| + \|g(a) - g(t)\| \quad .$$

For (2), $\|\tilde{g}^{(k)}(u)\| \leq B_k |u-a|^2 \leq B_k |u-t|^2$.

Proof of Theorem 1.7. Passing to a subset, and possibly translating it, let $E = \{\vec{x}_j : j \in \omega\}$, where the $\vec{x}_j$ converge to $\vec{0}$, and

a. $\|\vec{x}_0\| > \|\vec{x}_1\| > \|\vec{x}_2\| > \cdots$.
b. $\|\vec{x}_j\| \leq 2^{-j^2}$ for each j.

Let A be the set obtained by connecting each $\vec{x}_j$ to $\vec{x}_{j+1}$ by a straight line segment; so A is a "polygonal" arc, with ω steps. Moreover, the natural path which traverses it from $\vec{0}$ to $\vec{x}_0$ will be 1-1, because (a) guarantees that the line segments forming A meet only at the $\vec{x}_j$. Let $D = \{0\} \cup \{2^{-j} : j \in \omega\}$, and define $g : D \to \mathbb{R}^n$ by $g(0) = \vec{0}$ and $g(2^{-j}) = \vec{x}_j$. Then g is flat, by (b) (with $M_\alpha = 2^{1+\alpha+\alpha^2}$).

Let $\psi \in C^\infty(\mathbb{R})$ be such that

☞ $\psi(t) = 0$ when $t \leq 0$ and $\psi(t) = 1$ when $t \geq 1$.
☞ $\psi'(t) > 0$ for $0 < t < 1$.
☞ $\psi^{(k)}(0) = \psi^{(k)}(1) = 0$ for $k \geq 1$.

Such a ψ may be obtained by integrating a scalar multiple of the function described in Lemma 5.1. Let $\tilde{g} : [0, 1] \to \mathbb{R}^n$ be the ψ interpolation for g. Then, by Lemma 6.4, $\tilde{g} \in C^\infty([0, 1], \mathbb{R}^n)$. ☕

For the path $\tilde{g}$ in the preceding proof, all $\tilde{g}^{(k)}$ (for $k \geq 1$) will be $\vec{0}$ when passing through each $\vec{x}_j$, so that no acceleration is felt when rounding a corner. Also, each $\tilde{g}^{(k)}$ will be $\vec{0}$ at $t = 0$.

Now consider the perfect set version.

Theorem 6.5 *If $E \subseteq \mathbb{R}^n$ is Borel and uncountable, then E meets some weakly C^∞ arc in an uncountable set.*

Proof. Write elements of $\mathbb{R}^n$ as $\vec{x} = (x^1, \ldots, x^n)$. By shrinking and rotating E, we may assume that E is a Cantor set and the projection π^1 of E on the x^1 coordinate is 1-1. Shrinking E further, we may assume that $E = \bigcap_j (\bigcup \{F_\sigma : \sigma \in \{0, 2\}^j\})$, where the F_σ are compact and form a tree and each $\mathrm{diam}(F_\sigma) \leq 3^{-(\mathrm{lh}(\sigma))^2}$.

In $\mathbb{R}$, the "t–axis", let D be the usual middle-third Cantor set. Then $D = \bigcap_j (\bigcup \{I_\sigma : \sigma \in \{0, 2\}^j\})$, where I_σ is an interval of length $3^{-\mathrm{lh}(\sigma)}$. Let $g : D \twoheadrightarrow E$ be the natural homeomorphism. So, if $\alpha \in \{0, 2\}^\omega$, it determines the point $t_\alpha = \sum_{i \in \omega} (\alpha_i 3^{-i}) \in D$. Then $\bigcap_{i \in \omega} I_{\alpha \restriction i} = \{t_\alpha\}$ and $\bigcap_{i \in \omega} F_{\alpha \restriction i} = \{g(t_\alpha)\}$.

Note that g is flat. Let $\psi \in C^\infty(\mathbb{R})$ be as in the proof of Theorem 1.7, and let $\tilde{g}$ be the ψ interpolation for g. Then $\tilde{g} \in C^\infty([0, 1], \mathbb{R}^n)$.

Finally, in choosing E and the F_σ, make sure that if $\sigma < \tau$ lexicographically, then all elements of $\pi^1(F_\sigma)$ are less than all elements of $\pi^1(F_\tau)$. This will guarantee that $\pi^1 \circ g : D \to \mathbb{R}$ is order-preserving, so that $\tilde{g}$ is a 1-1 function. ☕

Under MA($\aleph_1$), if $E \subseteq \mathbb{R}^n$ has size $\aleph_1$, then E can be covered by $\aleph_0$ weakly C^∞ arcs. In particular, E can be covered by $\aleph_0$ copies, or rotated copies, of the perfect set $g(D)$ constructed in the preceding proof.

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