Measures on Corson Compact Spaces

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§**1. Introduction.** All spaces considered here are Hausdorff.

Suppose X is compact and μ is a Radon probability measure on X. We say that μ is *separable* if the measure algebra of $\langle X, \mu \rangle$ is separable (as a metric space; equivalently, $L^1(\mu)$ is separable). Haydon asked whether the existence of a non-separable Radon measure on X implies that X can be mapped continuously onto $[0,1]^{\omega_1}$. It is open whether a "yes" answer is consistent with *ZFC*, or even follows from $MA + \neg CH$; see Fremlin [4] for more discussion. Under *CH* or some other axioms of set theory, a number of counter-examples are known, due to Džamonja and Kunen [2,6]. These spaces have the additional properties of being either hereditarily Lindelöf (HL), or hereditarily separable (HS), or both. Either of these properties implies immediately that the space cannot be mapped continuously onto $[0,1]^{\omega_1}$ (since $[0,1]^{\omega_1}$ is neither HL nor HS, and both HL and HS are preserved under continuous maps).

There are many other classes of spaces that cannot be mapped continuously onto $[0,1]^{\omega_1}$ for some obvious reason. For such a class, say $\mathcal{K}$, one can ask whether there is a counter-example to Haydon's question that belongs to $\mathcal{K}$. In this paper we consider the class of all *Corson compact spaces*. Recall that a compact space X is called *Corson compact* if it can be embedded in a Σ-product of real lines. Since every separable subspace of a Corson compact space is second countable, it is easy to see that no Corson compact space can be mapped continuously onto $[0,1]^{\omega_1}$.

So, we ask, *can there be a non-separable Radon measure on a Corson compact space?*

It follows from results already known that the answer to this question is independent of *ZFC*. There is no such space under $MA + \neg CH$. To see this, let X be a Corson compact space with a Radon probability measure μ. By removing all open subsets of X of measure 0, we may assume without loss of generality that X itself is the *support* of μ – that is, all nonempty open subsets of X have positive measure. This implies that X satisfies the countable chain condition (ccc). But under $MA + \neg CH$, a Corson compact space that satisfies the ccc is second countable (see [4]), which implies that μ is separable. On the other hand, the HL space constructed in [6] under *CH* is easily seen to be Corson compact.

In this paper, we prove that the statement, "there is a Corson compact space with a non-separable Radon measure", is *equivalent* to a number of natural statements in set theory. Let $MA_{\rm ma}(\omega_1)$ denote $MA(\omega_1)$ restricted to measure algebras. We shall prove:

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1.1. Theorem. The following statements are equivalent:

(1) There is a Corson compact space which has a non-separable Radon probability measure.
(2) There is a first countable Corson compact space X which has a Radon probability measure μ such that the measure algebra of $\langle X, \mu \rangle$ is isomorphic to the measure algebra of 2^{ω_1} with the usual product measure.
(3) $MA_{\mathrm{ma}}(\omega_1)$ fails
(4) 2^{ω_1} with the usual product measure is the union of ω_1 nullsets.

Observe that $MA_{\mathrm{ma}}(\omega_1)$ is much weaker than full $MA(\omega_1)$. For example, $MA_{\mathrm{ma}}(\omega_1)$ is true in the random real model, or in any model with a real-valued measurable cardinal; in both of these models, most of the combinatorial consequences of *MA* fail. On the other hand, there are models in which $\mathfrak{p} = \mathfrak{c} = \omega_2$ holds (which implies most of the elementary combinatorial consequences of *MA*), but yet (1–4) of Theorem 1.1 hold also.

Note that any X satisfying (1) or (2) of Theorem 1.1 cannot be HS, since any separable Corson compact is second countable. Such an X could be an L-space (HL and not HS), however; such an example was constructed in [6] under *CH*. It is natural, then, to ask whether one can construct such an X just assuming the failure of $MA_{\mathrm{ma}}(\omega_1)$. We do not know the answer to this question. But we do know that a similar but stronger assumption, still weaker than *CH*, suffices.

1.2. Theorem. Suppose that there is a family $\mathcal{A}$ consisting of ω_1 nullsets in 2^{ω_1} such that every nullset $N \subseteq 2^{\omega_1}$ is contained in some member of $\mathcal{A}$. Then there is a Corson compact L-space X with a non-separable Radon probability measure μ.

The hypothesis of this theorem holds, for example, in any model obtained by adding any number of Sacks reals side-by-side over a model of *CH*.

Theorem 1.1 is proved in §3. Theorem 1.2 is proved in §4, where we also prove that the existence of an L-space with a "nice enough" measure implies the existence of a family $\mathcal{A}$ as in Theorem 1.2.

We conclude this Introduction with some additional remarks.

The notion of *Eberlein compact* is somewhat stronger than Corson compact. Every ccc Eberlein compact is second countable (Rosenthal [9]), so by the above argument, every Radon measure on an Eberlein compact is separable.

The *Borel* sets are the sets in the least σ-algebra containing the open sets. The *Baire* sets are the sets in the least σ-algebra containing the open F_σ sets; in a 0-dimensional compact space, this is the same as the least σ-algebra containing the clopen sets. For a second countable compact space (such as 2^α for $\alpha < \omega_1$), the Borel sets and Baire sets are the same, but they are not the same in 2^{ω_1}. In 2^{ω_1} every Baire set depends only on countably many co-ordinates, but this is not true for Borel sets.

The usual product measure on 2^{ω_1} is *completion regular*, as is every Haar measure on a compact group (see, e.g., Theorem 64.H of Halmos [5]). This means that for every Borel set E, there are Baire A, B such that $A \subseteq E \subseteq B$ and $B \backslash A$ is a nullset. This implies in particular that every nullset of 2^{ω_1} is contained in a Baire G_δ-nullset. This fact will be used in the proofs of Theorems 1.1 and 1.2.

The hypotheses about coverings by nullsets in Theorems 1.1 and 1.2 are most frequently studied on the space 2^ω (equivalently, $[0,1]$). If 2^ω is the union of ω_1 nullsets, then, by taking inverse projections, the same is true of 2^{ω_1}, but the converse need not hold; for example, it fails in the model obtained by iterating, with finite support, adding single random reals. However, by completion regularity, the existence of a family of ω_1 nullsets such that every nullset is covered by a nullset in the family is equivalent for 2^ω and 2^{ω_1}. (Let $\mathcal{A}$ be a family of ω_1 nullsets of 2^{ω_1} such that every nullset is covered by a nullset in the family. We may assume without loss of generality that every $A \in \mathcal{A}$ is a G_δ. Let $\pi : 2^{\omega_1} \to 2^\omega$ denote the projection. For every $A \in \mathcal{A}$ put $B_A = \{x \in 2^\omega : \pi^{-1}(\{x\}) \subseteq A\}$. Then the family $\{B_A : A \in \mathcal{A}\}$ is as required for 2^ω.)

By a result of Cichon, Kamburelis, and Pawlikowski [1], the existence of such a family has a surprising consequence for dense subsets of the measure algebra:

1.3. Lemma. Suppose that there is a family $\mathcal{A}$ consisting of ω_1 nullsets in 2^{ω_1} such that every nullset $N \subseteq 2^{\omega_1}$ is contained in some member of $\mathcal{A}$. Then there is a family $\mathcal{B}$ consisting of ω_1 closed positive measure G_δ sets in 2^{ω_1} such that every Borel set of positive measure contains some member of $\mathcal{B}$.

Proof. Such a family in 2^α for countable α follows immediately from [1], and the family in 2^{ω_1} now follows by completion regularity. □

§**2. Preliminaries.** We make some general remarks here on the construction of compact spaces with non-separable Radon measures.

A complete probability measure μ on a space X is said to be *Radon* if it is defined on the Borel subsets of X and has the property that the measure of each Borel set is the supremum of the measures of its compact subsets.

Our construction is patterned after the the inverse limit constructions of Федорчук, Kunen, and Džamonja [3,6,2]. Here, in order to utilize (4) of Theorem 1.1, we wish to keep track of an explicit measure isomorphism between our space and the usual product measure on 2^{ω_1}. To do this, it will be convenient to construct our X as a proper closed subspace of $(\omega+1)^{\omega_1}$. Then, the isomorphism will be induced by mapping each $n \in \omega$ to 0 and ω to 1.

Definition. For each ordinal α, $\varphi_\alpha : (\omega+1)^\alpha \to 2^\alpha$ is defined by: $\varphi_\alpha(f)(\xi)$ is 0 if $f(\xi) < \omega$ and 1 if $f(\xi) = \omega$. λ_α denotes the usual product measure on 2^α. For $\alpha \leq \beta$, define $\pi^\beta_\alpha : (\omega+1)^\beta \to (\omega+1)^\alpha$ by $\pi^\beta_\alpha(f) = f \restriction \alpha$; likewise, σ^β_α is the natural projection from 2^β onto 2^α. If $\alpha \leq \omega_1$ and $A \subseteq 2^\alpha$ then $\hat{A}$ denotes $(\sigma^{\omega_1}_\alpha)^{-1}(A) \subseteq 2^{\omega_1}$.

We shall see that φ_{ω_1} will be 1-1 on X, and will induce a measure isomorphism between X and 2^{ω_1}. Observe now that φ commutes with projection, in that $\sigma^\beta_\alpha \circ \varphi_\beta = \varphi_\alpha \circ \pi^\beta_\alpha$.

We now describe the construction of our space X. We shall choose X_α, for $\alpha \leq \omega_1$, so that (among other things):

R1. X_α is a closed subspace of $(\omega+1)^\alpha$, and $\pi^\beta_\alpha(X_\beta) = X_\alpha$ whenever $\alpha < \beta \leq \omega_1$.
R2. For every $n < \omega$, $X_n = \{\{0\} \cup \{\omega\}\}^n$.

Observe that X_γ is now determined from the earlier X_α at limit γ:

$$X_\gamma = \{f \in (\omega+1)^\gamma : \forall \alpha < \gamma (f \restriction \alpha \in X_\alpha)\} \quad .$$

Topologically, X_γ is the inverse limit of the previous X_α.

For $\alpha \le \beta$, define $\hat{\pi}^\beta_\alpha : X_\beta \to X_\alpha$ by $\hat{\pi}^\beta_\alpha = \pi^\beta_\alpha \restriction X_\beta$.

We also choose μ_α for $\omega \le \alpha \le \omega_1$ so that:

R3. μ_α is a finitely additive probability measure on the clopen subsets of X_α, and $\mu_\alpha = \mu_\beta(\hat{\pi}^\beta_\alpha)^{-1}$ whenever $\alpha < \beta \le \omega_1$. All non-empty clopen sets have positive measure.

For limit γ, μ_γ is determined from the earlier μ_α, since each clopen $C \subseteq X_\gamma$ is of the form $(\hat{\pi}^\gamma_\alpha)^{-1}(D)$ for some $\alpha < \gamma$ and some clopen $D \subseteq X_\alpha$.

The measures μ_α, $\alpha \le \omega_1$, have a unique extension to a Radon measure on X_α, which we denote by $\hat{\mu}_\alpha$ (see Fremlin [4, p. 279]).

Since $X_\omega = \{\{0\} \cup \{\omega\}\}^\omega$, we simply define $\mu_\omega(C)$ for every clopen subset $C \subseteq X_\omega$ by the formula $\mu_\omega(C) = \lambda_\omega(\varphi_\omega(C))$.

We will now describe how we construct $X_{\alpha+1}$ and $\mu_{\alpha+1}$ from X_α and μ_α for every $\omega \le \alpha < \omega_1$.

R4. For every $\omega \le \alpha < \omega_1$ there is a sequence $\langle A^\alpha_n : n < \omega \rangle$ of closed subsets of X_α so that

(1) If $n \ne m$ then $A^\alpha_n \cap A^\alpha_m = \emptyset$,
(2) $\sum_{n<\omega} \hat{\mu}_\alpha(A^\alpha_n) = 1$,
(3) For every $n < \omega$ and every relatively open set $U \subseteq A^\alpha_n$, $\hat{\mu}_\alpha(U) > 0$, and
(4) $X_{\alpha+1} = (X_\alpha \times \{\omega\}) \cup (\bigcup_{n<\omega} A^\alpha_n \times \{n\})$.

Here, we identify $(\omega+1)^{\alpha+1}$ with $(\omega+1)^\alpha \times (\omega+1)$. Observe that $X_{\alpha+1}$ is a closed subset of $(\omega+1)^{\alpha+1}$ and that $\pi^{\alpha+1}_\alpha(X_{\alpha+1}) = X_\alpha$. So the requirements **R4** and **R1** are consistent. We now define $\mu_{\alpha+1}$. Informally, $X_{\alpha+1}$ has two pieces; one is a copy of X_α and one is a copy of $\bigcup_{n<\omega} A^\alpha_n$, which equals X_α modulo a nullset. Then $\mu_{\alpha+1}$ gives each piece measure $\frac{1}{2}$, and distributes the measure μ_α equitably over the two pieces. Formally,

R5. For every $\omega \le \alpha < \omega_1$ and clopen $C \subseteq X_{\alpha+1}$,

$$\mu_{\alpha+1}(C) = \frac{1}{2}\left(\hat{\mu}_\alpha(\hat{\pi}^{\alpha+1}_\alpha(C \cap (X_\alpha \times \{\omega\}))) + \sum_{n<\omega} \hat{\mu}_\alpha(\hat{\pi}^{\alpha+1}_\alpha(C \cap (A^\alpha_n \times \{n\})))\right).$$

It is left as an exercise to the reader to verify that $\mu_{\alpha+1}$ is a finitely additive probability measure on the clopen subsets of $X_{\alpha+1}$ and that $\mu_\alpha = \mu_{\alpha+1}(\pi^{\alpha+1}_\alpha)^{-1}$. It is easy to see inductively that for every α, $\hat{\mu}_\alpha$ gives each point measure 0 and each non-empty clopen set positive measure. (For the latter statement, use Requirement **R4**(3).)

We remark that in this construction for every $\omega \le \alpha < \omega_1$ there are only three requirements for the sequence of closed sets $\langle A^{\alpha+1}_n \rangle_n$, namely, **R4**(1), (2) and (3). Modulo these requirements we have the freedom to pick the $\langle A^\alpha_n \rangle_n$ as we want. This will be exploited in the forthcoming sections.

Now put $X = X_{\omega_1}$, $\mu = \mu_{\omega_1}$, and $\varphi = \varphi_{\omega_1} \restriction X$. As in [2,6], the measure algebra $\langle \mathcal{B}, \hat{\mu} \rangle$ of $\langle X, \hat{\mu} \rangle$ is isomorphic to the usual measure algebra of 2^{ω_1}. In [2,6], the isomorphism was proved to exist using Maharam's Theorem [7], but here, the isomorphism is induced by the explicit function φ (by (4) of the next Lemma).

2.1. Lemma. For $\omega \le \alpha < \omega_1$:
(1) $\varphi_\alpha \restriction X_\alpha : X_\alpha \to 2^\alpha$ is 1-1.
(2) If $B \subseteq 2^\alpha$ is Borel then $X_\alpha \cap \varphi_\alpha^{-1}(B)$ is Borel in X_α and $\hat{\mu}_\alpha(X_\alpha \cap \varphi_\alpha^{-1}(B)) = \lambda_\alpha(B)$.
(3) If $B \subseteq X_\alpha$ is Borel then $\varphi_\alpha(B)$ is Borel in 2^α and $\hat{\mu}_\alpha(B) = \lambda_\alpha(\varphi_\alpha(B))$.
(4) φ induces an isomorphism between the measure algebras of $\langle X, \hat{\mu}\rangle$ and $\langle 2^{\omega_1}, \lambda_{\omega_1}\rangle$.

Proof. (1) is proved by induction, using the fact that the A_n^α are disjoint.

To prove (2) and (3), it is sufficient to consider the case when B is clopen. First, by induction, show that if $B \subseteq 2^\alpha$ is clopen then $X_\alpha \cap \varphi_\alpha^{-1}(B)$ is Borel in X_α, and if $B \subseteq X_\alpha$ is clopen, then $\varphi_\alpha(B)$ is Borel in 2^α. The fact that φ preserves the measure is likewise proved by induction, using the formula which defines $\mu_{\alpha+1}$ from μ_α.

For (4), we define a measure isomorphism, Φ, from the measure algebra of $\langle 2^{\omega_1}, \lambda_{\omega_1}\rangle$ onto the measure algebras of $\langle X, \hat{\mu}\rangle$. An element of the measure algebra of $\langle 2^{\omega_1}, \lambda_{\omega_1}\rangle$ is of the form $[B]$ (the equivalence class of B modulo null sets), where B is a Baire set in 2^{ω_1}. Choose an $\alpha \in (\omega, \omega_1)$ such that $B = \hat{E}$ for some Borel $E \subseteq 2^\alpha$, and let $\Phi([B]) = [\varphi_\alpha^{-1}(E)]$. Note that this is independent of the α chosen. By (2) and (3), Φ is a measure isomorphism. $\square$

§**3. Proof of Theorem 1.1.** We shall prove (4) $\Rightarrow$ (2) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Rightarrow$ (4). Note that (2) $\Rightarrow$ (1) is trivial, so there are only three things to prove.

Proof of (4) $\Rightarrow$ (2)**.** We aim at making X Corson compact by making sure that points are not being split too often. We assume that 2^{ω_1} is the union of ω_1 nullsets. Using the fact that the usual product measure on 2^{ω_1} is completion regular (see the Introduction), we may, for every $\alpha < \omega_1$, choose a G_δ-nullset $N_\alpha \subseteq 2^\alpha$ such that the collection

$$\{\hat{N}_\alpha : \alpha < \omega_1\}$$

covers 2^{ω_1}. We may additionally assume that for $\alpha \le \beta$ we have $\hat{N}_\alpha \subseteq \hat{N}_\beta$.

Now, we impose the additional requirement on the choice of the A_n^α in the inductive construction of the X_α:

R6. For every $\omega \le \alpha < \omega_1$, $\varphi_\alpha^{-1}(N_\alpha) \cap \bigcup_n A_n^\alpha = \emptyset$.

Since $\varphi_\alpha^{-1}(N_\alpha)$ is a nullset, we can achieve this without any problem. So, we are done if we can verify that X is a first countable Corson compact space.

Fix any $f \in X$. Then, fix $\alpha < \omega_1$ such that $\varphi(f) \in \hat{N}_\alpha$. Then for all $\beta \in [\alpha, \omega_1)$, $\varphi(f) \in \hat{N}_\beta$, so $f \restriction \beta \in \varphi_\alpha^{-1}(N_\beta)$, so, by Requirement **R6**, $(\hat{\pi}_\beta^{\beta+1})^{-1}(\{f \restriction \beta\})$ contains the point $\langle f \restriction \beta, \{\omega\}\rangle$ only. It follows that $(\hat{\pi}_\alpha^{\omega_1})^{-1}(\{f \restriction \alpha)\} = \{f\}$, so $\{f\}$ is a G_δ-subset of X. It also follows that $\forall \beta \ge \alpha(f(\beta) = \omega)$. Thus X is first countable (all points are G_δ sets), and X is a subset of the Σ-product

$$\{f \in (\omega+1)^{\omega_1} : (\exists \alpha < \omega_1)(\forall \beta \ge \alpha)(f(\beta) = \omega)\} \quad ,$$

and hence Corson compact. $\square$

Proof of (3) $\Rightarrow$ (4)**.** Let $\mathcal{B}$ be any abstract measure algebra, and suppose, for $\alpha < \omega_1$, D_α is dense in $\mathcal{B}$, but no ultrafilter meets all the D_α. We prove that 2^{ω_1} is the union of ω_1 nullsets. First, by Maharam's Theorem [7], we may assume that $\mathcal{B}$ is the measure algebra of some 2^κ with the usual product measure. Next, since $\mathcal{B}$ is ccc and the equivalence classes of closed G_δ sets are dense in $\mathcal{B}$, we may choose, for each α, an $A_\alpha \subseteq D_\alpha$ such that $A_\alpha = \{[K_\alpha^n] : n \in \omega\}$, A_α is a maximal antichain in $\mathcal{B}$, and each K_α^n is a closed G_δ. Since A_α is a maximal antichain, $N_\alpha = 2^\kappa \setminus \bigcup_{n\in\omega} K_\alpha^n$ is a nullset. Let $\{M_\gamma : \gamma < \omega_1\}$ list all finite intersections from $\{K_\alpha^n : n < \omega, \alpha < \omega_1\}$ which happen to be nullsets.

We claim that 2^κ is covered by the N_α, M_γ $(\alpha, \gamma < \omega_1)$. If not, pick a point p which is not covered. For each α, choose n_α such that $p \in K_\alpha^{n_\alpha}$. Then every finite intersection from $\mathcal{F} = \{[K_\alpha^{n_\alpha}] : \alpha < \omega_1\}$ has positive measure (since p is not in any M_γ), so $\mathcal{F}$ would extend to an ultrafilter which meets all the A_α, and hence all the D_α.

So, 2^κ is covered by ω_1 closed G_δ nullsets. Since each of these nullsets is a Baire set, and therefore has countable support, 2^{ω_1} is also covered by ω_1 nullsets. □

Proof of (1) $\Rightarrow$ (3)**.** We assume that $MA_{\mathrm{ma}}(\omega_1)$ holds, let μ be a Radon measure on the Corson compact X, and prove that μ is separable. Without loss of generality, we may assume that every non-empty open subset of X has positive measure. With this assumption, we now show that X must be second countable, which implies that μ is separable.

First, applying the definition of Corson compact, we assume that $X \subseteq [0,1]^\lambda$ and for each $f \in X$, $\{\alpha \in \lambda : f(\alpha) \neq 0\}$ is countable. Let $J = \{\alpha \in \lambda : \exists f \in X(f(\alpha) \neq 0)\}$. If J is countable, then X is second-countable, so we assume J is uncountable and derive a contradiction. Choose distinct $\alpha_\xi \in J$ for $\xi < \omega_1$. For each ξ, let π_ξ be projection onto coordinate α_ξ: $\pi_\xi(f) = f(\alpha_\xi)$. Choose ϵ_ξ such that $U_\xi = \pi_\xi^{-1}(\epsilon_\xi, 1] \neq \emptyset$.

Applying $MA_{\mathrm{ma}}(\omega_1)$, there is an uncountable $L \subseteq J$ such that $\{U_\xi : \xi \in L\}$ has the finite intersection property. L exists because $MA(\omega_1)$ for a ccc partial order implies that the order has ω_1 as a precaliber. Here the order in question is the measure algebra of X.

Now, choose $f \in \bigcap_{\xi\in L} \overline{U}_\xi$. Then $f(\alpha_\xi) > 0$ for all $\xi \in L$, contradicting that $\{\alpha \in \lambda : f(\alpha) \neq 0\}$ is countable. □

The referee points out a fifth equivalent to (1) – (4):

(5) There is a compact space X and a finite Radon measure μ on X such that all non-empty open subsets of X have positive measure and X does not have caliber ω_1.

To see the equivalence, note that (5) $\Rightarrow$ (3) is like (1) $\Rightarrow$ (3), and (4) $\Rightarrow$ (5) follows from the proof of (4) $\Rightarrow$ (2).

§4. **Proof of Theorem 1.2.** First, using ideas from [6], we state some abstract conditions on X, μ which will imply that X is an L-space.

4.1. Lemma. Suppose that X, μ satisfy:
(1) X is compact, and μ is a finite Radon measure on X.
(2) All non-empty open sets have positive measure.
(3) All points have measure 0.
(4) For all closed nowhere-dense G_δ sets $K \subseteq X$, $\mu(K) = 0$ and K is second countable.
Then:
(5) For all Borel $B \subseteq X$, the following are equivalent:
  (a) $\mu(B) = 0$.
  (b) B is second countable.
  (c) B is separable.
  (d) B is nowhere dense.
(6) X is an L-space.

Proof. X is ccc (by (2)), so every nowhere dense set is a subset of a closed nowhere-dense G_δ sets. Applying (4), we get $(d) \Rightarrow (a)$ and $(d) \Rightarrow (b)$. Also, $(b) \Rightarrow (c)$ is trivial.

To prove $(a) \Rightarrow (d)$: We may assume that B is a G_δ nullset, and let $U_n \searrow B$, where each U_n is open and $\mu(U_n) \searrow 0$. Suppose B were dense in some non-empty open set V. For each n, $V \backslash U_n$ is nowhere-dense, so $\mu(V \backslash U_n) = 0$ (by $(d) \Rightarrow (a)$), so $\mu(V \backslash B) = 0$, contradicting (2).

To prove $(c) \Rightarrow (d)$: Suppose that S is a countable subset of B and S is dense in B. By (3), $\mu(S) = 0$, so, by $(a) \Rightarrow (d)$, S is nowhere dense. Hence, B is nowhere dense.

To prove (6): X is not separable by (5). To prove X is HL, let K be any closed set, and we prove K is a G_δ. Since the measure is Radon, there is some closed G_δ $H \supseteq K$ with $\mu(H) = \mu(K)$. But then $H \backslash K$ is a nullset, and hence second countable by (5), so K is a G_δ. □

Proof of 1.2. As in the proof of (4) $\Rightarrow$ (2) in §3, we let $N_\alpha \subseteq 2^\alpha$ be a G_δ-nullset such that $\hat{N}_\alpha \subseteq \hat{N}_\beta$ whenever $\alpha \le \beta$. Now we can assume that the N_α cover all nullsets – not just points. So, assume that whenever $M \subseteq 2^{\omega_1}$ is a nullset, there is an $\alpha \in (\omega, \omega_1)$ such that $M \subseteq \hat{N}_\alpha$.

Furthermore, by Lemma 1.3, we may fix closed positive measure $B_\alpha \subseteq 2^\alpha$ for $\omega \le \alpha < \omega_1$ such that whenever $S \subseteq 2^{\omega_1}$ is a Baire set of positive measure, there are unboundedly many $\alpha \in (\omega, \omega_1)$ such that $\hat{B}_\alpha \subseteq S$. Now we add one more requirement to our construction:

R7. Whenever $\omega \le \alpha < \omega_1$, $A_0^\alpha \subseteq \varphi_\alpha^{-1}(B_\alpha)$.

There is no problem with this, since B_α has positive measure. We are now done if we can verify that the X that we construct satisfies condition (4) of Lemma 4.1. So, fix a closed nowhere-dense G_δ set, $K \subseteq X$. Then, fix a $\gamma < \omega_1$ such that $K = (\hat{\pi}_\gamma^{\omega_1})^{-1}(H)$ for some $H \subseteq X_\gamma$.

We first verify that $\mu(K) = 0$. If not, then $\mu_\gamma(H) > 0$, so $\lambda_\gamma(\varphi_\gamma(H)) > 0$, so we may fix an $\alpha \in (\gamma, \omega_1)$ such that $B_\alpha \subseteq (\sigma_\gamma^\alpha)^{-1}(\varphi_\gamma(H))$, so, by **R7**, $A_0^\alpha \subseteq \varphi_\alpha^{-1}(\sigma_\gamma^\alpha)^{-1}(\varphi_\gamma(H)) = (\hat{\pi}_\gamma^\alpha)^{-1}(H)$. But then $(\hat{\pi}_\alpha^{\omega_1})^{-1}(A_0^\alpha) \subseteq K$, which is a contradiction, since $(\hat{\pi}_\alpha^{\omega_1})^{-1}(A_0^\alpha)$ has non-empty interior.

Now, since H and K are nullsets, $\lambda_\gamma(\varphi_\gamma(H)) = 0$, so we may fix an $\delta \in (\gamma, \omega_1)$ such that $(\sigma_\gamma^\delta)^{-1}(\varphi_\gamma(H)) \subseteq N_\delta$, and hence $(\sigma_\gamma^\alpha)^{-1}(\varphi_\gamma(H)) \subseteq N_\alpha$ for all $\alpha \in (\delta, \omega_1)$. But then, applying **R6**, $\hat{\pi}^{\omega_1}_{\delta}$ is 1–1 on K, so K is homeomorphic to H, and hence is second countable. $\square$

We proceed now to prove a partial converse to Theorem 1.2 – namely, the existence of an L-space with the properties of Lemma 4.1 implies a family of ω_1 nullsets covering all nullsets. Recall that the *weight* of X, $\mathrm{w}(X)$ is the least cardinality of a basis for X. As a first preliminary, we prove:

4.2. Lemma. Suppose that X, μ satisfy (1) – (4) of Lemma 4.1. Then $\mathrm{w}(X) = \omega_1$.

Proof. Clearly, $\mathrm{w}(X) \geq \omega_1$. Let $\mathcal{U}$ be the family of all open $U \subseteq X$ such that $\mathrm{w}(U) \leq \omega_1$. If $\bigcup \mathcal{U}$ is dense in X, then by HL plus (4), $\mathrm{w}(X) = \omega_1$, so we assume that $\bigcup \mathcal{U}$ is not dense and derive a contradiction. Let V be a non-empty open set such that $\overline{V}$ is disjoint from $\bigcup \mathcal{U}$. Since separable sets are nowhere dense, there is a left-separated ω_1-sequence S such that $K = \overline{S} \subseteq V$. Since K is not second countable, there is a non-empty open $W \subseteq K$. So, W is disjoint from $\bigcup \mathcal{U}$. Say $S = \{s_\alpha : \alpha < \omega_1\}$. For $\beta < \omega_1$, let $K_\beta = \overline{\{s_\alpha : \alpha < \beta\}}$. Then K_β is second countable, so (applying the Tietze Extension Theorem), there is a countable $\mathcal{F}_\beta \subseteq C(X, [0,1])$ which separates points in K_β. Then $\bigcup_{\beta<\omega_1} \mathcal{F}_\beta$ is a family of ω_1 functions which separates points in $K = \bigcup_{\beta<\omega_1} K_\beta$, so $\mathrm{w}(K) = \omega_1$. But then $W \in \mathcal{U}$, a contradiction. $\square$

As a second preliminary, we prove:

4.3. Lemma. Suppose that X is completely regular and that μ is a Radon probability measure on X such that $\mu(\{x\}) = 0$ for every $x \in X$. Let K be any compact subset of X such that $\mu(K) > 0$. Then there is a continuous $f : X \to [0,1]$ such that μf^{-1} is Lebesgue measure and $f(K) = [0,1]$.

Proof. We may assume without loss of generality that X is compact. If it is not compact, replace it by βX, with the same measure μ (supported by X).

Let $\mathcal{B}$ denote the collection of all open subsets B of X such that $\mu(\overline{B} \setminus B) = 0$. First note that $\mathcal{B}$ is a base at every closed set H. To see this, fix a neighborhood U of H. Then there is function $\xi : X \to [0,1]$ such that $\xi(x) = 0$ for all $x \in H$ and $\xi(y) = 1$ for all $y \notin U$. Now, fix a $t \in (0,1)$ such that $\xi^{-1}(\{t\})$ is a nullset (this must be true for all but countably many $t \in (0,1)$). Then $\xi^{-1}([0,t))$ is a neighborhood of H in $\mathcal{B}$ which is a subset of U.

Now, we shall construct a countable dense set D in $[0,1]$ and for every $d \in D$ an element $B_d \in \mathcal{B}$ such that:
(1) If $d, e \in D$ and $d < e$ then $\overline{B}_d \subseteq B_e$.
(2) For every $d \in D$, $\mu(B_d) = d$.
(3) If $d, e \in D$ and $d < e$ then $\mu(B_d \cap K) < \mu(B_e \cap K)$.
Assuming this can be done, define $f : X \to [0,1]$, as in the proof of Urysohn's Lemma, by the formula:

$$f(x) = \inf\{d \in D : x \in B_d\} \quad .$$

By (1), f is continuous. For every $d \in D$, $f^{-1}([0,d)) = \bigcup_{e<d} B_e$, so $\mu(f^{-1}([0,d)) = d$ by (2). This implies that μf^{-1} is Lebesgue measure. We next claim that $f(K)$ is dense in $[0,1]$. To this end, pick arbitrary $d, e \in D$ with $d < e$. By (3), there exists $x \in K$ such that $x \in B_e \setminus B_d$. For this x we clearly have $d \leq f(x) \leq e$. As a consequence, $f(K)$ is dense because D is. By compactness, $f(K) = [0,1]$.

Note that the Lemma makes no claim about the measure induced by $f \restriction K$, and all we needed from (3) was that $B_d \cap K$ is a proper subset of $B_e \cap K$. The stronger assumption in (3) just facilitates the inductive construction of D and the B_d, which we construct together, in ω steps, as follows. Suppose that we already constructed B_d and B_e, where $d < e$, while moreover no B_c is constructed for any element c between d and e. We aim at finding c in the middle third subinterval of $[d, e]$ and $B_c \in \mathcal{B}$ so that (1–3) are satisfied. Since the measure is non-atomic and $\overline{B}_d \setminus B_d$ is a nullset, there is a Borel set E such that $\overline{B}_d \subseteq E \subseteq B_e$ and

$$\mu(E) = \frac{1}{2}(\mu(B_d) + \mu(B_e)) = \frac{1}{2}(d + e) \quad ;$$
$$\mu(E \cap K) = \frac{1}{2}(\mu(B_d \cap K) + \mu(B_e \cap K)) \quad .$$

Using the fact that μ is inner and outer regular, we may now find a compact H with $B_d \subseteq H \subseteq E$, and then an open U with $H \subseteq U \subseteq \overline{U} \subseteq B_e$ such that $E \setminus H$ and $U \setminus H$ have arbitrarily small measures. In particular, we may ensure that:

$$\frac{2}{3}d + \frac{1}{3}e < \mu(U) < \frac{1}{3}d + \frac{2}{3}e \quad ;$$
$$\mu(B_d \cap K) < \mu(U \cap K) < \mu(B_e \cap K) \quad .$$

Also, since $\mathcal{B}$ is a base at H, we may assume that $U \in \mathcal{B}$. So, we add $c = \mu(U)$ to D, and set $B_c = U$. □

We do not know whether Lemma 4.3 is new, but there are related results in the literature, see e.g., Mauldin [8].

4.4. Theorem. Suppose that X, μ satisfy (1) – (4) of Lemma 4.1. Then there is a family $\mathcal{A}$ consisting of ω_1 nullsets in 2^{ω_1} such that every nullset $N \subseteq 2^{\omega_1}$ is contained in some member of $\mathcal{A}$.

Proof. We shall in fact find such a family of nullsets in $[0, 1]$ with ordinary Lebesgue measure. This is equivalent to finding such a family in 2^{ω} or (as pointed out in the Introduction), in 2^{ω_1}.

Since the measure on X is nonatomic, fix a continuous $f : X \to [0, 1]$ such that μf^{-1} is Lebesgue measure (Lemma 4.3).

By Lemma 4.2, $\mathrm{w}(X) = \omega_1$, so let $\{s_\alpha : \alpha < \omega_1\}$ be a dense subset of X. Let $K_\beta = \overline{\{s_\alpha : \alpha < \beta\}}$. Then, by Lemma 4.1, each K_β is a nullset. For $\beta < \omega_1$, let

$$N_\beta = \{x \in [0, 1] : f^{-1}(\{x\}) \subseteq K_\beta\} \quad ;$$

observe that N_β is a nullset since K_β is.

We claim that $\mathcal{A} = \{N_\beta : \beta < \omega_1\}$ is as required. To this end, let $N \subseteq [0, 1]$ be a nullset. We may assume without loss of generality that N is Borel. Then $f^{-1}(N)$ is a nullset and is Borel, and hence is second countable by (5) of Lemma 4.1. But this implies that for some $\beta < \omega_1$, $f^{-1}(N) \subseteq K_\beta$, and hence $N \subseteq N_\beta$. □

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