Limits in Function Spaces and Compact Groups*

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Abstract

For B an infinite subset of ω and X a topological group, let $\mathcal{C}_B^X$ be the set of all $x \in X$ such that $\langle x^n : n \in B \rangle$ converges to 1. $C_B^{\mathbb{T}}$ always has measure 0 in the circle group $\mathbb{T}$. If $\mathcal{F}$ is a filter of infinite sets, let $\mathcal{D}_{\mathcal{F}}^X = \bigcup\{\mathcal{C}_B^X : B \in \mathcal{F}\}$. Then $\mathcal{C}_B^X$ and $\mathcal{D}_{\mathcal{F}}^X$ are subgroups of X when X is abelian. We show that there is a filter $\mathcal{F}$ such that $\mathcal{D}_{\mathcal{F}}^{\mathbb{T}}$ has measure 0 but is not contained in any $\mathcal{C}_B^{\mathbb{T}}$. In contrast, for any compact metric group X, there is a filter $\mathcal{G}$ such that $\mathcal{D}_{\mathcal{G}}^X = X$; this follows from a more general result in this paper on limits in function spaces. Also, we show that some of the properties of $\mathcal{D}_{\mathcal{F}}^X$, for arbitrary compact groups X, are determined by the special cases $X = \mathbb{T}$ or $X = \mathbb{T}^\omega$.

1 Introduction

In this paper, we answer a question on topological groups asked by Barbieri, Dikranjan, Milan, and Weber [2], and we relate this question to some general facts in C_p theory. All spaces considered here are Hausdorff. We recall the following standard definition; see Arkhangel'skii [1] for further details on such function spaces.

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Definition 1.1 *If X, Y are topological spaces, then $C(X, Y)$ is the set of continuous functions from X to Y, and $C_p(X, Y)$ denotes $C(X, Y)$ given the topology of* pointwise convergence *(i.e., regarding $C_p(X, Y)$ as a subset of Y^X with the usual product topology).*

An obvious question is:

❀ Suppose that each $f_n \in C_p(X, Y)$ and g is a limit point of the sequence $\langle f_n : n \in \omega \rangle$; is there an infinite $B \subseteq \omega$ such that the subsequence $\langle f_n : n \in B \rangle$ converges to g pointwise (i.e., in the topology of $C_p(X, Y)$)?

The answer is well-known to be "no" in general, even when X and Y are both compact metric spaces. If Y is metric, then for each $x \in X$, there is an infinite B_x such that $\langle f_n(x) : n \in B_x \rangle \to g(x)$. B_x may need to vary with x, but if X is compact, then one can choose the B_x from a filter; in Section 2 we show:

Proposition 1.2 *Suppose that X is compact, Y is metric, and g is a limit point of $\langle f_n : n \in \omega \rangle$ in $C_p(X, Y)$. Then there is a filter $\mathcal{F} \subset [\omega]^\omega$ such that for each $x \in X$ there is $B \in \mathcal{F}$ such that $\langle f_n(x) : n \in B \rangle \to g(x)$. Furthermore, $\mathcal{F}$ can be chosen to be an F_σ subset of $\mathcal{P}(\omega)$ (identifying $\mathcal{P}(\omega)$ with $2^\omega = \{0, 1\}^\omega$ in the standard way).*

As usual, $\mathcal{P}(\omega)$ denotes the family of all subsets of ω and $[\omega]^\omega$ denotes the family of infinite subsets of ω.

One cannot replace "metric" by first countable here, since the proposition fails when X, Y are both the lexicographically ordered square; see Example 4.2.

Observe that, in the notation of Proposition 1.2, g must be a limit point of $\langle f_n : n \in B \rangle$ in $C_p(X, Y)$ for each $B \in \mathcal{F}$ (see also Lemma 2.2). So, the $\mathcal{F}$ depends on the f_n and g.

Now, turning to compact groups, consider:

Definition 1.3 *In a group X, define $E_n(x) = x^n$ for each $n \in \omega$.*

Then $E_0(x) = 1$ for all x. If X is a compact topological group, then it is well-known that 1 is a limit point of the sequence $\langle x^n : n \in \omega \rangle$ for each fixed $x \in X$. Applying this fact in each X^k, it follows that E_0 is a limit point of $\langle E_n : n \in \omega \rangle$ in $C_p(X, X)$. So if X is also metric (equivalently, second countable), then Proposition 1.2 applies to yield a filter. In fact, the same filter $\mathcal{F}$ can be chosen to work for every compact metric group, since any $\mathcal{F}$ which works for the group $\mathbb{T}^\omega$ works for all compact metric groups (see Lemma 3.2). As usual, $\mathbb{T}$ denotes the circle group.

To discuss question (❀) for $\langle E_n : n \in \omega \rangle$, we use the following notation:

Definition 1.4 *If $B \in [\omega]^\omega$ and X is a topological group with identity 1, let $\mathcal{C}_B^X$ be the set of all $x \in X$ such that $\langle x^n : n \in B\rangle$ converges to 1. If $\mathcal{F} \subset [\omega]^\omega$ is a filter, let $\mathcal{D}_\mathcal{F}^X = \bigcup\{\mathcal{C}_B^X : B \in \mathcal{F}\}$.*

By Proposition 1.2, for X compact metric, one can choose $\mathcal{F}$ so that $\mathcal{D}_\mathcal{F}^X = X$. The following, proved in Section 3, answers question (❀) for $\langle E_n : n \in \omega\rangle$:

Proposition 1.5 *Let X be any non-trivial compact group. If X is totally disconnected, then there is a $B \in [\omega]^\omega$ such that $\langle E_n : n \in B\rangle \to E_0$ (equivalently, $\mathcal{C}_B^X = X$). If X is not totally disconnected, then $\mathcal{C}_B^X \neq X$ for all $B \in [\omega]^\omega$.*

Note that if X is abelian, then $\mathcal{C}_B^X$ and $\mathcal{D}_\mathcal{F}^X$ are subgroups of X, but for non-abelian X they need not be. The notion of $\mathcal{C}_B^X$ was considered in [2, 3], where they point out that $\mathcal{C}_B^\mathbb{T}$ is a Haar nullset in the circle group $\mathbb{T}$. Paper [2] shows that under Martin's Axiom, there is a Haar null subgroup of $\mathbb{T}$ which is not contained in any $\mathcal{C}_B^\mathbb{T}$, and also asks whether Martin's Axiom is needed to guarantee existence of such a subgroup. Our next result answers that question and describes a concrete $\mathcal{F}$ for which $\mathcal{D}_\mathcal{F}^\mathbb{T}$ does the job. More generally:

Theorem 1.6 *Let $\mathcal{F} \subset [\omega]^\omega$ be the filter generated by all sets of the form $\{k! + 1 : k \in D\}$, where $D \subseteq \omega$ has asymptotic density 1. Then:*

1. *$\mathcal{F}$ is a Borel subset of $\mathcal{P}(\omega) \cong 2^\omega$.*
2. *Whenever X is an infinite compact group:*
    - a. *$\mathcal{D}_\mathcal{F}^X$ is a Haar nullset.*
    - b. *If X is not totally disconnected, then $\mathcal{D}_\mathcal{F}^X$ is not a subset of $\mathcal{C}_B^X$ for any infinite B.*

In particular, when X is abelian and not totally disconnected, $\mathcal{D}_\mathcal{F}^X$ is a Haar null subgroup not contained in any $\mathcal{C}_B^X$. Part (1) of this theorem follows easily from the fact that membership in $\mathcal{F}$ is defined by quantifying over ω. Part (2) is proved in Section 3. By Proposition 1.5, we cannot delete the "not totally disconnected" from (2b).

Further information about properties of the $\mathcal{C}_B^X$ is contained in Section 3. In many cases, these properties for an arbitrary X can be inferred directly from the special cases $X = \mathbb{T}$ or $X = \mathbb{T}^\omega$.

We conclude this Introduction with some basic facts and definitions which will simplify the notation in the next two sections.

Definition 1.7 *If $\mathcal{F} \subset [\omega]^\omega$ is a filter, then $\widetilde{\mathcal{F}}$ is the filter generated by $\mathcal{F}$ and all cofinite sets. If $B \in [\omega]^\omega$ then $B\!\uparrow$ is the filter $\{W \subseteq \omega : B \subseteq W\}$.*

So, $\widetilde{B\!\uparrow} = \{W \subseteq \omega : B \subseteq^* W\}$. Since the convergence of a sequence $\langle f_n(x) : n \in B\rangle$ does not change if one modifies B on a finite set, we can always deal with $\widetilde{\mathcal{F}}$ rather than $\mathcal{F}$. In particular,

Lemma 1.8 $\mathcal{D}^X_{\mathcal{F}} = \mathcal{D}^X_{\widetilde{\mathcal{F}}}$.

Any general lemma about all $\mathcal{D}^X_{\mathcal{F}}$ will also apply to the $\mathcal{C}^X_B$ by:

Lemma 1.9 $\mathcal{C}^X_B = \mathcal{D}^X_{B\uparrow} = \bigcup\{\mathcal{C}^X_W : W \supseteq B\}$.

2 Filters

Proof of Proposition 1.2. Let d be a metric on Y. Choose (inductively) disjoint finite $S_k \subset \omega$ such that $S_0 = \emptyset$, and for $k \geq 1$,

$$\forall x_0, \ldots x_{k-1} \in X \; \exists n \in S_k \; \forall i < k \; [d(f_n(x_i), g(x_i)) < 1/k] \quad . \qquad (\star)$$

To see that such a finite S_k exists, let $A_k = \omega \setminus \bigcup_{j<k} S_j$, and let

$$U_{n,k} = \{(x_0, \ldots x_{k-1}) \in X^k : \forall i < k \; [d(f_n(x_i), g(x_i)) < 1/k]\} \quad .$$

Then $\bigcup\{U_{n,k} : n \in A_k\} = X^k$ (since g is a limit point of $\langle f_n : n \in A_k\rangle$), so by compactness of X^k, there is a finite $S_k \subset A_k$ such that $\bigcup\{U_{n,k} : n \in S_k\} = X^k$.

For $x \in X$, let $B_x = \bigcup_{k\in\omega}\{n \in S_k : d(f_n(x), g(x)) < 1/k\}$. Applying ($\star$), each $B_{x_0} \cap \cdots \cap B_{x_{\ell-1}}$ meets S_k for all $k \geq \ell$, and is hence an infinite set. Let $\mathcal{F}$ be the filter generated by $\{B_x : x \in X\}$. This $\mathcal{F}$ satisfies the theorem because $\langle f_n(x) : n \in B_x\rangle \to g(x)$.

To see that $\mathcal{F}$ is an F_σ in $\mathcal{P}(\omega) \cong 2^\omega$, let

$$\mathcal{G}_\ell = \{D \in \mathcal{P}(\omega) : \exists x_0, \ldots x_{\ell-1} \in X \; [B_{x_0} \cap \cdots \cap B_{x_{\ell-1}} \subseteq D]\} \quad .$$

Then $\mathcal{F} = \bigcup_\ell \mathcal{G}_\ell$. To prove that $\mathcal{G}_\ell$ is closed, let

$$\mathcal{H}_\ell = \big\{(x_0, \ldots, x_{\ell-1}, D) \in X^\ell \times \mathcal{P}(\omega) \;:\; \forall k \in \omega \forall n \in S_k$$
$$[d(f_n(x_0), g(x_0)) < 1/k \;\wedge\; \ldots \;\wedge\; d(f_n(x_{\ell-1}), g(x_{\ell-1})) < 1/k \implies n \in D]\big\} \quad .$$

Then $\mathcal{H}_\ell$ is closed in $X^\ell \times \mathcal{P}(\omega)$. Hence, $\mathcal{G}_\ell \subseteq \mathcal{P}(\omega)$, which is the projection of $\mathcal{H}_\ell$, is closed as well. □

The idea for obtaining the S_k above is taken from the proof that $C_p(X, Y)$ has countable tightness (Kelley and Namioka [6], Lemma 8.19; see also [1], Ch. II§1). We do not have, in general, a simpler description of a filter $\mathcal{F}$ satisfying Proposition 1.2, although $\mathcal{F}$ is related to the neighborhood filter in $C_p(X, Y)$:

Definition 2.1 *If g is a limit point of $\langle f_n : n \in \omega \rangle$ in $C_p(X, Y)$, then the* induced neighborhood filter *is the filter generated by all subsets of ω of the form $\{n \in \omega : f_n \in U\}$, where U is a neighborhood of g in $C_p(X, Y)$.*

Lemma 2.2 *If $\mathcal{F}$ is as in Proposition 1.2 and $\mathcal{N}$ is the induced neighborhood filter, then $\mathcal{N} \subseteq \widetilde{\mathcal{F}}$ (see Definition 1.7).*

Proof. $\mathcal{N}$ is generated by sets of the form $A = \{n \in \omega : f_n(x) \in V\}$, where $x \in X$ and V is a neighborhood of $g(x)$ in Y. Fix $B \in \mathcal{F}$ with $\langle f_n(x) : n \in B \rangle$ converging to $g(x)$. Then $B \subseteq^* A$, so $A \in \widetilde{\mathcal{F}}$. □

In some cases, one can simply take $\mathcal{F} = \mathcal{N}$ in Proposition 1.2; for example, this will always work if Y is finite. However, this does not work in the case $X = Y = \mathbb{T}$, with $f_n = E_n$ and $g = E_0$; see Proposition 4.3. Note that $\mathcal{N}$ is always an F_σ, by a proof similar to that of Proposition 1.2.

3 Groups

We consider now in more detail the $\mathcal{C}_B^X$ and $\mathcal{D}_\mathcal{F}^X$ for compact groups X. Basic facts about such groups can be found in Hofmann and Morris [5].

Lemma 3.1 *If X is a totally disconnected compact group, and $B = \{k! : k \in \omega\}$, then $\mathcal{C}_B^X = X$.*

Proof. Such an X is an inverse limit of finite groups, or, equivalently, a closed subgroup of a product of finite groups (see [5], Theorem 1.34). □

For the other groups, we shall show that $\mathcal{C}_B^X$ is never all of X, while $\mathcal{D}_\mathcal{F}^X$ may or may not be all of X, depending on $\mathcal{F}$ and X.

First, consider the "all of X" case:

Lemma 3.2 *If* $\mathcal{D}_{\mathcal{F}}^{\mathbb{T}^\omega} = \mathbb{T}^\omega$ *then* $\mathcal{D}_{\mathcal{F}}^{X} = X$ *for all compact metric groups* X.

Proof. Since each $x \in X$ generates a compact abelian subgroup, we may assume that X is abelian, in which case X is continuously isomorphic to a subgroup of $\mathbb{T}^\omega$. □

For the "not all of X" case, there is a similar reduction to $\mathbb{T}$. We shall show:

Theorem 3.3 *Let* $\mathcal{F} \subset [\omega]^\omega$ *be a filter which is analytic as a subset of* $\mathcal{P}(\omega) \cong 2^\omega$*, and assume that* $\mathcal{D}_{\mathcal{F}}^{\mathbb{T}} \neq \mathbb{T}$*. Let* X *be any infinite compact group. Then* $\mathcal{D}_{\mathcal{F}}^{X}$ *is a Haar nullset if at least one of the following holds:*

1. X *is abelian and not totally disconnected.*
2. X *is connected.*
3. *For all* $B \in \mathcal{F}$ *and all* $m \geq 2$*:* $\{n \in B : m \nmid n\}$ *is infinite.*

Furthermore, $\mathcal{D}_{\mathcal{F}}^{X} \neq X$ *whenever* X *is not totally disconnected.*

This in particular applies when $\mathcal{F} = B\uparrow$ (see Definition 1.7 and Lemmas 1.8 and 1.9). Observe that

Lemma 3.4 $\mathcal{C}_B^{\mathbb{T}}$ *is a Haar nullset for all infinite* $B \subseteq \omega$.

See [3] (Lemma 3.10) and [2] §4 for proofs; yet another proof is given below; see Remark 3.14. It follows that $\mathcal{C}_B^{X}$ is a Haar nullset if one of (1,2,3) from Theorem 3.3 hold. Note that these conditions cannot be dropped. For example, let $X = O(2)$. Then the component of 1 in X is $SO(2) \cong \mathbb{T}$, which has index 2 in X. The two cosets of $SO(2)$ are $SO(2)$ and R (the reflections), and the elements of R all have order 2. Now, let $B = \{2k : k \in \omega\}$. Then $\mathcal{C}_B^{X}$ contains all of R plus two elements of $SO(2)$, so $\mathcal{C}_B^{X}$ has Haar measure $1/2$.

Observe that if $\mathcal{F}$ is analytic, then $\mathcal{D}_{\mathcal{F}}^{\mathbb{T}}$ is also analytic, and hence Haar measurable. Since a measurable subgroup is null iff it has infinite index, the following lemma, called the Steinhaus-Weil Theorem in [3], is relevant:

Lemma 3.5 *If* X *is a compact group and* H *is a measurable subgroup of finite index, then* H *is clopen in* X.

Proof. H has positive measure, so $H^{-1}H = H$ has non-empty interior (see [4], Cor. 20.17), so H is open (since it is a group), and hence clopen (since it has finite index). □

In particular, if X is connected, then either H is null or $H = X$. Thus,

Corollary 3.6 *If $\mathcal{F}$ is an analytic filter and $\mathcal{D}_{\mathcal{F}}^{\mathbb{T}} \neq \mathbb{T}$, then $\mathcal{D}_{\mathcal{F}}^{\mathbb{T}}$ is a Haar nullset.*

Now, to prove Theorem 3.3, we prove a few lemmas which reduce the situation for a general X to the case $X = \mathbb{T}$.

Lemma 3.7 *If X, Y are compact groups and φ is a continuous homomorphism from X onto Y, then $\mathcal{D}_{\mathcal{F}}^{X} \subseteq \varphi^{-1}(\mathcal{D}_{\mathcal{F}}^{Y})$.*

Since φ^{-1} also preserves Haar measure, one may prove that $\mathcal{D}_{\mathcal{F}}^{X}$ is null by proving that $\mathcal{D}_{\mathcal{F}}^{Y}$ is null. Using this remark, the reduction for abelian groups is easy:

Lemma 3.8 *If $\mathcal{D}_{\mathcal{F}}^{\mathbb{T}}$ is null, then $\mathcal{D}_{\mathcal{F}}^{X}$ is null for all infinite compact abelian groups X which are not totally disconnected.*

Proof. X is continuously isomorphic to some compact subgroup of $\mathbb{T}^{\theta}$ for some cardinal θ, so we may assume that $X \subseteq \mathbb{T}^{\theta}$. If π_α is the projection onto the α^{th} coordinate, then $\pi_\alpha(X)$ is a compact subgroup of $\mathbb{T}$. Some of the $\pi_\alpha(X)$ may be finite, but they cannot all be finite unless X is totally disconnected. However, if $\pi_\alpha(X)$ is infinite, then it must be all of $\mathbb{T}$, so the lemma follows by using Lemma 3.7 with $\varphi = \pi_\alpha$. □

To handle non-abelian X, we shall replace $\mathbb{T}$ in the above proof by a compact Lie group. For this paper, we can take as a definition that X is a *compact Lie group* iff X is continuously isomorphic to a compact subgroup of the unitary group $U(n)$ for some finite n. Many other equivalents are known; see [5, 7]. Observe that all finite groups are compact Lie groups by this definition.

Lemma 3.9 *If X is a compact group and is not totally disconnected, then there is a continuous homomorphism π from X onto some infinite compact Lie group Y.*

Proof. By standard representation theory (see [4, 5]), we may assume that $X \subseteq \prod_{\alpha<\theta} U(n_\alpha)$. Then each $\pi_\alpha(X)$ is a Lie group, and at least one of the $\pi_\alpha(X)$ is infinite. □

Then, as in Lemma 3.8, if $\mathcal{D}_{\mathcal{F}}^{Y}$ is null then $\mathcal{D}_{\mathcal{F}}^{X}$ is null. However, in proving that $\mathcal{D}_{\mathcal{F}}^{Y}$ is null from the assumption that $\mathcal{D}_{\mathcal{F}}^{\mathbb{T}}$ is null, we cannot use a similar argument, since Y need not have a homomorphism onto $\mathbb{T}$. Rather, we use the fact that $\mathbb{T}$ is contained in Y. We shall apply Lemma 3.8 to the maximal tori in Y (i.e., maximal connected abelian subgroups; see [5, 7]) to get:

Lemma 3.10 *If $\mathcal{D}_{\mathcal{F}}^{\mathbb{T}}$ is null, then $\mathcal{D}_{\mathcal{F}}^{Y}$ is null for all non-trivial connected compact Lie groups Y.*

Proof. Let f be the characteristic function of $\mathcal{D}_{\mathcal{F}}^{Y}$; so we wish to show that $\int_Y f\, d\lambda_Y = 0$, where λ_Y is normalized Haar measure on Y. By Lemma 3.8, we know that $\int_H f\, d\lambda_H = 0$ whenever H is a maximal torus in Y. Now, observe that f is a class function; that is $f(x^{-1}yx) = f(y)$. It follows that we may integrate f by the Weyl Integration Formula (see [7], eqn. (8.62)):

$$\int_Y f(x)\, d\lambda_Y(x) = \frac{1}{|W|}\int_H f(t)|D(t)|^2\, d\lambda_H(t) \quad ,$$

where W is a finite group and $D(t)$ is a finite function of t. Since $f(t) = 0$ for λ_H – almost every t, we get $\int_Y f\, d\lambda_Y = 0$. □

Note that Y must be connected for this Weyl Integration Formula to be true, since all the maximal tori are contained in the identity component of Y; furthermore, as pointed out above, the lemma fails for $Y = O(2)$.

Proof of Theorem 3.3. $\mathcal{D}_{\mathcal{F}}^{\mathbb{T}}$ is null by Corollary 3.6. Now, assume that X is not totally disconnected. Then $\mathcal{D}_{\mathcal{F}}^{X}$ is null if X is abelian by Lemma 3.8, so we must handle the non-abelian case. If X is connected, then we can map X onto a non-trivial connected compact Lie group Y by Lemma 3.9, so that $\mathcal{D}_{\mathcal{F}}^{X}$ is null by Lemmas 3.10 and 3.7. If X is not connected, then Y may fail to be connected. If Y_0 is the identity component of Y, then $\mathcal{D}_{\mathcal{F}}^{Y} \cap Y_0 = \mathcal{D}_{\mathcal{F}}^{Y_0}$ will still be null by Lemma 3.10, which proves that $\mathcal{D}_{\mathcal{F}}^{X} \neq X$.

Now, we must prove that $\mathcal{D}_{\mathcal{F}}^{X}$ is null in Case (3) of Theorem 3.3. First, assume that X is not totally disconnected. Using the same Y, it is again sufficient to show that $\mathcal{D}_{\mathcal{F}}^{Y}$ is null. But we already know that $\mathcal{D}_{\mathcal{F}}^{Y} \cap Y_0$ is null, and in Case (3), $\mathcal{D}_{\mathcal{F}}^{Y} \subseteq Y_0$. To see this, fix $y \in Y \backslash Y_0$, and let m be the order of $[y]$ in the quotient Y/Y_0. If y were in $\mathcal{D}_{\mathcal{F}}^{Y}$, we could fix $B \in \mathcal{F}$ such that $\langle y^n : n \in B \rangle \to 1$. But $C = \{n \in B : m \nmid n\}$ is infinite, Y_0 is a neighborhood of 1, and $y^n \notin Y_0$ for all $n \in C$, a contradiction.

Finally if X is totally disconnected in Case (3), then $\mathcal{D}_{\mathcal{F}}^{X} = \{1\}$, as in the proof of Lemma 3.1. □

Proof of Proposition 1.5. By Lemmas 3.1 and 3.4 and Theorem 3.3. □

We now give some examples of $\mathcal{F}$ for which $\mathcal{D}_{\mathcal{F}}^{\mathbb{T}}$ is null, using some well-known facts about Hadamard sets and the Bohr topology on $\mathbb{Z}$; see [8] for definitions and references to the earlier literature.

Definition 3.11 *$\mathcal{N}_{\mathsf{b}} \subseteq \mathcal{P}(\omega)$ denotes the neighborhood filter at 0 in the topology ω inherits as a subset of $\mathbb{Z}^{\#}$ (that is, the group $\mathbb{Z}$ with its Bohr topology).*

Equivalently, $\mathcal{N}_{\mathsf{b}}$ is the induced neighborhood filter in the sense of Definition 2.1, taking $f_n = E_n$, $g = E_0$, and $X = Y = \mathbb{T}$. Applying Lemma 2.2,

Lemma 3.12 *If $\mathcal{D}^{\mathbb{T}}_{\mathcal{F}} = \mathbb{T}$ then $\mathcal{N}_{\mathsf{b}} \subseteq \widetilde{\mathcal{F}}$.*

Note that $\mathcal{D}^{\mathbb{T}}_{\mathcal{N}_{\mathsf{b}}}$ is countable (see Proposition 4.3), so to get $\mathcal{D}^{\mathbb{T}}_{\mathcal{F}} = \mathbb{T}$ (as in Proposition 1.2), $\mathcal{F}$ must properly extend $\mathcal{N}_{\mathsf{b}}$.

Applying Lemma 3.12 and Corollary 3.6,

Lemma 3.13 *Suppose that $\mathcal{N}_{\mathsf{b}} \not\subseteq \widetilde{\mathcal{F}}$, and $\mathcal{F}$ is analytic as a subset of $\mathcal{P}(\omega) \cong 2^{\omega}$. Then $\mathcal{D}^{\mathbb{T}}_{\mathcal{F}}$ is a Haar nullset.*

Remark 3.14 This yields another proof of Lemma 3.4: apply Lemma 3.13 to $\mathcal{F} = B\!\uparrow$, and note that $\mathcal{N}_{\mathsf{b}} \not\subseteq \widetilde{\mathcal{F}}$ because $\mathbb{Z}^{\#}$ has no convergent ω-sequences.

Proof of Theorem 1.6(2a). $H = \{k! + 1 : k \in \omega\}$ is a Hadamard set, and hence closed and discrete in $\mathbb{Z}^{\#}$. Thus, $\omega \backslash H \in \mathcal{N}_{\mathsf{b}}$, so that $\mathcal{N}_{\mathsf{b}} \not\subseteq \mathcal{F}$. Then, (2a) follows by Lemma 3.13 and Theorem 3.3 (Case (3)). □

Now, let us turn to a proof of Theorem 1.6(2b). Here, most of the work will be done on the *solenoid* $\widehat{\mathbb{Q}}$ (see [4]):

Definition 3.15 *$\widehat{\mathbb{Q}}$ denotes the dual of the discrete group of rationals. We shall realize $\widehat{\mathbb{Q}}$ concretely as $\{\vec{z} \in \mathbb{T}^{\omega} : \forall \alpha < \omega[(z_{\alpha+1})^{\alpha+1} = z_{\alpha}]\}$.*

This differs slightly from the notation in [4]. We can identify $\vec{z}$ with the character of $\mathbb{Q}$ which takes $1/\alpha!$ to z_{α}.

Lemma 3.16 *If X is an infinite compact group and is not totally disconnected, then X has a non-trivial closed subgroup which is a continuous homomorphic image of $\widehat{\mathbb{Q}}$.*

Proof. First, WLOG, X is abelian. To see this, let $\pi : X \to Y$ be a continuous homomorphism onto an infinite compact Lie group Y (see Lemma 3.9). Choose $x \in X$ such that $y := \pi(x)$ has infinite order. Then π maps $\overline{\langle x \rangle}$ onto $\overline{\langle y \rangle}$, which is an infinite compact Lie group, and hence not totally disconnected. It follows that $\overline{\langle x\rangle}$ is not totally disconnected (see [5], Exercise E1.13), so we may replace X by $\overline{\langle x\rangle}$.

Second, WLOG, X is connected, since we may replace X by the component of 1.

Now, let $G = \widehat{X}$, which is a discrete torsion-free abelian group. Let $V \supseteq G$ be the divisible hull of G; then V is torsion-free and divisible, so we may regard V as a vector space over $\mathbb{Q}$. Choose a basis $\{v_\alpha : \alpha < \kappa\}$ for V with $v_0 \in G$, let W be the vector subspace generated by v_0, and let ψ be the canonical homomorphism of V onto W. Let $H = \psi(G) \subseteq W$. Then H is torsion-free and non-trivial (since $v_0 \in G$), and $\psi \restriction G$ maps G onto H, so $Y := \widehat{H}$ is a non-trivial closed subgroup of $X = \widehat{G}$. But also, H is isomorphic to a subgroup of $\mathbb{Q} \cong W$, so Y is a quotient of $\widehat{\mathbb{Q}}$. □

Definition 3.17 *$A \subseteq \omega$ is* thin *iff A is of the form $\{a_k : k \in \omega\}$, where $0 < a_0 < a_1 < \cdots$ and $\lim_k a_k/a_{k+1} = 0$.*

Lemma 3.18 *If $A \subseteq \omega$ is thin, A is partitioned into disjoint infinite subsets B, C, and $\vec{v}, \vec{w} \in \widehat{\mathbb{Q}}$, then there is a $\vec{z} \in \widehat{\mathbb{Q}}$ such that $\langle (\vec{z})^n : n \in B\rangle$ converges to $\vec{v}$ and $\langle (\vec{z})^n : n \in C\rangle$ converges to $\vec{w}$.*

Proof. List A in increasing order as $\{a_j : j \in \omega\}$. Let $\varepsilon_j = \sup_{\ell \geq j}(a_\ell/a_{\ell+1})$. Then $\varepsilon_j \searrow 0$, each $a_j/a_{j+1} \leq \varepsilon_j$, and $\varepsilon_j \leq 1$.

Let $\gamma_0 = 0$, and let γ_{j+1} be the largest integer γ such that $\gamma! \leq 1/\sqrt{\varepsilon_j}$. Note that $\gamma_j \nearrow \infty$ and

$$(a_j/a_{j+1})(\gamma_{j+1}!) \leq \varepsilon_j(\gamma_{j+1}!) \leq \sqrt{\varepsilon_j} \quad .$$

Hence, $(a_j/a_{j+2})(\gamma_{j+2}!) \leq (a_j/a_{j+1})(\gamma_{j+1}!)\cdot(a_{j+1}/a_{j+2})(\gamma_{j+2}!) \leq \sqrt{\varepsilon_j\varepsilon_{j+1}}$. Continuing in this way,

$$j < k \implies (a_j/a_k)(\gamma_k!) \leq \sqrt{\varepsilon_j\varepsilon_{j+1}\cdots\varepsilon_{k-1}} \leq (\sqrt{\varepsilon_j})^{k-j} \quad . \qquad (\circledast)$$

Our $\vec{z}$ will be $\lim_j \vec{z}_j$, where each $\vec{z}_j \in \widehat{\mathbb{Q}}$; the $\vec{z}_j$ will be chosen by induction on j. Use $z_{j,\alpha}$ for the components of $\vec{z}_j \in \widehat{\mathbb{Q}} \subset \mathbb{T}^\omega$. Let d be the metric on $\mathbb{T}$ obtained by identifying $\mathbb{T}$ with $\mathbb{R}/\mathbb{Z}$, so that the circumference of $\mathbb{T}$ is 1, and $d(e^{i\theta}, 1) = |\theta|/2\pi$ when $|\theta| \leq \pi$. We shall get, for each $k \in \omega$:

1. For $\alpha \leq \gamma_k$: $(z_{k,\alpha})^{a_k}$ equals v_α when $a_k \in B$ and w_α when $a_k \in C$.
2. For $\alpha \leq \gamma_k$, when $k > 0$: $d(z_{k,\alpha}, z_{k-1,\alpha}) \leq (\gamma_k!)/a_k$.

To ensure (1) for a given k, it is sufficient to choose z_{k,γ_k} so that (1) holds for $\alpha = \gamma_k$; then (1) will hold for $\alpha < \gamma_k$ by our definition (3.15) of $\widehat{\mathbb{Q}}$. For $\alpha > \gamma_k$, we just choose the $z_{k,\alpha}$ so that the point $\vec{z}_k$ lies in $\widehat{\mathbb{Q}}$. To get (2) along with (1): We are given $\vec{z}_{k-1}$ and we must define $\vec{z}_k$ by choosing z_{k,γ_k} so that $(z_{k,\gamma_k})^{a_k}$ is v_{γ_k} when $a_k \in B$ and w_{γ_k} when $a_k \in C$. There are a_k possible choices for z_{k,γ_k}, spaced evenly around the circle, at distance $1/a_k$ apart, so we can make the choice so that $d(z_{k,\gamma_k}, z_{k-1,\gamma_k}) \le 1/a_k$. Then (2) follows by our definition of $\widehat{\mathbb{Q}}$.

Now, if we fix j and $\alpha \le \gamma_j$, then whenever $j < k$, we may apply (2) and (✾) to get $a_j d(z_{k,\alpha}, z_{k-1,\alpha}) \le (a_j/a_k)(\gamma_k!) \le (\sqrt{\varepsilon_j})^{k-j}$. Thus, whenever $\ell > j$:

$$a_j d(z_{\ell,\alpha}, z_{j,\alpha}) \le \sum_{k=j+1}^{\ell} (\sqrt{\varepsilon_j})^{k-j} \le \sqrt{\varepsilon_j}/(1-\sqrt{\varepsilon_j}) \quad . \qquad (†)$$

In particular, the sequence $\langle z_{j,\alpha} : \alpha \in \omega\rangle$ is Cauchy, so we can define $\vec{z} = \lim_j \vec{z}_j$. Then, (†) yields $d(z_\alpha^{a_j}, z_{j,\alpha}^{a_j}) \le \sqrt{\varepsilon_j}/(1-\sqrt{\varepsilon_j})$ whenever $\alpha \le \gamma_j$. Applying this for the $a_j \in B$, when $(z_{j,\alpha})^{a_j} = v_\alpha$, we get $\langle (\vec{z})^n : n \in B\rangle \to \vec{v}$, and applying it for the $a_j \in C$ yields $\langle (\vec{z})^n : n \in C\rangle \to \vec{w}$. □

Lemma 3.19 *Let X be any infinite compact group which is not totally disconnected. Let $A \subseteq \omega$ be thin, with A partitioned into disjoint infinite subsets B, C. Then there are $x, y \in X$ with $y \ne 1$ such that $\langle x^n : n \in B\rangle$ converges to y and $\langle x^n : n \in C\rangle$ converges to 1.*

Proof. By Lemma 3.16, we may assume that there is a continuous homomorphism φ from $\widehat{\mathbb{Q}}$ onto X. Then y can be any element of $X\backslash\{1\}$. Now, in $\widehat{\mathbb{Q}}$, let $\vec{w} = 1$, and choose $\vec{v}$ with $\varphi(\vec{v}) = y$, and apply Lemma 3.18. □

Consider this in particular with $X = \mathbb{T}$, where the argument of Lemma 3.18 can be done (considerably simplified) in $\mathbb{T}$ directly, and resembles the proof that Hadamard sets are I_0 sets. However, Lemma 3.19 fails if we only assume that A is a Hadamard set. For example, let $A = \{2^j : j \in \omega\}$, $B = \{2^{2k+1} : k \in \omega\}$, and $C = \{2^{2k} : k \in \omega\}$, and consider $x \in \mathbb{T}$. If $\langle x^n : n \in C\rangle \to 1$, then $x^{2^j} = 1$ for some j, but then also $\langle x^n : n \in B\rangle \to 1$.

Proof of Theorem 1.6(2b). Fix an infinite B; we need to show that $\mathcal{D}_{\mathcal{F}}^X \not\subseteq \mathcal{C}_B^X$. Since $\mathcal{C}_B^X$ gets bigger as B gets smaller, we may assume that B is small enough so that for some $C \in \mathcal{F}$: $C \cap B = \emptyset$ and $A := C \cup B$ is thin. Then, applying Lemma 3.19, we may fix $x \in X$ so that $\langle x^n : n \in C\rangle$ converges to 1 and $\langle x^n : n \in B\rangle$ converges to $y \ne 1$. Then $x \in \mathcal{D}_{\mathcal{F}}^X$ and $x \notin \mathcal{C}_B^X$. □

4 Examples

First, we point out that Proposition 1.2 can fail if X is not assumed to be compact.

Example 4.1 *Let X be the cardinal $\mathfrak{c} = 2^{\aleph_0}$ with the discrete topology, and let $Y = \omega + 1$ with the order topology. Then in $C_p(X, Y)$, there is a sequence $\langle f_n : n \in \omega \rangle$ with limit point g such that there is no filter $\mathcal{F} \subset [\omega]^\omega$ satisfying:*

$$\forall x \in X \exists B \in \mathcal{F}[\langle f_n(x) : n \in B \rangle \to g(x)] \qquad (✈)$$

Proof. Let $\mathcal{U}$ be a non-principal ultrafilter on ω which is not a P-point. List $\mathcal{U}$ as $\{B_\alpha : 0 < \alpha < \mathfrak{c}\}$. Also, partition ω into infinite sets A_k (for $k \in \omega$) so that each $A_k \notin \mathcal{U}$ and each $B \in \mathcal{U}$ meets some A_k in an infinite set.

Let $g(\alpha) = \omega$ for all α. For $\alpha > 0$, let $f_n(\alpha)$ be n for $n \in B_\alpha$ and 0 for $n \notin B_\alpha$. Let $f_n(0)$ be the k such that $n \in A_k$.

Now, suppose that $\mathcal{F}$ satisfied (✈). Fix $C \in \mathcal{F}$ with $\langle f_n(0) : n \in C \rangle \to \omega$. Then each $C \cap A_k$ is finite, so $C \notin \mathcal{U}$, so fix $\alpha > 0$ with $B_\alpha \cap C = \emptyset$. Now fix $D \in \mathcal{F}$ with $\langle f_n(\alpha) : n \in D \rangle \to \omega$. Then $D \subseteq^* B_\alpha$, so $D \cap C$ is finite. But $D \cap C \in \mathcal{F}$, contradicting $\mathcal{F} \subset [\omega]^\omega$. □

This is the simplest possible Y for such an example, since Proposition 1.2 does hold for arbitrary X whenever Y is finite, taking $\mathcal{F}$ to be the induced neighborhood filter $\mathcal{N}$ (as in Definition 2.1). Also, under Martin's Axiom, Proposition 1.2 holds for all X of size less than $\mathfrak{c}$ whenever Y is first countable. When $|X| = \mathfrak{c}$, taking Y to be first countable is not enough, even when X compact, since Proposition 1.2 fails when X and Y are the lexicographically ordered square. This is the space $[0, 1] \times [0, 1]$, ordered lexicographically, and given the usual order topology; note that it is compact and first countable.

Example 4.2 *Let X be the lexicographically ordered square. Then there is a sequence $\langle f_n : n \in \omega \rangle$ in $C_p(X, X)$ with limit point g such that there is no filter $\mathcal{F} \subset [\omega]^\omega$ satisfying (✈) above.*

Proof. Let $\langle \widetilde{f}_n : n \in \omega \rangle$ and $\widetilde{g}$ be the functions in $C_p([0, 1]_d, \omega + 1)$ obtained in Example 4.1, identifying the ordinal $\mathfrak{c}$ with the discrete $[0, 1]$. We shall encode this example into $C_p(X, X)$.

To embed $\omega + 1$ into $[0, 1]$, let $u_\omega = 1/2$ and fix u_n with $0 < u_n < 1/2$ and $u_n \nearrow u_\omega$. Next, in X, let $I_r = \{r\} \times [0, 1]$; then I_r is a homeomorphic copy of

$[0,1]$ for each $r \in [0,1]$. We shall have $f_n : X \to X$ map each I_r onto I_r, and encode the $\widetilde{f_n}$ as follows: Let

$$f_n(r, 1/2) = (r, u_{\widetilde{f_n}(r)}) \ , \ f_n(r,0) = (r,0) \ , \ f_n(r,1) = (r,1) \ .$$

Then, fill in the rest of the values $f_n(r,y)$ for $y \in [0,1]$ by linear interpolation, so that the graph of f_n intersected with $I_r \times I_r$ is the union of two line segments.

Likewise, we get g from $\widetilde{g}$, but then g is the identity function. We can now apply the same argument as in Example 4.1. $\square$

We remark on the relevance of P-points here. For arbitrary X and first countable Y, let $\mathcal{N}$ be the induced neighborhood filter (as in Definition 2.1). If there is a P-point $\mathcal{U}$ such that $\mathcal{N}$ is contained in $\mathcal{U}$, then taking $\mathcal{F} = \mathcal{U}$ will work in Proposition 1.2. In Examples 4.1 and 4.2 we have $\mathcal{N} = \mathcal{U}$, but that $\mathcal{U}$ was chosen to be a *non*-P-point. For X compact and Y metric, if we assume Martin's Axiom, then we can produce a P-point ultrafilter $\mathcal{F} \supseteq \mathcal{N}$ which gives the convergence described in Proposition 1.2, but, of course, this $\mathcal{F}$ will not be a Borel set. The proof of Proposition 1.2 amounts to adding to $\mathcal{N}$ some diagonal intersections of sets in $\mathcal{N}$, which is part of the P-point construction, although we do not actually build an ultrafilter. Note that $\mathcal{N}$ itself will not in general satisfy Proposition 1.2; for example, to get a filter $\mathcal{F}$ with $\mathcal{D}^{\mathbb{T}}_{\mathcal{F}} = \mathbb{T}$, we cannot just take $\mathcal{F} = \mathcal{N}_{\mathsf{b}}$ (see Definition 3.11):

Proposition 4.3 *$\mathcal{D}^{\mathbb{T}}_{\mathcal{N}_{\mathsf{b}}}$ is the set of all $w \in \mathbb{T}$ of finite order.*

Proof. If $w^k = 1$, then $w^n = 1$ for all $n \in \{jk : j \in \omega\} \in \mathcal{N}_{\mathsf{b}}$.

Conversely, suppose w has infinite order. Consider $\mathbb{Z} \subseteq \mathsf{b}\mathbb{Z} = \mathrm{Hom}(\mathbb{T}_d, \mathbb{T})$ in the standard way, and let λ be Haar measure on $\mathsf{b}\mathbb{Z}$. Observe that if $B \in \mathcal{N}_{\mathsf{b}}$, then $\lambda(\overline{B}) > 0$, whereas if $\langle w^n : n \in B \rangle \to 1$, then $\lambda(\overline{B}) = 0$. $\square$

The following shows that Lemma 3.2 can fail if X is non-metrizable:

Example 4.4 *If $\mathcal{D}^{\mathbb{T}^\omega}_{\mathcal{F}} = \mathbb{T}^\omega$, then $\mathcal{D}^{\mathbb{T}^{\mathfrak{c}}}_{\mathcal{F}}$ has inner Haar measure 0 and outer Haar measure 1.*

Proof. Let P be the set of all $f \in \mathbb{T}^{\mathfrak{c}}$ such that f maps $\mathfrak{c}$ onto $\mathbb{T}$, and let Q be the set of elements of $f \in \mathbb{T}^{\mathfrak{c}}$ such that $\{\alpha : f(\alpha) \neq 1\}$ is countable. Then Q has outer measure 1 and $Q \subseteq \mathcal{D}^{\mathbb{T}^{\mathfrak{c}}}_{\mathcal{F}}$ (since $\mathcal{D}^{\mathbb{T}^\omega} = \mathbb{T}^\omega$), so $\mathcal{D}^{\mathbb{T}^{\mathfrak{c}}}_{\mathcal{F}}$ has outer measure 1. Also, P has outer measure 1 and $\mathcal{D}^{\mathbb{T}^{\mathfrak{c}}}_{\mathcal{F}} \cap P = \emptyset$ (since $\mathcal{C}^{\mathbb{T}}_B \neq \mathbb{T}$ for all B), so $\mathcal{D}^{\mathbb{T}^{\mathfrak{c}}}_{\mathcal{F}}$ has inner measure 0. $\square$

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5 Appendix

Here we collect a few proofs which don't seem worth putting in the published part of the paper, since they just verify some remarks in Section 4, which is itself essentially an appendix.

First, as remarked in Section 4, one can get a P-point in a modified version of Proposition 1.2.

Proposition 5.1 *Assume Martin's Axiom (or, just $\mathfrak{p} = \mathfrak{c}$). Suppose that X is compact, Y is metric, and g is a limit point of $\langle f_n : n \in \omega \rangle$ in $C_p(X, Y)$. Then there is a P-point ultrafilter $\mathcal{F} \subset [\omega]^\omega$ such that for each $x \in X$ there is $B \in \mathcal{F}$ such that $\langle f_n(x) : n \in B \rangle \to g(x)$.*

Proof. As remarked in Section 4, it is sufficient to construct a P-point extending the induced neighborhood filter, $\mathcal{N}$. As usual, $\mathcal{F}$ will be generated by sets $\{B_\alpha : \alpha < \mathfrak{c}\}$, where $\alpha < \beta \to B_\beta \subseteq^* B_\alpha$. Make sure, inductively, that

$$g \text{ is a limit point of } \langle f_n : n \in B_\alpha \rangle \quad . \qquad (\circledcirc)$$

To make $\mathcal{F}$ an ultrafilter, list $[\omega]^\omega$ as $\{A_\alpha : \alpha < \mathfrak{c}\}$, and let $B_{\alpha+1}$ be either $B_\alpha \cap A_\alpha$ or $B_\alpha \backslash A_\alpha$, preserving $(\circledcirc)$, which ensures in particular that if $A_\alpha \in \mathcal{N}$ then $A_\alpha \in \mathcal{F}$.

Now, say γ is a limit. We need $B_\gamma \subseteq^* B_\alpha$ for all $\alpha < \gamma$, along with $(\circledcirc)$ for B_γ. When $1 \le k < \omega$, let $\mathcal{S}_k$ be the set of all finite $S \subseteq \omega$ such that $\min(S) > k$ and

$$\forall x_0, \ldots x_{k-1} \in X \; \exists n \in S \; \forall i < k \; [d(f_n(x_i), g(x_i)) < 1/k] \quad . \qquad (\star)$$

Observe that $(\circledcirc)$ will hold if for all k, there is an $S \subseteq B_\gamma$ with $S \in \mathcal{S}_k$. As in the proof of Proposition 1.2, there is an $S \subseteq B_{\alpha_1} \cap \cdots \cap B_{\alpha_\ell}$ with $S \in \mathcal{S}_k$ whenever $\alpha_1 < \cdots < \alpha_\ell < \gamma$. Now, we obtain an appropriate B_γ by a standard application of Martin's Axiom. $\square$

Next, the argument in Proposition 4.3 requires:

Lemma 5.2 *Consider $\omega \subset \mathbb{Z} \subseteq \mathsf{b}\mathbb{Z} = \mathrm{Hom}(\mathbb{T}_d, \mathbb{T})$, and let λ be Haar measure on $\mathsf{b}\mathbb{Z}$. Fix $B \subseteq \omega$. Then:*

1. *If $B \in \mathcal{N}_\mathsf{b}$, then $\lambda(\overline{B}) > 0$.*
2. *If $w \in \mathbb{T}$ has infinite order and $\langle w^n : n \in B \rangle \to 1$, then $\lambda(\overline{B}) = 0$.*

Proof. For (1): We can assume that $B = U_1 \cap \cdots \cap U_k \cap \omega$, where $U_j = \{n \in \mathbb{Z} : d(z_j^n, 1) < \varepsilon\}$. Then $B \cup -B = U_1 \cap \cdots \cap U_k$, a basic open set in $\mathbb{Z}^{\#}$. So, $\lambda(\overline{B \cup -B}) > 0$ because $\mathbb{Z}$ is dense in $\mathsf{b}\mathbb{Z}$. But $\overline{B \cup -B} = \overline{B} \cup \overline{-B}$ and $\lambda(\overline{B}) = \lambda(\overline{-B})$, so $\lambda(\overline{B}) > 0$.

For (2): Fix $\varepsilon > 0$; we show that $\lambda(\overline{B}) \leq 2\varepsilon$. Let d be the metric on $\mathbb{T}$ used in the proof of Lemma 3.18. By removing finitely many elements of B, we can assume, WLOG, that $d(w^n, 1) \leq \varepsilon$ for all $n \in B$. Identifying $\mathsf{b}\mathbb{Z}$ with $\mathrm{Hom}(\mathbb{T}_d, \mathbb{T})$, let $F = \{\varphi \in \mathsf{b}\mathbb{Z} : d(\varphi(w), 1) \leq \varepsilon\}$. Applying duality (viewing $\mathbb{T}_d$ as $\widehat{\mathsf{b}\mathbb{Z}}$), let $E_w(\varphi) = \varphi(w)$; then E_w is a continuous homomorphism from $\mathsf{b}\mathbb{Z}$ into $\mathbb{T}$, and $F = \{\varphi \in \mathsf{b}\mathbb{Z} : d(E_w(\varphi), 1) \leq \varepsilon\}$, which is a closed subset of $\mathsf{b}\mathbb{Z}$. Also, identifying $\omega \subset \mathbb{Z} \subseteq \mathsf{b}\mathbb{Z}$, we have $B \subseteq F$, so it is sufficient to show that $\lambda(F) = 2\varepsilon$ if $\varepsilon \leq 1/2$. But this follows from the fact that E_w maps *onto* $\mathbb{T}$ (since w has infinite order), since $\{x \in \mathbb{T} : d(x, 1) \leq \varepsilon\}$ has measure 2ε. $\square$