Chromatic Numbers and Bohr Topologies*

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Abstract

We use chromatic numbers of hypergraphs to study the Bohr topology $G^{\#}$ on discrete abelian groups. In particular, if K is an infinite abelian group of a given prime exponent, we show that $G^{\#}$ and $K^{\#}$ are homeomorphic iff G is the product of K and some finite group. Also, if K is of finite exponent and G is not of finite exponent, then $G^{\#}$ is never topologically embeddable in $K^{\#}$.

1 Introduction

If G is a (discrete) abelian group, then $G^{\#}$ denotes G with its Bohr topology. This is the coarsest topology on G which makes each $\varphi \in \mathrm{Hom}(G, \mathbb{T})$ continuous. Here, $\mathbb{T}$ is the circle group and $\mathrm{Hom}(G, \mathbb{T}) = \widehat{G}$ is the group of characters of G. If H is a subgroup of G, then H is closed in $G^{\#}$, and $H^{\#}$ is the same as the subspace topology which H inherits from $G^{\#}$. $\widehat{G}$ is described in texts on harmonic analysis [6, 9, 13]. For more on $G^{\#}$, see van Douwen [4] and Dikranjan [1, 2].

It is an open question, first raised by van Douwen, whether one can characterize those G, K such that $G^{\#}$ and $K^{\#}$ are homeomorphic (just as topological spaces), by using only basic algebraic properties of G and K. In this paper, we give such a characterization in the case that G is arbitrary and K is an infinite abelian group of a given prime exponent (see Corollary 1.4). See [1]

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for a discussion of what else is known about this question; for example, it is still unknown whether $\mathbb{Z}^\#$ and $\mathbb{Q}^\#$ are homeomorphic.

To describe our results in more detail, we introduce some terminology:

Definition 1.1 *$\bigoplus^\kappa H$ is the direct sum of κ many copies of H. $\mathbb{V}_n^\kappa = \bigoplus^\kappa \mathbb{Z}_n$. $\mathbb{V}_n^0$ is the one-element group.*

Here, $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. If p is prime and κ is infinite, then $\mathbb{V}_p^\kappa$ is the unique abelian group of order κ and exponent p; it may be viewed as the vector space of dimension κ over the field $\mathbb{Z}_p$. It is already known [3, 11] that $(\mathbb{V}_p^\kappa)^\#$ and $(\mathbb{V}_q^\kappa)^\#$ are not homeomorphic whenever p, q are distinct primes. In this paper, we characterize those G such that $G^\#$ is embeddable (topologically) into $(\mathbb{V}_p^\kappa)^\#$. In Section 5, we prove:

Theorem 1.2 *Let G be any abelian group, κ any infinite cardinal, and p any prime. Then $G^\#$ is homeomorphic to a subset of $(\mathbb{V}_p^\kappa)^\#$ iff G is isomorphic to $\mathbb{V}_p^\lambda \times F$ for some finite F and some $\lambda \le \kappa$.*

Actually, only the $\rightarrow$ direction of this theorem requires proof, since the $\leftarrow$ direction is immediate from the fact that $\mathbb{V}_p^\lambda$ is a subgroup of $\mathbb{V}_p^\kappa$ when $\lambda \le \kappa$, along with:

Lemma 1.3 *For abelian group K, F, with K infinite and F finite: $K^\#$ is homeomorphic to $(K \times F)^\#$.*

For K countable, this lemma is an easy consequence of homogeneity, but in fact it is true for arbitrary infinite K by Trigos-Arrieta [14], Theorem 6.33 (see also [7], Lemma 3.3.3).

Corollary 1.4 *If κ is any infinite cardinal, G any abelian group of order κ, and p any prime, then the following are equivalent:*

1. *$G^\#$ is homeomorphic to a subset of $(\mathbb{V}_p^\kappa)^\#$.*
2. *$G^\#$ is homeomorphic to $(\mathbb{V}_p^\kappa)^\#$.*
3. *G is isomorphic to $\mathbb{V}_p^\kappa \times F$ for some finite F.*

Proof. (2) $\rightarrow$ (1) is obvious, (3) $\rightarrow$ (2) is immediate from Lemma 1.3, and (1) $\rightarrow$ (3) follows from Theorem 1.2. $\square$

In Section 2, we describe a class of abstract topological spaces constructed using chromatic numbers of hypergraphs. Some of these hypergraph spaces are embeddable into many $G^\#$ (see Section 3), but not into $(\mathbb{V}_p^\kappa)^\#$ (see Section 4).

This establishes that many $G^\#$ are not embeddable into $(\mathbb{V}_p^\kappa)^\#$, which will prove the $\to$ direction of Theorem 1.2 (see Section 5).

A special case of a hypergraph is an ordinary undirected graph, and in this case, the graph topologies and their embeddings into $G^\#$ are described in [12].

In Section 5, we also prove some extensions of Theorem 1.2 involving embedding $G^\#$ into $K^\#$ in the case that K is an arbitrary abelian group of finite exponent (see Theorems 5.1 and 5.3). Groups of the form $\mathbb{V}_n^\kappa$ are key to understanding general groups of finite exponent by the following well-known structure theorem:

Theorem 1.5 *If K is an abelian group of finite exponent, then K is isomorphic to a product of the form:*

$$\prod_{p\in P\,;\,0<i<\omega} \mathbb{V}_{p^i}^{\kappa_{p,i}} \quad ,$$

where P is the set of primes, and all but finitely many of the $\kappa_{p,i}$ are zero. Furthermore, the sequence of cardinals, $\langle \kappa_{p,i} : p \in P\,;\, 0 < i < \omega\rangle$, is uniquely determined by K.

Note that this product is essentially a finite product (= direct sum). The fact that K can be written as such a direct sum is Theorem 6, p. 17, of Kaplansky [10]; then, the $\kappa_{p,i}$ are unique because they can be determined from the Ulm invariants of the primary components of K (see [10], p. 27).

2 Hypergraph Topologies

We use the standard Erdös notation, $[A]^n$, for the collections of all n-element subsets of the set A. If $\Gamma \subseteq [A]^n$, we may view Γ as a hypergraph, with node set A and edges $\mathfrak{e} \in \Gamma$. We may assign Γ a chromatic number, but for $n > 2$, there are various possible definitions of "chromatic number" other than the "standard" one, from Erdös and Hajnal [5], which required only that each edge get at least two colors. Our definition depends on an ordering of A and a sequence of coefficients.

Notation 2.1 *If A is totally ordered by $<$ and $n \ge 2$ then we use* $\mathfrak{e} = \{e_0, \ldots, e_{n-1}\}$ (always written in increasing order) *for elements of* $[A]^n$.

Definition 2.2 *A* sequence of coefficients *is a sequence* $\vec{b} = (b_0, \ldots, b_{n-1})$ *of elements from some fixed abelian group, where* $2 \le n < \omega$, $\sum_{j<n} b_j = 0$, *and all* $b_j \ne 0$.

Definition 2.3 *Given a set A totally ordered by $<$, a sequence of coefficients $\vec{b} = (b_0, \ldots, b_{n-1})$, and $\Gamma \subseteq [A]^n$:*

- ☞ *A* successful coloring *for Γ in θ colors is a partition of A into θ disjoint sets, $A = \bigcup_{\alpha<\theta} C_\alpha$, such that for all $\mathfrak{e} = \{e_0, \ldots, e_{n-1}\} \in \Gamma$, there is an $\alpha < \theta$ such that:*
$$\sum \{b_j : e_j \in C_\alpha\} \neq 0 \quad .$$
- ☞ *$\chi_{\vec{b}}(\Gamma)$, the $\vec{b}$–*chromatic number *of Γ, is the least θ such that there is successful coloring for Γ in θ colors.*
- ☞ *If the coefficients are in $\mathbb{Z}$, and all $b_j \neq 0 \bmod m$, then $\chi^m_{\vec{b}}(\Gamma)$ is the chromatic number computed by regarding each b_j as representing an element of $\mathbb{Z}_m$.*

Definition 2.4 *Given $(A, <)$ and coefficients $\vec{b} = (b_0, \ldots, b_{n-1})$:*

- ➡ *$\mathcal{T}_{\vec{b}}$ is the topology on the set $[A]^n \cup \{\infty\}$ defined so that:*
  - *the subset $[A]^n$ is open and discrete, and*
  - *for all $\Gamma \subseteq [A]^n$: $\infty \in \mathrm{cl}(\Gamma)$ iff $\chi_{\vec{b}}(\Gamma)$ is infinite.*
- ➡ *If the coefficients are in $\mathbb{Z}$, then $\mathcal{T}^m_{\vec{b}}$ is the topology obtained by regarding each b_j as representing an element of $\mathbb{Z}_m$ (using $\chi^m_{\vec{b}}(\Gamma)$ instead of $\chi_{\vec{b}}(\Gamma)$).*

Lemma 2.5 *If $\Gamma \neq \emptyset$, then $2 \leq \chi_{\vec{b}}(\Gamma) \leq |A|$. If A is infinite, then $\chi_{\vec{b}}([A]^n) = |A|$. If the coefficients are in $\mathbb{Z}$, then $\chi^m_{\vec{b}}(\Gamma) \geq \chi_{\vec{b}}(\Gamma)$ and $\mathcal{T}^m_{\vec{b}}$ is coarser than $\mathcal{T}_{\vec{b}}$.*

$\mathbb{Z}_m$ is often represented concretely as $\{0, 1, \ldots, m-1\}$, so to avoid confusion when the coefficients are in $\mathbb{Z}_m$, we sometimes write $\chi^m_{\vec{b}}(\Gamma)$ instead of $\chi_{\vec{b}}(\Gamma)$.

We conclude this section with some examples illustrating the specific cases of these notions most relevant for the rest of this paper.

Definition 2.6 *For a natural number σ, the* $\mp$-coefficients *of length 2σ is the sequence $\vec{b} = (-1, \ldots, -1, 1, \ldots, 1)$ from $\mathbb{Z}$, where $b_j = -1$ for $0 \leq j < \sigma$ and $b_j = 1$ for $\sigma \leq j < 2\sigma$.*

Now, with this $\vec{b}$, we can compute $\chi_{\vec{b}}(\Gamma)$, and also $\chi^m_{\vec{b}}(\Gamma) \geq \chi_{\vec{b}}(\Gamma)$, obtained by viewing the ± 1 as elements of $\mathbb{Z}_m$

Example 2.7 *Assume that A is ordered by $<$ in type ω and that $D \subset A$ with D and $A \backslash D$ both infinite. Fix μ, ν with $0 \le \mu < \sigma$ and $\sigma \le \nu < 2\sigma$. Let:*

$$\begin{aligned} \Gamma_1 &= \{\mathfrak{e} \in [A]^{2\sigma} : e_\mu \in D \ \& \ e_\nu \in D \ \& \ \forall j \notin \{\mu, \nu\}[e_j \in A \backslash D]\} \\ \Gamma_2 &= \{\mathfrak{e} \in [A]^{2\sigma} : \forall j < \sigma[e_j \in D] \ \& \ \forall j \ge \sigma[e_j \in A \backslash D]\} \ . \end{aligned}$$

Let $\vec{b}$ be the $\mp$-coefficients of length 2σ. Then $\chi^m_{\vec{b}}(\Gamma_1) = \chi_{\vec{b}}(\Gamma_1) = \aleph_0$ and $\chi_{\vec{b}}(\Gamma_2) = 2$. $\chi^m_{\vec{b}}(\Gamma_2) = 2$ when $m \nmid \sigma$ and $\chi^m_{\vec{b}}(\Gamma_2) = \aleph_0$ when $m \mid \sigma$.

Proof. If $A = \bigcup_{\alpha<\theta} C_\alpha$, with θ finite, then one can fix α, β with $C_\alpha \cap D$ and $C_\beta \cap (A \backslash D)$ both infinite. We can then choose $\mathfrak{e} \in \Gamma_1$ with $e_\mu, e_\nu \in C_\alpha \cap D$ and $e_j \in C_\beta \cap (A \backslash D)$ for all $j \ne \mu, \nu$. This $\mathfrak{e}$ is not successfully colored. Hence, $\chi^m_{\vec{b}}(\Gamma_1) = \chi_{\vec{b}}(\Gamma_1) = \aleph_0$.

Likewise, with the same α, β, we can choose $\mathfrak{e} \in \Gamma_2$ with $e_j \in C_\alpha \cap D$ for all $j < \sigma$ and $e_j \in C_\beta \cap (A \backslash D)$ for all $j \ge \sigma$. If $m \mid \sigma$, then $\sigma = 0 \bmod m$, so that this $\mathfrak{e}$ is not successfully colored $\bmod m$. Hence, $\chi^m_{\vec{b}}(\Gamma_2) = \aleph_0$.

However, if $m \nmid \sigma$, then $A = D \cup (A \backslash D)$ is a successful coloring for Γ in 2 colors, so that $\chi^m_{\vec{b}}(\Gamma_2) = 2$. $\square$

3 Embeddings into Groups

Throughout, G is an abelian group. We use our coefficients (Definition 2.2) to map edges in a hypergraph to elements of G as follows:

Definition 3.1 *Suppose that $\vec{b} = (b_0, \ldots, b_{n-1})$ is a sequence of coefficients and $A \subseteq G$ is totally ordered by $<$. Fix $\mathfrak{e} = \{e_0, \ldots, e_{n-1}\} \in [A]^n$. We define $\vec{b} \cdot \mathfrak{e} = b_0 e_0 + \cdots + b_{n-1} e_{n-1} \in G$ whenever either:*

a. The coefficients are in $\mathbb{Z}$ or

b. The coefficients are in $\mathbb{Z}_m$ and $mA = \{0\}$.

In either of these two cases, define $\vec{b} \cdot \Gamma = \{\vec{b} \cdot \mathfrak{e} : \mathfrak{e} \in \Gamma\}$ whenever $\Gamma \subseteq [A]^n$, and define $\Psi_{\vec{b}} : [A]^n \cup \{\infty\} \to G$ by: $\Psi_{\vec{b}}(\mathfrak{e}) = \vec{b} \cdot \mathfrak{e}$ and $\Psi_{\vec{b}}(\infty) = 0$.

Note that there is no presumed relationship between the order $<$ and the group structure; the order is just used to define the notions $\vec{b} \cdot \mathfrak{e}$ and $\vec{b} \cdot \Gamma$ (which depend on $<$). We proceed to show that for certain A, this $\Psi_{\vec{b}}$ is a topological embedding with respect to the Bohr topology on G and the topology $\mathcal{T}_{\vec{b}}$ of Definition 2.4.

Definition 3.2 *If all $b_i \in \mathbb{Z}$, then $\|\vec{b}\| = \|\vec{b}\|_1 = \Sigma_{i=0}^{n-1} |b_i|$. If $b_i \in \mathbb{Z}_m$, we may view $\mathbb{Z}_m$ as $\{0, 1, \ldots, m-1\}$ and define $\|\vec{b}\|$ in the same way.*

Note that even if $b_i \in \mathbb{Z}_m$, it is possible that $\|\vec{b}\| > m$.

Lemma 3.3 *In either of the two cases of Definition 3.1, the map $\Psi_{\vec{b}} : [A]^n \cup \{\infty\} \to G^\#$ is continuous with respect to the topology $\mathcal{T}_{\vec{b}}$ on $[A]^n \cup \{\infty\}$.*

Proof. It is only necessary to prove continuity at the point ∞, since the other points are isolated. To do this, it is sufficient to fix $\Gamma \subseteq [A]^n$, assume that $0 \notin \mathrm{cl}(\vec{b} \cdot \Gamma)$, and show that $\chi_{\vec{b}}(\Gamma)$ is finite.

First, consider Case (a) of Definition 3.1. If $V \subseteq G$ and r is a positive integer, let $V^r = \{v_1 + \cdots + v_r : v_1, \cdots, v_r \in V\}$. Fix an open neighborhood U of 0 with $\vec{b} \cdot \Gamma \cap U = \emptyset$. Then fix an open $V \ni 0$ with $V = -V$ and $V^{\|\vec{b}\|} \subseteq U$. Since the Bohr topology is totally bounded, G can be covered by finitely many translates of V; say $G = \bigcup_{\alpha<\theta}(V + x_\alpha)$, where θ is finite. The sets $(V + x_\alpha) \cap A$ may not be disjoint, but by shrinking them if necessary, we may partition A into $\bigcup_{\alpha<\theta} C_\alpha$ so that each $C_\alpha \subseteq V + x_\alpha$. We shall show that this coloring is successful:

If the coloring is unsuccessful, then fix $\mathfrak{e} = \{e_0, \ldots, e_{n-1}\} \in \Gamma$ such that for every $\alpha < \theta$, $\sum\{b_j : e_j \in C_\alpha\} = 0$. For each α, let $r_\alpha = \sum\{|b_j| : e_j \in C_\alpha\}$, so that $\|\vec{b}\| = \sum_{\alpha<\theta} r_\alpha$. Then, using $C_\alpha \subseteq V + x_\alpha$:

$$\sum\{b_j e_j : e_j \in C_\alpha\} \ \in \ V^{r_\alpha} + \sum\{b_j x_\alpha : e_j \in C_\alpha\} \ = \ V^{r_\alpha} \quad . \qquad (\star)$$

Now, summing over α: $\vec{b} \cdot \mathfrak{e} \in V^{\|\vec{b}\|} \subseteq U$, contradicting $\vec{b} \cdot \Gamma \cap U = \emptyset$.

In Case (b), the proof is essentially the same. We may assume that $G = \langle A \rangle$, so that $mG = \{0\}$. Then, for equation ($\star$), it is sufficient that each $\sum\{b_j : e_j \in C_\alpha\} = 0 \bmod m$, which will hold if the coloring is unsuccessful viewing the b_j as representing elements of $\mathbb{Z}_m$. $\square$

In general, one cannot assert that $\Psi_{\vec{b}}$ is a topological embedding (i.e., a homeomorphism onto its range). It is easy to construct examples where $\Psi_{\vec{b}}$ is not 1-1, or where $\Psi_{\vec{b}}$ is 1-1 but is not a homeomorphism. It is an embedding when A is suitably independent. We consider Case (b) of Definition 3.1 first. Recall that $A \subseteq G$ is called *independent* iff for all n, all $\{e_0, \ldots, e_{n-1}\} \in [A]^n$, and all $c_0, \ldots, c_{n-1} \in \mathbb{Z}$: if $\sum_j c_j e_j = 0$, then every $c_j e_j = 0$.

Notation 3.4 *We represent the circle group, $\mathbb{T}$, as $\mathbb{R}/\mathbb{Z}$, so all elements of $\mathbb{T}$ are of the form $[x]$ for $x \in \mathbb{R}$. We use "d" for linear distance, so that $d([x],[y]) = \min(|x-y|, 1-|x-y|)$ for $x, y \in [0,1]$.*

Lemma 3.5 *Suppose that $A \subseteq G$ is independent and all elements of A have order exactly m, and the coefficients $\vec{b}$ are in $\mathbb{Z}_m$. Then $\Psi_{\vec{b}} : [A]^n \cup \{\infty\} \to G^\#$ is a topological embedding with respect to the topology $\mathcal{T}_{\vec{b}}^m$ on $[A]^n \cup \{\infty\}$.*

Proof. Independence implies that $\Psi_{\vec{b}}$ is 1-1. Also by independence, for each $a \in A$, there is a character δ_a such that $\delta_a(a) = [1/m] \in \mathbb{T}$, and $\delta_a(x) = 0$ for all $x \in A \backslash \{a\}$. Then for each $\mathfrak{e} = \{e_0, \ldots, e_{n-1}\} \in [A]^n$, $\Psi_{\vec{b}}(\mathfrak{e}) = \vec{b} \cdot \mathfrak{e}$ is isolated in $\mathrm{ran}(\Psi_{\vec{b}})$ because

$$\{y \in G^\# : \forall j < n \, [d(\delta_{e_j}(y) - \delta_{e_j}(\vec{b} \cdot \mathfrak{e})) < 1/m]\}$$

is a neighborhood of $\vec{b} \cdot \mathfrak{e}$ containing no other element of $\mathrm{ran}(\Psi_{\vec{b}})$.

Thus, to prove that $\Psi_{\vec{b}}$ is an embedding, it is sufficient to fix $\Gamma \subseteq [A]^n$ such that $\infty \notin \mathrm{cl}(\Gamma)$ and prove that $0 \notin \mathrm{cl}(\vec{b} \cdot \Gamma)$. Applying the definition of $\mathcal{T}_{\vec{b}}^m$, let $\langle C_\alpha : \alpha < \theta \rangle$ be a finite successful coloring for Γ. For each $\alpha < \theta$, choose a character φ_α so that $\varphi_\alpha(x) = [1/m]$ if $x \in C_\alpha$ and $\varphi_\alpha(x) = 0$ if $x \in A \backslash C_\alpha$. For each $\mathfrak{e} \in \Gamma$, there is an α such that $\sum \{b_j : e_j \in C_\alpha\} \neq 0$ in $\mathbb{Z}_m$, so that $d(\varphi_\alpha(\vec{b} \cdot \mathfrak{e}), 0) \geq 1/m$ in $\mathbb{T}$. Thus, there is a neighborhood of 0 in $G^\#$ containing no elements of $\vec{b} \cdot \Gamma$. □

When the coefficients are in $\mathbb{Z}$, a similar proof will work if the elements of A have infinite order and are independent. However, in proving Theorem 1.2, we will need to get embeddings into groups such as $\mathbb{Q}$ and $\mathbb{Z}$, where there are no independent sets. To handle this, we introduce a variant of the notion of independence:

Definition 3.6 *If $\varepsilon > 0$, then $A \subseteq G$ is ε–*free *iff whenever $\tau : A \to \mathbb{T}$ is given, there is a $\varphi \in \mathrm{Hom}(G, \mathbb{T})$ such that $d(\tau(a), \varphi(a)) < \varepsilon$ for all $a \in A$.*

Observe that A is ε–free for *all* $\varepsilon > 0$ iff A is independent and all elements of A have infinite order. However, a weaker condition suffices to make $\Psi_{\vec{b}}$ an embedding:

Lemma 3.7 *Suppose that the coefficients $\vec{b}$ are in $\mathbb{Z}$ and $A \subseteq G$ is $1/(4\|\vec{b}\|)$–free. Then $\Psi_{\vec{b}} : [A]^n \cup \{\infty\} \to G^\#$ is a topological embedding with respect to the topology $\mathcal{T}_{\vec{b}}$ on $[A]^n \cup \{\infty\}$.*

Proof. We plan to show two things:

1. If $\Gamma \subseteq [A]^n$ and $\infty \notin \mathrm{cl}(\Gamma)$, then $0 \notin \mathrm{cl}(\vec{b} \cdot \Gamma)$.
2. If $\mathfrak{e} \in [A]^n$, then there is a neighborhood U of $\vec{b} \cdot \mathfrak{e}$ in $G^\#$ which does not contain $\vec{b} \cdot \mathfrak{w}$ for any $\mathfrak{w} \in [A]^n \backslash \{\mathfrak{e}\}$.

By (1) for singleton Γ along with (2), $\Psi_{\vec{b}}$ is 1-1. Then (2) implies that each $\vec{b}\cdot\mathfrak{e}$ is isolated in $\mathrm{ran}(\Psi_{\vec{b}})$, and then (1) implies that $\Psi_{\vec{b}}$ is a homeomorphism onto its range.

Whenever $\tau : A \to \mathbb{T}$, define $\widetilde{\tau} : [A]^n \to \mathbb{T}$ by $\widetilde{\tau}(\mathfrak{e}) = \sum_j b_j \tau(e_j)$; then $\widetilde{\tau}(\mathfrak{e}) = \bar{\tau}(\vec{b}\cdot\mathfrak{e})$ in the case that τ is the restriction of some $\bar{\tau} \in \mathrm{Hom}(G,\mathbb{T})$. Note that if $\varphi \in \mathrm{Hom}(G,\mathbb{T})$ and $d(\tau(a),\varphi(a)) < \varepsilon$ for all $a \in A$, then $d(\widetilde{\tau}(\mathfrak{e}),\varphi(\vec{b}\cdot\mathfrak{e})) < \|\vec{b}\|\varepsilon$ for all $\mathfrak{e} \in [A]^n$.

For (2): For each $i < n$, define $\tau_i(e_i) = [1/(2b_i)] \in \mathbb{T}$ and $\tau_i(x) = 0$ for $x \in A\backslash\{e_i\}$. Thus, $\widetilde{\tau_i}(\mathfrak{e}) = [1/2]$. Choose $\varphi_i \in \mathrm{Hom}(G,\mathbb{T})$ such that $d(\tau_i(x),\varphi_i(x)) < 1/(4\|\vec{b}\|)$ for all $x \in A$. Then $d(\widetilde{\tau_i}(\mathfrak{w}),\varphi_i(\vec{b}\cdot\mathfrak{w})) < 1/4$ for all $\mathfrak{w} \in [A]^n$, so that $d(\varphi_i(\vec{b}\cdot\mathfrak{e}),[1/2]) < 1/4$. Let

$$U = \{y \in G^{\#} \ : \ \forall i < n\,[d(\varphi_i(y),[1/2]) < 1/4]\} \ .$$

Fix $\mathfrak{w} \in [A]^n\backslash\{\mathfrak{e}\}$, and then fix i with $e_i \notin \mathfrak{w}$. Then $\widetilde{\tau_i}(\mathfrak{w}) = 0$, so that $d(\varphi_i(\vec{b}\cdot\mathfrak{w}),0) < 1/4$, so $\vec{b}\cdot\mathfrak{w} \notin U$.

For (1): let $\langle C_\alpha : \alpha < \theta\rangle$ be a finite successful coloring for Γ. We shall produce characters φ^r_β, for $\beta < \theta$ and $r = \pm 1, \pm 2, \ldots, \pm\|\vec{b}\|$, so that for each $\mathfrak{e} \in \Gamma$ there are some β, r with $d(\varphi^r_\beta(\vec{b}\cdot\mathfrak{e}),0) > 1/4$.

For each β, r, define $\tau^r_\beta : A \to \mathbb{T}$ so that $\tau^r_\beta(a) = [1/(2r)]$ for $a \in C_\beta$ and $\tau^r_\beta(a) = 0$ for $a \notin C_\beta$. Then fix a character $\varphi^r_\beta : G \to \mathbb{T}$ such that $d(\tau^r_\beta(a),\varphi^r_\beta(a)) < 1/(4\|\vec{b}\|)$ for all $a \in A$.

Now fix $\mathfrak{e} = \{e_0,\ldots,e_{n-1}\} \in \Gamma$. Say $e_j \in C_{\alpha_j}$ for $j < n$. Since the coloring is successful, fix β with $r := \sum\{b_j : \alpha_j = \beta\} \neq 0$. Then

$$\widetilde{\tau^r_\beta}(\mathfrak{e}) = \sum\{b_j : \alpha_j = \beta\}\,[1/(2r)] = [1/2] \ \ ,$$

and $d(\widetilde{\tau^r_\beta}(\mathfrak{e}),\varphi^r_\beta(\vec{b}\cdot\mathfrak{e})) < \|\vec{b}\|/(4\|\vec{b}\|) = 1/4$, so that $d(\varphi^r_\beta(\vec{b}\cdot\mathfrak{e}),0) > 1/4$. ☐

The following lemma is proved by a case analysis patterned after the proof that every infinite abelian group contains an infinite I_0-set (Hartman and Ryll-Nardzewski [8], Theorem 5). Such analyses are further described in [12].

Lemma 3.8 *The following are equivalent for any infinite abelian group G:*

1. *G is not of finite exponent.*
2. *For all $\varepsilon > 0$, G contains an infinite ε–free subset.*

Proof. For (2) $\to$ (1), observe that if $mG = \{0\}$, then all character values are of the form $[k/m]$, so that G cannot have a non-empty $1/(2m)$–free subset.

Now, assume (1). If G is not a torsion group, then $\mathbb{Z}$ is a subgroup of G, and then a suitably thin Hadamard set in $\mathbb{Z}$ will be ε–free (see [12], §2).

We are left with the case that G is a torsion group but contains elements of arbitrarily large order. Inductively choose x_i for $i \in \omega$, and let H_i be the subgroup generated by $\{x_j : j < i\}$; so $H_0 = \{0\}$. Given H_i, choose x_i such that if n_i is the least n with $nx_i \in H_i$, then $n_i > 1/(2\varepsilon)$.

We now show that $A = \{x_i : i \in \omega\}$ is ε–free. Fix $\tau : A \to \mathbb{T}$. We shall find a $\varphi \in \mathrm{Hom}(G, \mathbb{T})$ such that that each $d(\tau(x_i), \varphi(x_i)) < \varepsilon$ by inductively defining $\varphi{\restriction}H_i$. Fix i, and assume we have $\varphi{\restriction}H_i$. Let $U = \{u \in \mathbb{T} : n_i u = \varphi(n_i x_i)\}$. We may obtain a homomorphism on H_{i+1} by setting $\varphi(x_i)$ to be any element of U. Now $|U| = n_i$ and adjacent elements of U are distance $1/n_i$ apart, so we can fix $u \in U$ with $d(\tau(x_i), u) \le 1/(2n_i) < \varepsilon$, and set $\varphi(x_i) = u$. Finally, extend φ arbitrarily from $\langle\{x_i : i < \omega\}\rangle$ to all of G. □

Note that the notion of "free" does not involve any order on A, although the topology $\mathcal{T}_{\vec{b}}$ does depend on the order. We may apply Lemma 3.8 to get an infinite ε-free A, and then order A arbitrarily in applying Lemma 3.7, yielding:

Corollary 3.9 *Suppose that G is any abelian group which is not of finite exponent and $\vec{b}$ is a sequence of coefficients in $\mathbb{Z}$. Then there is an infinite $A \subseteq G$ and an ordering $<$ of A in type ω such that $\Psi_{\vec{b}} : [A]^n \cup \{\infty\} \to G^{\#}$ is a topological embedding with respect to the topology $\mathcal{T}_{\vec{b}}$ on $[A]^n \cup \{\infty\}$.*

4 Non-Embeddability of Hypergraph Spaces

We begin by proving (Lemma 4.8) that some hypergraph spaces are not embeddable in $(\mathbb{V}_p^\kappa)^{\#}$, where p is prime. To do this, we use some results from [11] on *normal forms* of functions from $[A]^n$ to $\mathbb{V}_p^\kappa$.

Definition 4.1 *A indexed* sequence $\langle w_j : j \in J\rangle$ *from an abelian group is* independent *iff the w_j are all non-0, and are distinct (as j varies), and the* set $\{w_j : j \in J\}$ *is independent in the usual sense.*

Definition 4.2 *If A is totally ordered by $<$, $\mathfrak{e} = \{e_0, \ldots, e_{n-1}\} \in [A]^n$, and $s \in [n]^r$, then $\mathfrak{e}{\restriction}s = \{e_i : i \in s\}$.*

Definition 4.3 *Suppose that A is totally ordered by $<$ and $f : [A]^n \to \mathbb{V}_p^\kappa$. Then f is* in normal form *iff there is an indexed sequence $\langle w_t^i : i < \ell\,;\, t \in [A]^{r_i}\rangle$ from $\mathbb{V}_p^\kappa$ (where $\ell < \omega$ and each $r_i \le n$) and elements $k_s^i \in \mathbb{Z}_p$ (for $i < \ell$ and $s \in [n]^{r_i}$) such that:*

- *The sequence* $\langle w^i_t : i < \ell\,;\ t \in [A]^{r_i}\rangle$ *is independent.*
- *Each* $f(\mathfrak{e}) = \sum_{i<\ell} \sum_{s\in[n]^{r_i}} k^i_s w^i_{\mathfrak{e}\restriction s}$.

Note that r_i may be 0, so that for this i, there is only the non-zero element $w^i_\emptyset \in \mathbb{V}^\kappa_p$, and the sum (for this i) reduces to just $k^i_\emptyset w^i_\emptyset$. Thus, every constant function is in normal form.

By [11], Theorem 3.4, we have:

Lemma 4.4 *Suppose that* p *is prime,* A *is totally ordered by* $<$ *in type* ω*, and* $f : [A]^n \to \mathbb{V}^\kappa_p$. *Then there is an infinite* $A' \subseteq A$ *such that* $f \restriction [A']^n$ *is in normal form.*

Now, say we wish to prove that some hypergraph space $[A]^n \cup \{\infty\}$ is not embeddable into $(\mathbb{V}^\kappa_p)^\#$, where A is ordered in type ω. So, we fix $f : [A]^n \cup \{\infty\} \to (\mathbb{V}^\kappa_p)^\#$ and try to prove that it is not an embedding. By translating, we may also assume that $f(\infty) = 0$. Further, by Lemma 4.4, we may assume that f is in normal form. Then, we show (Lemma 4.6) that if f is not identically 0, then continuity alone allows us to simplify the normal form to the case where all the $r_i = 1$. To do this, we need the following from [11], Lemma 4.3:

Lemma 4.5 *Fix* $r \geq 2$*, and an infinite* A *ordered by* $<$ *in type* ω*. Then there is a* $\Delta \subseteq [A]^r$ *with the following property: Whenever* $n \geq r$, $\emptyset \neq S \subseteq [n]^r$*, and* $A = \bigcup_{\alpha<\theta} C_\alpha$ *with* θ *finite, there is some* α *and an* $\mathfrak{e} \in [C_\alpha]^n$ *such that exactly one* $s \in S$ *satisfies* $\mathfrak{e}\restriction s \in \Delta$.

Lemma 4.6 *Suppose that* A *is infinite and totally ordered by* $<$*. Let* $\vec{b} = (b_0, \ldots, b_{n-1})$ *be a sequence of coefficients from some fixed abelian group. Assume that* $f : [A]^n \cup \{\infty\} \to (\mathbb{V}^\kappa_p)^\#$ *is such that* $f(\infty) = 0$ *and* $f \restriction [A]^n$ *is in normal form, using the notation of Definition 4.3. Assume that* f *is continuous with respect to* $\mathcal{T}_{\vec{b}}$*. Then* $k^i_s = 0$ *for all* i *such that* $r_i \neq 1$.

Proof. *Fix* i such that $r = r_i \neq 1$. Let $\pi : \mathbb{V}^\kappa_p \to \mathbb{V}^\kappa_p$ be some linear map such that each $\pi(w^i_t) = w^i_t$ and each $\pi(w^j_u) = 0$ whenever $j \neq i$. Then $\pi(f(\infty)) = 0$, and each $\pi(f(\mathfrak{e})) = \sum_{s\in[n]^r} k^i_s w^i_{\mathfrak{e}\restriction s}$. Also, $\pi \circ f$ is continuous because all group homomorphisms are continuous with respect to the Bohr topology.

From now on, we drop the superscript "i".

If $r = 0$, then each $\pi(f(\mathfrak{e})) = k_\emptyset w_\emptyset$, and then continuity forces $k_\emptyset = 0$.

Now, assume that $r \geq 2$. Let $S = \{s \in [n]^r : k_s \neq 0\}$. Assume that $S \neq \emptyset$; we shall derive a contradiction.

Fix $\Delta \subseteq [A]^r$ as in Lemma 4.5. Let $\Gamma = \{\mathfrak{e} \in [A]^n : \sum\{k_s : \mathfrak{e}\restriction s \in \Delta\} \neq 0\}$. We contradict continuity by showing that $0 \notin \mathrm{cl}(\pi(f(\Gamma)))$ but $\infty \in \mathrm{cl}(\Gamma)$.

Fix $\varphi \in \mathrm{Hom}(\mathbb{V}_p^\kappa, \mathbb{T})$ so that $\varphi(w_t) = [1/p]$ if $t \in \Delta$ and $\varphi(w_t) = 0$ if $t \notin \Delta$. By the definition of Γ, $\varphi(\pi(f(\mathfrak{e}))) \neq 0$ for all $\mathfrak{e} \in \Gamma$. Since the range of φ is finite, this proves that $0 \notin \mathrm{cl}(\pi(f(\Gamma)))$.

Now, suppose that $\infty \notin \mathrm{cl}(\Gamma)$. Then by the definition of $\mathcal{T}_{\vec{b}}$, there is a finite successful coloring for Γ, $A = \bigcup_{\alpha<\theta} C_\alpha$. Applying the property of Δ from Lemma 4.5, we can fix $\alpha < \theta$ and $\mathfrak{e} \in [C_\alpha]^n$ such that exactly one $s \in S$ satisfies $\mathfrak{e}\restriction s \in \Delta$. Then $\sum\{k_s : \mathfrak{e}\restriction s \in \Delta\} \neq 0$ since this sum contains exactly one nonzero term, so that $\mathfrak{e} \in \Gamma$. But all elements of $\mathfrak{e}$ have the same color, so the coloring could not be successful. ☐

We remark that $\vec{b}$ did not figure explicitly in this proof; we only used the fact that in a successful coloring, there are no monochromatic edges.

Now, we cannot in general claim that the $k_s^i = 0$ when $r_i = 1$, since we know by Lemma 3.5 that hypergraph spaces formed with coefficients from $\mathbb{Z}_p$ do embed into $(\mathbb{V}_p^\kappa)^\#$. However, in the case that $\vec{b}$ is the $\mp$-coefficient sequence (see Definition 2.6), we can simplify the normal form for continuous functions into $(\mathbb{V}_p^\kappa)^\#$:

Lemma 4.7 *Let $n = 2\sigma$, and let $\vec{b}$ be the $\mp$-coefficients of length 2σ. Let $\mathcal{T}$ be either the topology $\mathcal{T}_{\vec{b}}$ on $[A]^n \cup \{\infty\}$ or the topology $\mathcal{T}_{\vec{b}}^m$ for some $m \geq 2$. Assume that $f : [A]^n \cup \{\infty\} \to (\mathbb{V}_p^\kappa)^\#$, with $f(\infty) = 0$, $f \restriction [A]^n$ in normal form, and f continuous with respect to $\mathcal{T}$. Then either*

1. *f is identically 0, or*
2. *there is an independent sequence $\langle v_a : a \in A\rangle$ from $\mathbb{V}_p^\kappa$ such that $f(\mathfrak{e}) = \sum_{j<n} b_j v_{e_j}$ for all $\mathfrak{e} = \{e_0, \ldots, e_{n-1}\} \in [A]^n$.*

Proof. Applying Lemma 4.6 and deleting zero summands, we may assume that each $f(\mathfrak{e}) = \sum_{i<\ell}\sum_{j<n} k_j^i w_{e_j}^i$, where the $k_j^i \in \mathbb{Z}_p$ (for $k < \ell$ and $j < n$), and the sequence $\langle w_a^i : i < \ell\,;\ a \in A\rangle$ is independent. Identify $\mathbb{Z}_p$ with $\{[z/p] : z \in \mathbb{Z}\} \subset \mathbb{R}/\mathbb{Z} = \mathbb{T}$, so that we can identify $\mathrm{Hom}(\mathbb{V}_p^\kappa, \mathbb{T})$ with $\mathrm{Hom}(\mathbb{V}_p^\kappa, \mathbb{Z}_p)$.

Next, we show that for each i, $k_\mu^i + k_\nu^i = 0$ whenever $0 \leq \mu < \sigma$ and $\sigma \leq \nu < 2\sigma$. If this fails, fix i and fix some $\mu < \sigma$ and $\nu \geq \sigma$ with $k_\mu^i + k_\nu^i \neq 0$. Fix $D \subset A$ so that both D and $A\backslash D$ are infinite. Let Γ_1 be as in Example 2.7. Then both $\chi_{\vec{b}}(\Gamma_1)$ and $\chi_{\vec{b}}^m(\Gamma_1)$ are infinite, so $\infty \in \mathrm{cl}(\Gamma_1)$. Now, fix $\varphi \in \mathrm{Hom}(\mathbb{V}_p^\kappa, \mathbb{Z}_p)$ so that $\varphi(w_a^i) = 1$ if $a \in D$ and $\varphi(w_a^{i'}) = 0$ whenever $a \notin D$ or $i' \neq i$. Then for $\mathfrak{e} \in \Gamma_1$, $\varphi(f(\mathfrak{e})) = k_\mu^i + k_\nu^i \neq 0$. Thus $0 \notin \mathrm{cl}(f(\Gamma_1))$, contradicting the continuity of f.

Thus, for each i, there is a $c^i \in \mathbb{Z}_p$ such that $k^i_j = -c^i$ when $j < \sigma$ and $k^i_j = c^i$ when $j \geq \sigma$, so $k^i_j = c^i b_j$ for each j. We now have

$$f(\mathfrak{e}) = \sum_{i<\ell} \sum_{j<n} c^i b_j w^i_{e_j} = \sum_{j<n} b_j v_{e_j} \quad ,$$

where each $v_a = \sum_{i<\ell} c^i w^i_a$. Furthermore, $\langle v_a : a \in A \rangle$ is independent unless all the $c^i = 0$, in which case f is identically 0. $\square$

Lemma 4.8 *Suppose that A is infinite and ordered by $<$ in type ω. Let p be prime, and let $\vec{b}$ be the $\mp$-coefficients of length $n = 2\sigma$, where $p \mid \sigma$. Let $\mathcal{T}$ be either the topology $\mathcal{T}_{\vec{b}}$ on $[A]^n \cup \{\infty\}$ or the topology $\mathcal{T}^m_{\vec{b}}$, where $m \nmid \sigma$. Then $[A]^n \cup \{\infty\}$ with topology $\mathcal{T}$ cannot be embedded topologically into $(\mathbb{V}^\kappa_p)^\#$.*

Proof. Assume that $f : [A]^n \cup \{\infty\} \to (\mathbb{V}^\kappa_p)^\#$ is such an embedding. Translating in $\mathbb{V}^\kappa_p$ by $-f(\infty)$, we may assume that $f(\infty) = 0$. Applying Lemma 4.4, we may assume that $f \restriction [A]^n$ is in normal form. Then, applying Lemma 4.7, we have an independent $\langle v_a : a \in A \rangle$ from $\mathbb{V}^\kappa_p$ such that each $f(\mathfrak{e}) = \sum_{j<n} b_j v_{e_j}$.

We now show that f cannot be an embedding. Again, choose $D \subset A$ such that D and $A \backslash D$ are infinite, and let Γ_2 be as in Example 2.7. Then $\chi_{\vec{b}}(\Gamma_2) = 2$ and $\chi^m_{\vec{b}}(\Gamma_2) = 2$ (since $m \nmid \sigma$). Hence, $\infty \notin \mathrm{cl}(\Gamma_2)$ in the topology $\mathcal{T}$. However, $\infty \in \mathrm{cl}(\Gamma_2)$ with respect to the topology $\mathcal{T}^p_{\vec{b}}$ because $\chi^p_{\vec{b}}(\Gamma_2) = \aleph_0$ (since $p \mid \sigma$). It follows by Lemma 3.5 (applied with $G = \mathbb{V}^\kappa_p$ and $m = p$) that $0 \in \mathrm{cl}(f(\Gamma_2))$ (since one may identify $a \in A$ with v_a). Hence, f is not an embedding. $\square$

This lemma is sufficient to prove Theorem 1.2, which involves only embeddings into $(\mathbb{V}^\kappa_p)^\#$. However, we can pursue this analysis further and prove the non-existence of certain embeddings into arbitrary groups of finite exponent. These groups are related to groups of the form $(\mathbb{V}^\kappa_{p^\ell})^\#$ by Theorem 1.5. Now, for $\ell > 1$, we do not have a normal form result like Lemma 4.4 for functions into $\mathbb{V}^\kappa_{p^\ell}$. However, if $f : [A]^n \cup \{\infty\} \to (\mathbb{V}^\kappa_{p^\ell})^\#$ is continuous, we may compose f with the canonical projection into $(\mathbb{V}^\kappa_{p^\ell})/p(\mathbb{V}^\kappa_{p^\ell})$, which is isomorphic to $\mathbb{V}^\kappa_p$, and apply Lemma 4.4 to the composition. In doing this, the following lemma will be useful in showing that "independent" lifts:

Lemma 4.9 *Let $K = \mathbb{V}^\kappa_{p^\ell}$, and for $x \in K$, let $[x]$ be the equivalence class of x in K/pK. Suppose that $\langle [v_a] : a \in A \rangle$ is independent in K/pK. Then each v_a has order p^ℓ and the sequence $\langle v_a : a \in A \rangle$ is independent in K.*

Proof. As usual in a group, $p^r \mid x$ means that $\exists y [p^r y = x]$. Observe that for $x \in K$ and $r \leq \ell$: $p^r \mid x$ iff $p^{\ell - r} x = 0$. In particular, v_a has order p^ℓ, because otherwise $p^{\ell-1}v_a = 0$, and hence $p \mid v_a$, so that $[v_a] = 0$, contradicting independence of $\langle [v_a] : a \in A\rangle$ (Definition 4.1).

Now, to prove independence of $\langle v_a : a \in A\rangle$, assume that $\sum_i c_i v_{a_i} = 0$, where each $c_i \in \mathbb{Z}$. By induction on $k \leq \ell$, we show that $p^k \mid c_i$ for each i; then, the case $k = \ell$ implies that each $c_i v_{a_i} = 0$.

The case $k = 0$ is trivial. For the induction step, fix $k \leq \ell$, and assume that $p^{k-1} \mid c_i$ for each i, so we can let $c_i = p^{k-1}d_i$. Then $p^{k-1}\sum_i d_i v_{a_i} = 0$ in K, so $p^{\ell-k+1} \mid \sum_i d_i v_{a_i}$. Then $\sum_i d_i [v_{a_i}] = 0$ in K/pK. By independence, $p \mid d_i$, and hence $p^k \mid c_i$, for each i. □

We can now prove a version of Lemma 4.7 which applies to maps into arbitrary groups of finite exponent.

Lemma 4.10 *Let A be totally ordered by $<$ in type ω. Let $n = 2\sigma$, and let $\vec{b}$ be the $\mp$-coefficients of length 2σ. Let $\mathcal{T}$ be either the topology $\mathcal{T}_{\vec{b}}$ on $[A]^n \cup \{\infty\}$ or the topology $\mathcal{T}_{\vec{b}}^m$ for some $m \geq 2$. Let K be any abelian group of finite exponent. Let $f : [A]^n \cup \{\infty\} \to K^\#$ be continuous with respect to $\mathcal{T}$, with $f(\infty) = 0$. Then there is an infinite $A' \subseteq A$ such that either*

1. *$f \restriction [A']^n$ is identically 0, or*
2. *there is an independent sequence $\langle v_a : a \in A'\rangle$ of elements of the same order such that $f(\mathfrak{e}) = \sum_{j<n} b_j v_{e_j}$ for all $\mathfrak{e} = \{e_0, \ldots, e_{n-1}\} \in [A']^n$.*

Proof. By Theorem 1.5, it is sufficient to prove this in the case that $K = \mathbb{V}^\kappa_{p^\ell}$. We now induct on ℓ. For $\ell = 1$, the result follows from Lemmas 4.4 and 4.7, so assume that $\ell > 1$. Let $\pi : K \to K/pK$ be the canonical projection. Since K/pK is isomorphic to $\mathbb{V}^\kappa_p$, we may apply the $\ell = 1$ case to $\pi \circ f$, obtaining A' as above.

If $\pi \circ f \restriction [A']^n$ is identically 0, then each $f(\mathfrak{e}) \in pK \cong \mathbb{V}^\kappa_{p^{\ell-1}}$ (for $\mathfrak{e} \in [A']^n$), so the result follows by the inductive hypothesis. Thus, we may assume that we have a sequence $\langle v_a : a \in A'\rangle$ in K such that each $\pi(f(\mathfrak{e})) = \sum_{j<n} b_j [v_{e_j}]$ and such that $\langle [v_a] : a \in A'\rangle$ is independent in K/pK.

Let $g(\mathfrak{e}) = f(\mathfrak{e}) - \sum_{j<n} b_j v_{e_j}$ and $g(\infty) = 0$. Then $g : [A']^n \cup \{\infty\} \to (pK)^\#$ is continuous by Lemma 3.3. Applying the inductive hypothesis, we get an $A'' \subseteq A'$ and a sequence of elements $\langle w_a : a \in A''\rangle$ from pK such that $g(\mathfrak{e}) = \sum_{j<n} b_j w_{e_j}$ for all $\mathfrak{e} \in [A'']^n$, so that $f(\mathfrak{e}) = \sum_{j<n} b_j (v_{e_j} + w_{e_j})$. Since $[v_a + w_a] = [v_a]$ in K/pK, $\langle [v_a + w_a] : a \in A''\rangle$ is independent in K/pK, so that by Lemma 4.9, $\langle v_a + w_a : a \in A''\rangle$ is independent in K, and each $v_a + w_a$ has order p^ℓ. □

Using this, we can prove a result along the lines of Lemma 4.8, refuting some embeddings of hypergraph spaces into groups of finite exponent.

Lemma 4.11 *Suppose that A is infinite and ordered by $<$ in type ω. Let $\vec{b}$ be the $\mp$-coefficients of length $n = 2\sigma$. Let K be any abelian group such that $\sigma K = \{0\}$. Let $\mathcal{T}$ be either the topology $\mathcal{T}_{\vec{b}}$ on $[A]^n \cup \{\infty\}$ or the topology $\mathcal{T}_{\vec{b}}^m$, where $m \nmid \sigma$. Then $[A]^n \cup \{\infty\}$ with topology $\mathcal{T}$ cannot be embedded topologically into $K^\#$.*

Proof. Assume that $f : [A]^n \cup \{\infty\} \to K^\#$ is such an embedding. Translating, we may assume that $f(\infty) = 0$. Applying Lemma 4.10, we may assume that there is an independent sequence $\langle v_a : a \in A\rangle$ of elements of K of some fixed order r such that $f(\mathfrak{e}) = \sum_{j<n} b_j v_{e_j}$ for all $\mathfrak{e} \in [A]^n$.

As in the proof of Lemma 4.8, choose $D \subset A$ such that D and $A \backslash D$ are infinite, and let Γ_2 be as in Example 2.7. Then $\chi_{\vec{b}}(\Gamma_2) = 2$ and $\chi_{\vec{b}}^m(\Gamma_2) = 2$ (since $m \nmid \sigma$). Hence, $\infty \notin \mathrm{cl}(\Gamma_2)$ in the topology $\mathcal{T}$. However, $\infty \in \mathrm{cl}(\Gamma_2)$ with respect to the topology $\mathcal{T}_{\vec{b}}^r$ because $\chi_{\vec{b}}^r(\Gamma_2) = \aleph_0$ (since $r \mid \sigma$). Thus, by Lemma 3.5, $0 \in \mathrm{cl}(f(\Gamma_2))$, so that f is not an embedding. □

5 Non-Embeddability of Groups

Proof of Theorem 1.2. For the $\to$ direction, assume that $G^\#$ embeds into $(\mathbb{V}_p^\kappa)^\#$, and assume that G is infinite (since otherwise the result is trivial, taking $\lambda = 0$ and $F = G$).

If G is not of finite exponent, then Corollary 3.9 plus Lemma 4.8 shows that there is a hypergraph space (using the topology $\mathcal{T}_{\vec{b}}$, where $\vec{b}$ is the $\mp$ coefficient sequence of length $2p$) which embeds into $G^\#$ and not into $(\mathbb{V}_p^\kappa)^\#$.

So, fix r with $rG = \{0\}$. By Theorem 1.5, $G = \bigoplus_{\xi<\lambda} \mathbb{Z}_{n_\xi}$, where each $n_\xi \mid r$ and $\lambda = |G| \le \kappa$. If all but finitely many $n_\xi = p$, then G is isomorphic to $\mathbb{V}_p^\lambda \times F$ for some finite F. If not, then fix $m \neq p$ such that $n_\xi = m$ for infinitely many ξ. Then $(\mathbb{V}_m^\omega)^\#$ embeds into $G^\#$ and hence into $(\mathbb{V}_p^\kappa)^\#$. However, Lemma 3.5 plus Lemma 4.8 shows that there is a hypergraph space (now using the topology $\mathcal{T}_{\vec{b}}^m$) which embeds into $(V_m^\omega)^\#$ and not into $(\mathbb{V}_p^\kappa)^\#$, so we have a contradiction. □

A similar proof yields:

Theorem 5.1 *If G, K are any abelian groups, where K is of finite exponent and G is not of finite exponent, then $G^\#$ is not embeddable into $K^\#$.*

Proof. Say $\sigma K = \{0\}$ and let $\vec{b}$ be as in Lemma 4.11. The topology $\mathcal{T}_{\vec{b}}$ does not embed into $K^\#$, but it does embed into $G^\#$ by Corollary 3.9. □

It is now natural to ask when $G^{\#}$ embeds into $K^{\#}$ in the case that both groups have finite exponent. The next theorem gives a complete answer for countable groups.

Definition 5.2 *If K is an infinite abelian group of finite exponent, let $eo(K)$, the* essential order *of K, be the least σ such that K is of the form $F \times H$, where F is finite and H is of exponent σ.*

In the notation of Theorem 1.5, $eo(K) = \mathrm{lcm}\{p^i : \kappa_{p,i} \geq \aleph_0\}$.

Theorem 5.3 *If G, K are any countably infinite abelian groups of finite exponent, then the following are equivalent:*

1. *$G^{\#}$ is homeomorphic to a subset of $K^{\#}$*
2. *$eo(G) \mid eo(K)$.*

Proof. For (2) $\to$ (1): By Theorem 1.5, we have $G \cong G_1 \times G_2$ and $K \cong K_1 \times K_2$, where G_1 and K_1 are finite and G_2 is isomorphic to a subgroup of K_2. The result now follows by Lemma 1.3.

For (1) $\to$ (2): By Theorem 1.5, it is sufficient to show that $(\mathbb{V}_{p^\ell}^{\omega})^{\#}$ is not homeomorphic to a subset of $K^{\#}$ when $\sigma K = \{0\}$ and $p^\ell \nmid \sigma$. Let $m = p^\ell$, let A be infinite and ordered by $<$ in type ω, and let $\vec{b}$ be the $\mp$-coefficients of length $n = 2\sigma$. By Lemma 4.11, $[A]^n \cup \{\infty\}$ with topology $\mathcal{T}_{\vec{b}}^m$ does not embed into $K^{\#}$, whereas it does embed into $(\mathbb{V}_{p^\ell}^{\omega})^{\#}$ by Lemma 3.5. $\square$

6 Conclusion

The following three kinds of questions about abelian groups are not answered by the results of this paper:

First, we have no results refuting the embeddability of any $G^{\#}$ into any $K^{\#}$, where K is not of finite exponent.

Second, when G, K are both of finite exponent, Theorem 5.3 leaves open many questions about the embeddability of $G^{\#}$ into $K^{\#}$ when G is uncountable. For example, it is not clear whether $(\mathbb{V}_2^{\omega_1})^{\#}$ embeds into $(\mathbb{V}_2^{\omega} \times \mathbb{V}_3^{\omega_1})^{\#}$.

Third, all proofs of non-homeomorphism actually prove non-embeddability. We know of no example where $G^{\#}$ and $K^{\#}$ are not homeomorphic but each is embeddable into the other. Specifically, let $G = \mathbb{V}_4^{\omega}$ and $K = \mathbb{V}_2^{\omega} \times \mathbb{V}_4^{\omega}$. Then each is isomorphic to a subgroup of the other, but we do not know whether $G^{\#}$ and $K^{\#}$ are homeomorphic.

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