Dissipated Compacta*

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Abstract

The dissipated spaces form a class of compacta which contains both the scattered compacta and the compact LOTSes (linearly ordered topological spaces), and a number of theorems true for these latter two classes are true more generally for the dissipated spaces. For example, every regular Borel measure on a dissipated space is separable.

The standard Fedorčuk S-space (constructed under $\diamondsuit$) is dissipated. A dissipated compact L-space exists iff there is a Suslin line.

A product of two compact LOTSes is usually not dissipated, but it may satisfy a weakening of that property. In fact, the degree of dissipation of a space can be used to distinguish topologically a product of n LOTSes from a product of m LOTSes.

1 Introduction

All topologies discussed in this paper are assumed to be Hausdorff. As usual, a subset of a space is *perfect* iff it is closed and non-empty and has no isolated points, so X is *scattered* iff X has no perfect subsets.

There are many constructions in the literature which build a compactum X as an inverse limit of metric compacta X_α for $\alpha < \omega_1$, with the bonding maps $\pi^\beta_\alpha : X_\beta \twoheadrightarrow X_\alpha$ for $\alpha < \beta < \omega_1$. In some cases, as in [7, 11, 12], the construction has the property that for each α, β, $(\pi^\beta_\alpha)^{-1}\{x\}$ is a singleton for all but countably many $x \in X_\alpha$. We shall call such π^β_α *tight maps*; these are discussed in greater detail in Section 2. The spaces X so constructed are examples of *dissipated compacta*;

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these are discussed in Section 3. Section 7 shows that the property of tightness is absolute for transitive models of set theory.

The precise definition of "dissipated" in Section 3 will be that there are "sufficiently many" tight maps onto metric compacta; so the definition will not mention inverse limits. Then, Section 6 will relate this definition to inverse limits.

Dissipated compacta include the scattered compacta, the metric compacta, and the compact LOTSes (totally ordered spaces with the order topology). Section 3 also describes the more general notion of κ–dissipated, which gets weaker as κ gets bigger; "dissipated" is the same as as "2–dissipated", while "1–dissipated" is the same as "scattered". Every regular Borel measure on a $2^{\aleph_0}$–dissipated compactum is separable (see Section 5).

If X is the double arrow space of Alexandroff and Urysohn, then X is a non-scattered LOTS and hence is 2–dissipated but not 1–dissipated, while X^{n+1} is $(2^n + 1)$–dissipated but not 2^n–dissipated. Considerations of this sort can be used to distinguish topologically a product of n LOTSes from a product of m LOTSes; see Section 4.

2 Tight Maps

As usual, $f : X \to Y$ means that f is a *continuous* map from X to Y, and $f : X \twoheadrightarrow Y$ means that f is a continuous map from X *onto* Y.

Definition 2.1 *Assume that X, Y are compact and $f : X \to Y$.*

- ☞ *A* loose family *for f is a disjoint family $\mathcal{P}$ of closed subsets of X such that for some non-scattered $Q \subseteq Y$, $Q = f(P)$ for all $P \in \mathcal{P}$.*
- ☞ *f is κ–*tight *iff there are no loose families for f of size κ.*
- ☞ *f is* tight *iff f is 2–tight.*

This notion gets weaker as κ gets bigger. f is 1–tight iff $f(X)$ is scattered, so that "2–tight" is the first non-trivial case. f is trivially $|X|^+$–tight. The usual projection from $[0, 1]^2$ onto $[0, 1]$ is not $2^{\aleph_0}$–tight.

Some easy equivalents to "κ–tight":

Lemma 2.2 *Assume that X, Y are compact and $f : X \to Y$. Then* (1) $\leftrightarrow$ (2)*. If κ is finite, then* (1) $\leftrightarrow$ (3)*; if also Y is metric, then all five of the following are equivalent:*

1. *There is a loose family of size κ.*
2. *There is a disjoint family $\mathcal{P}$ of perfect subsets of X with $|\mathcal{P}| = \kappa$ and a perfect $Q \subseteq Y$ such that $Q = f(P)$ for all $P \in \mathcal{P}$.*

3. *There are distinct $a_i \in X$ for $i < \kappa$ with all $f(a_i) = b \in Y$ such that whenever U_i is a neighborhood of a_i for $i < \kappa$, $\bigcap_{i<\kappa} f(\overline{U_i})$ is not scattered.*
4. *For some metric M and $\varphi \in C(X, M)$, $\{y \in Y : |\varphi(f^{-1}\{y\})| \geq \kappa\}$ is uncountable.*
5. *Statement (4), with $M = [0, 1]$.*

Proof. (2) → (1) is obvious. Now, assume (1), and let $\mathcal{P}$ be a loose family of size κ, with $Q = f(P)$ for $P \in \mathcal{P}$. Let Q' be a perfect subset of Q, and, for $P \in \mathcal{P}$, let P' be a closed subset of $P \cap f^{-1}(Q')$ such that $f\restriction P' : P' \twoheadrightarrow Q'$ is irreducible. Then $\{P' : P \in \mathcal{P}\}$ satisfies (2).

From now on assume that κ is finite.

(3) → (1) and (5) → (4) are obvious.

For (1) → (3), use compactness of $\prod_i P_i$ and the fact that a finite union of scattered spaces is scattered.

For (1) → (5): If $\mathcal{P} = \{P_i : i < \kappa\}$ is a loose family, with $Q = f(P_i)$, apply the Tietze Theorem to get $\varphi \in C(X, [0, 1])$ such that $\varphi(x) = 2^{-i}$ for all $x \in P_i$.

Now, we prove (4) → (1) when Y is metric. Fix φ as in (4). We may assume that $M = \varphi(X)$, so that M is compact. Let $\mathcal{B}$ be a countable base for M. Then we can find $B_i \in \mathcal{B}$ for $i < \kappa$ such that the $\overline{B_i}$ are disjoint and such that $Q := \{y \in Y : \forall i < \kappa\, [\varphi(f^{-1}\{y\}) \cap \overline{B_i} \neq \emptyset]\}$ is uncountable, and hence not scattered (since Y is metric). Q is also closed. Let $P_i = f^{-1}(Q) \cap \varphi^{-1}(\overline{B_i})$. Then $\{P_i : i < \kappa\}$ is a loose family. ☕

Lemma 2.3 *If X, Y are compact LOTSes and $f : X \to Y$ is order-preserving ($x_1 < x_2 \to f(x_1) \leq f(x_2)$), then f is tight.*

Proof. If not, we would have $a_0 < a_1$ and b as in (3) of Lemma 2.2. Let U_0, U_1 be open intervals in X with disjoint closures such that each $a_i \in U_i$. But then $f(\overline{U_0}) \cap f(\overline{U_1}) = \{b\}$, a contradiction. ☕

In many cases, the loose family will be defined uniformly via a continuous function, and we may replace the cardinal κ in Definition 2.1 by some compact space K of size κ:

Definition 2.4 *Assume that X, Y, K are compact spaces and $f : X \to Y$. Then a* K–loose function *for f is a $\varphi : \mathrm{dom}(\varphi) \to K$ such that*: $\mathrm{dom}(\varphi)$ *is closed in X, and for some non-scattered $Q \subseteq Y$, $\varphi(f^{-1}\{y\}) = K$ for all $y \in Q$.*

Note that we then have a loose family $\mathcal{P} = \{P_z : z \in K\}$ of size $|K|$, where $P_z = f^{-1}(Q) \cap \varphi^{-1}\{z\}$. For finite n, we may view the ordinal n as a discrete topological space, so an n–loose *function* is equivalent to a loose *family* $\mathcal{P} = \{P_i : i < n\}$, since φ can map P_i to $i \in n$. The same phenomenon holds for $\aleph_0$, but seems harder to prove:

Theorem 2.5 *If X, Y are compact and $f : X \to Y$, then there is an infinite loose family iff there is an $(\omega + 1)$–loose function.*

This will be proved in Section 7. Beyond $\aleph_0$, there is no simple equivalence between the cardinal version and the topological version of looseness. At $2^{\aleph_0}$, we shall use the following terminology to avoid possible confusion between the Cantor set 2^ω and the cardinal $\mathfrak{c} = 2^{\aleph_0}$:

Definition 2.6 *Assume that X, Y are compact and $f : X \to Y$.*

- ☞ *A* strongly $\mathfrak{c}$–loose family *for f is a K–loose function $\varphi : \mathrm{dom}(\varphi) \to K$, where K is the Cantor set 2^ω.*
- ☞ *f is* weakly $\mathfrak{c}$–tight *iff there is no strongly $\mathfrak{c}$–loose function for f.*

In this paper, whenever we produce a loose family of size $2^{\aleph_0}$, it will usually be strongly $\mathfrak{c}$–loose. However, if we view $\mathfrak{c} + 1$ as a compact ordinal and let $X = Y \times (\mathfrak{c}+1)$, then assuming that Y is not scattered, the usual projection $f : X \twoheadrightarrow Y$ has an obvious loose family of size $\mathfrak{c}$ but no strongly $\mathfrak{c}$–loose family.

When X, Y are both metric, the κ-tightness of f is related to the sizes of the sets $f^{-1}\{y\}$ by:

Theorem 2.7 *If X, Y are compact metric and $f : X \to Y$, then f is κ–tight iff $\{y \in Y : |f^{-1}\{y\}| \geq \kappa\}$ is countable. f is weakly $\mathfrak{c}$–tight iff f is $\mathfrak{c}$–tight.*

In particular, if $f : X \twoheadrightarrow Y$, then f is tight iff $f^{-1}\{y\}$ is a singleton for all but countably many y, as we said in the Introduction.

For both "iff"s, the $\leftarrow$ direction is trivial and is true for any X, Y. For $\kappa = 3$, say, the proof of the $\to$ direction will show that if there are uncountably many $y \in Y$ such that $f^{-1}\{y\}$ contains three or more points, then for some perfect $Q \subseteq Y$, we can, on Q, choose three of these points continuously, producing disjoint perfect $P_0, P_1, P_2 \subseteq X$ which f maps homeomorphically onto Q, so $\{P_0, P_1, P_2\}$ is a loose family of size 3.

Since X is second countable, each $f^{-1}\{y\}$ is either countable or of size $2^{\aleph_0}$, so it is sufficient to prove the theorem for the cases $\kappa \leq \aleph_0$ and $\kappa = 2^{\aleph_0}$. However, for $\kappa = 2^{\aleph_0}$, we can get more detailed results. For example, if there are uncountably many $y \in Y$ such that $f^{-1}\{y\}$ contains a Klein bottle, then we can choose the bottle continuously on a perfect set (see Theorem 2.9). This "continuous selector" result follows easily from standard descriptive set theory. First, observe:

Lemma 2.8 *Suppose that $g : \Phi \to Y$, where Y is a Polish space, Φ is an analytic subset of some Polish space, and $g(\Phi)$ is uncountable. Then there is a Cantor subset $C \subseteq \Phi$ such that g is 1-1 on C.*

Proof. Let $h : \omega^\omega \twoheadrightarrow \Phi$, apply the classical argument of Suslin to obtain a Cantor subset $D \subseteq \omega^\omega$ such that $g \circ h$ is 1-1 on D, and let $C = h(D)$.

Theorem 2.9 *Assume that X, Y, Z are compact metric, $f : X \to Y$, and there are uncountably many $y \in Y$ such that $f^{-1}\{y\}$ contains a homeomorphic copy of Z. Then there is a perfect $Q \subseteq Y$ and a 1-1 map $i : Q \times Z \to X$ such that $f(i(q, u)) = q$ for all $(q, u) \in Q \times Z$.*

Proof. Assume that $Z \neq \emptyset$. Fix metrics d_Z, d_X on Z, X, and give $C(Z, X)$ the usual uniform metric, which makes it a Polish space. Let Φ be the set of all $\varphi \in C(Z, X)$ such that φ is 1-1 and $\varphi(Z) \subseteq f^{-1}\{y\}$ for some (unique) $y \in Y$. Observe that Φ is an $F_{\sigma\delta}$ set, since the "φ is 1-1" can be expressed as:

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall u, v \in Z \, [d_Z(u, v) \geq \varepsilon \to d_X(\varphi(u), \varphi(v)) \geq \delta] \ .$$

Define $g : \Phi \to Y$ so that $g(\varphi)$ is the $y \in Y$ such that $\varphi(Z) \subseteq f^{-1}\{y\}$. Using Lemma 2.8, let $C \subseteq \Phi$ be a Cantor subset with g 1-1 on C, let $Q = g(C)$, and let $i(g(\varphi), u) = \varphi(u)$.

Proof of Theorem 2.7. To prove the $\to$ direction of the first "iff" in the three cases $\kappa < \aleph_0$, $\kappa = \aleph_0$, and $\kappa = \mathfrak{c}$, apply Theorem 2.9 respectively with Z the space κ (with the discrete topology), $\omega + 1$, and 2^ω. This also yields the $\to$ direction of the second "iff".

Of course, we are using the fact that every uncountable metric compactum contains a copy of the Cantor set. One could also prove Theorem 2.7 using the following, plus the fact that every uncountable metric compactum maps onto $[0, 1]$:

Theorem 2.10 *Assume that X, Y, K are compact metric with $f : X \to Y$, and assume that for uncountably many $y \in Y$, there is a closed subset of $f^{-1}\{y\}$ which can be mapped onto K. Then there is a K–loose function for f.*

Proof. Let H be the Hilbert cube, $[0, 1]^\omega$. We may assume that $K \subseteq H$. Then, for uncountably many $y \in Y$, there is a $\psi \in C(X, H)$ such that $\psi(f^{-1}\{y\}) \supseteq K$. Let $\Psi = \{(y, \psi) \in Y \times C(X, H) : \psi(f^{-1}\{y\}) \supseteq K\}$, and let $g(y, \psi) = y$. Applying Lemma 2.8, let $C \subseteq \Psi$ be a Cantor set on which g is 1-1, and let $Q = g(C) \subseteq Y$. For $(y, \psi) \in C$, let $E_y = \{x \in X : \psi(x) \in K\}$. Define φ so that $\mathrm{dom}(\varphi) = \bigcup\{E_y : y \in Q\}$, and $\varphi(x) = \psi(x)$ whenever $x \in \mathrm{dom}(\varphi)$ and $(y, \psi) \in C$. Then φ is a K–loose function.

Theorems 2.7, 2.9, and 2.10 can fail when X is not metric; counter-examples are provided by the double arrow space and some related spaces described by:

Definition 2.11 $I = [0,1]$. *If* $S \subseteq (0,1)$, *then* I_S *is the compact LOTS which results by replacing each* $x \in S$ *by a pair of neighboring points,* $x^- < x^+$. *The* double arrow space *is* $I_{(0,1)}$.

I_S has no isolated points because $0, 1 \notin S$. The double arrow space is obtained by splitting all points other than $0, 1$. $I_\emptyset = I$, and $I_{\mathbb{Q}\cap(0,1)}$ is homeomorphic to the Cantor set.

Lemma 2.12 *For each* $S \subseteq (0,1)$, I_S *is a compact separable LOTS with no isolated points.* I_S *is second countable iff* S *is countable.*

Now, let $Y = [0,1]$, let $S \subseteq (0,1)$, let $X = I_S$ and let $f : X \twoheadrightarrow Y$ be the natural map. Then f is 2–tight by Lemma 2.3, but $S = \{y \in Y : |f^{-1}\{y\}| \geq 2\}$ need not be countable, so Theorems 2.7, 2.9, and 2.10 fail here when S is uncountable (and hence X is not metric). However, one can apply these theorems in some generic extension, to get a (perhaps strange) alternate proof that f is 2–tight. Roughly, if $V[G]$ makes S countable, then X, Y will both be compact metric in $V[G]$, so Theorem 2.7 implies that f is 2–tight in $V[G]$ (because S is countable); but then by absoluteness, f is 2–tight in V. Absoluteness of tightness is discussed more precisely in Section 7.

The composition properties of tight maps are given by:

Lemma 2.13 *Assume that* X, Y, Z *are compact,* m, n *are finite,* $f : X \twoheadrightarrow Y$, *and* $g : Y \twoheadrightarrow Z$. *Then:*

1. *If* $g \circ f$ *is* n*–tight then* g *is* n*–tight.*
2. *If* f *and* g *are tight, then* $g \circ f$ *is tight.*
3. *If* f *is* $m+1$*–tight and* g *is* $n+1$*–tight, then* $g \circ f$ *is* $mn+1$*–tight.*

Proof. (1) is trivial, and (2) is a special case of (3).

For (3), assume that f is $m+1$–tight, g is $n+1$–tight, and $g \circ f$ is not $mn+1$–tight; we shall derive a contradiction. Fix disjoint closed $P_0, P_1, \ldots, P_{mn} \subseteq X$ with $g(f(P_0)) \cap g(f(P_1)) \cap \cdots \cap g(f(P_{mn}))$ not scattered. Shrinking X, Y, Z, and the P_i, we may assume WLOG that $X = P_0 \cup P_1 \cup \cdots \cup P_{mn}$ and that $g(f(P_i)) = Z$ for each i. For each $s \subseteq \{0, 1, \ldots, mn\}$, let $Q_s = \bigcap_{i \in s} f(P_i)$. Shrinking the P_i, we may assume WLOG that each $Q_s \subseteq Y$ is either empty or not scattered; to see this, for a fixed s: If Q_s is scattered, then so is $g(Q_s)$; if R is a perfect subset of $Z \backslash g(Q_s)$, then we may replace Z by R and each P_i by $P_i \cap f^{-1}(g^{-1}(R))$.

Now, using compactness of $P_0 \times P_1 \times \cdots \times P_{mn}$, as in the proof of Lemma 2.2, fix $a_i \in P_i$ for $i \leq mn$ such that $g(f(\overline{U_0})) \cap \cdots \cap g(f(\overline{U_{mn}}))$ is not scattered whenever each U_i is a neighborhood of a_i. Then at least one of the following two cases holds:

Case I. Some $n+1$ of the $f(a_0), \ldots, f(a_{mn})$ are different. WLOG, these are $f(a_0), f(a_1), \ldots, f(a_n)$. Choose the U_i so that the $f(\overline{U_0}), f(\overline{U_1}), \ldots, f(\overline{U_n})$ are all disjoint. But then $g(f(\overline{U_0})) \cap \cdots \cap g(f(\overline{U_n})) \supseteq g(f(\overline{U_0})) \cap \cdots \cap g(f(\overline{U_{mn}}))$ is not scattered, contradicting the $n+1$–tightness of g.

Case II. Some $m+1$ of the $f(a_0), \ldots, f(a_{mn})$ are the same. WLOG, $f(a_0) = f(a_1) = \cdots = f(a_m)$. Let $s = \{0, 1, \ldots, m\}$. Then $Q_s \neq \emptyset$, so $Q_s = \bigcap_{i \leq m} f(P_i)$ is not scattered, contradicting the $m+1$–tightness of f.

The "$mn+1$" in (3) cannot be reduced; for example, let $Y = Z \times n$ and $X = Y \times m$, with f, g the natural projection maps.

There is a similar result, with a similar proof, involving products:

Lemma 2.14 *Assume that for $i = 0, 1$: X_i, Y_i are compact, $f_i : X_i \to Y_i$ is $(m_i + 1)$–tight, $m_i \leq n_i < \omega$, and $|f_i^{-1}\{y\}| \leq n_i$ for* all $y \in Y_i$. *Then $f_0 \times f_1 : X_0 \times X_1 \to Y_0 \times Y_1$ is* $(\max(m_0 n_1, m_1 n_0) + 1)$*– tight.*

Proof. Let $L = \max(m_0 n_1, m_1 n_0)$, and let $f = f_0 \times f_1$. In view of Lemma 2.2, it is sufficient to fix any $L+1$ distinct points $a^0, a^1, \ldots, a^L \in X_0 \times X_1$ with all $f(a^\alpha) = b \in Y_0 \times Y_1$, and show that one can find neighborhoods U^α of a^α for $\alpha = 0, 1, \ldots, L$ such that $\bigcap_\alpha f(\overline{U^\alpha})$ is scattered.

Let $b = (b_0, b_1)$ and $a^\alpha = (a_0^\alpha, a_1^\alpha)$.

Note that although the a^α are all distinct points, the a_0^α need not be all different and the a_1^α need not be all different. However, $|\{a_0^\alpha : 0 \leq \alpha \leq L\}| \geq m_0 + 1$: If not, then using $f(a^\alpha) = b$ and $|f_1^{-1}\{b_1\}| \leq n_1$, we would have $L + 1 \leq m_0 n_1$, a contradiction. Likewise, $|\{a_1^\alpha : 0 \leq \alpha \leq L\}| \geq m_1 + 1$.

Now, using Lemma 2.2 and the fact that each $f_i : X_i \to Y_i$ is (m_i+1)–tight, choose neighborhoods U_i^α of a_i^α such that $\bigcap_\alpha f(\overline{U_i^\alpha})$ is scattered for $i = 0, 1$. The U_i^α can depend just on the value of a_i^α (that is $a_i^\alpha = a_i^\beta \to U_i^\alpha = U_i^\beta$). Finally, let $U^\alpha = U_0^\alpha \times U_1^\alpha$

The bound on the $|f_i^{-1}\{y\}|$ cannot be removed here. For example, for each cardinal κ, one can find compact perfect LOTSes X_0, X_1, Y_0, Y_1 with order-preserving $f_i : X_i \twoheadrightarrow Y_i$ such that all point inverses have size at least κ. Then the f_i are tight by Lemma 2.3, but $f_0 \times f_1$ is not κ–tight.

A variant of the product of maps is much simpler to analyze:

Lemma 2.15 *Assume that $\ell \in \omega$ and $f_i : X \to Y_i$ is κ–tight for each $i < \ell$, where X and the Y_i are compact. Then the map $x \mapsto (f_0(x), \ldots, f_{\ell-1}(x))$ from X to $\prod_{i<\ell} Y_i$ is also κ–tight.*

We now consider the opposite of tight maps:

Definition 2.16 *If X, Y are compact and $f : X \to Y$, then f is* nowhere tight *iff $f(X)$ is not scattered and there is no closed $P \subseteq X$ such that $f{\restriction}P$ is tight and $f(P)$ is not scattered.*

Note also that if X, Y are metric compacta with $f : X \twoheadrightarrow Y$ and Y not scattered, then there is a Cantor set $P \subseteq X$ such that $f{\restriction}P$ is 1-1, so

Lemma 2.17 *If X, Y are compact and $f : X \to Y$ is nowhere tight, then X is not second countable.*

A further limitation on nowhere tight maps:

Lemma 2.18 *If $f : X \to Y$ is nowhere tight, then f is not weakly $\mathfrak{c}$–tight.*

Proof. We shall get a non-scattered $Q \subseteq Y$ and disjoint non-scattered sets $P^k \subseteq X$ for $k \in 2^\omega$ so that each $f(P^k) = Q$. We shall build the P^k and Q by a tree argument. Each P^k will be non-scattered because it will be formed using a Cantor tree of closed sets, so we shall actually get a doubly indexed family. So, we build $Q_s \subseteq Y$ for $s \in 2^{<\omega}$ and $P^t_s \subseteq X$ for $s, t \in 2^{<\omega}$ with $\mathrm{lh}(s) = \mathrm{lh}(t)$ satisfying:

1. P^t_s is closed, $f(P^t_s) = Q_s$, and Q_s is not scattered.
2. The sets $Q_{s\frown 0}, Q_{s\frown 1}$ are disjoint subsets of Q_s.
3. The sets $P^{t\frown 0}_{s\frown 0}$, $P^{t\frown 1}_{s\frown 0}$, $P^{t\frown 0}_{s\frown 1}$, $P^{t\frown 1}_{s\frown 1}$ are disjoint subsets of P^t_s.

We construct these inductively. $P^{\mathbb{1}}_{\mathbb{1}}$ and $Q_{\mathbb{1}}$ exist (where $\mathbb{1}$ is the empty sequence) because $f(X)$ is not scattered. Now, say we have Q_s and P^t_s for all s, t with $\mathrm{lh}(s) = \mathrm{lh}(t) = n$. Fix s.

First, get disjoint closed non-scattered $\widetilde{Q}_{s\frown 0}, \widetilde{Q}_{s\frown 1} \subseteq Q_s$, and let $\widetilde{P}^t_{s\frown\mu} = P^t_s \cap f^{-1}(\widetilde{Q}_{s\frown\mu})$ for each t of length n and each $\mu = 0, 1$. Then, use "nowhere tight" 2^n times to get $Q_{s\frown\mu} \subseteq \widetilde{Q}_{s\frown\mu}$ and $P^{t\frown\nu}_{s\frown\mu} \subseteq \widetilde{P}^t_{s\frown\mu}$ for each $\mu, \nu = 0, 1$ and each t of length n so that each $f(P^{t\frown\nu}_{s\frown\mu}) = Q_{s\frown\mu}$ and each $Q_{s\frown\mu}$ is non-scattered.

For $h, k \in 2^\omega$, define $Q_h = \bigcap_{n\in\omega} Q_{h{\restriction}n}$ and $P^k_h = \bigcap_{n\in\omega} P^{k{\restriction}n}_{h{\restriction}n}$, let $Q = \bigcup\{Q_h : h \in 2^\omega\}$, and let $P_h = \bigcup\{P^k_h : k \in 2^\omega\}$ and $P^k = \bigcup\{P^k_h : h \in 2^\omega\}$. Then $f(P_h) = Q_h$ and $f(P^k) = Q$, and the φ of Definition 2.6 sends P^k to $k \in 2^\omega$, with $\mathrm{dom}(\varphi) = \bigcup_k P_k$.

Corollary 2.19 *If X, Y are compact, $f : X \twoheadrightarrow Y$, $w(X) < \mathfrak{c}$, Y is metric and not scattered, and f is weakly $\mathfrak{c}$–tight, then X has a Cantor subset.*

Proof. Since f is not nowhere tight, we may assume, shrinking X and Y, that f is tight. Let $\kappa = w(X)$, and let $\mathcal{B}$ be a base for X with $|\mathcal{B}| = \kappa$. Whenever $B_0, B_1 \in \mathcal{B}$ with $\overline{B_0} \cap \overline{B_1} = \emptyset$, let $S(B_0, B_1) = f(\overline{B_0}) \cap f(\overline{B_1})$. Each $S(B_0, B_1)$ is scattered, and hence countable, so at most κ points of Y are in some $S(B_0, B_1)$, so there is a $K \subseteq Y$ homeomorphic to the Cantor set with K is disjoint from all $S(B_0, B_1)$. $|f^{-1}\{y\}| = 1$ for all $y \in K$, so $f^{-1}(K)$ is homeomorphic to K. ☕

Note that we have not yet given any examples of nowhere tight maps. The argument of Corollary 2.19 shows that one class of examples is given by:

Example 2.20 *If X, Y are compact, $f : X \twoheadrightarrow Y$, $w(X) < \mathfrak{c}$, Y is metric and not scattered, and X has no Cantor subset, then f is nowhere tight.*

Of course, under CH, this class of examples is empty. More generally, the class is empty under MA (or just the assumption that $\mathbb{R}$ is not the union of $< \mathfrak{c}$ meager sets), since then every non-scattered compactum of weight less than $\mathfrak{c}$ contains a Cantor subset (see [12]). However, by Dow and Fremlin [5], it is consistent to have a non-scattered compactum X of weight $\aleph_1 < \mathfrak{c}$ with no convergent ω–sequences, and hence with no Cantor subsets; in the ground model, CH holds, and X is any compact F-space (so $w(X)$ can be $\aleph_1$); then, the extension adds any number of random reals.

A class of ZFC examples of nowhere tight maps with $w(X) = \mathfrak{c}$ is given by:

Example 2.21 *If X, Y are compact, $f : X \twoheadrightarrow Y$, X is a compact F–space and Y is metric and not scattered, then f is nowhere tight.*

Proof. Here, it is sufficient to prove that f is not tight, since any $f \restriction P : P \twoheadrightarrow f(P)$ will have the same properties. Also, shrinking Y, we may assume that Y has no isolated points.

First, choose a perfect $Q \subseteq Y$ which is nowhere dense in Y. Then, choose a discrete set $D = \{d_n : n \in \omega\} \subseteq Y \backslash Q$ with $\overline{D} = D \cup Q$ and each $f^{-1}\{d_n\}$ not a singleton. Then, choose $x_n, z_n \in f^{-1}\{d_n\}$ with $x_n \neq z_n$. Now, since X is an F–space, $\mathrm{cl}\{x_n : n \in \omega\}$ and $\mathrm{cl}\{z_n : n \in \omega\}$ are two disjoint copies of $\beta\mathbb{N}$ in X which map onto $\overline{D}$. ☕

3 Dissipated Spaces

Only a scattered compactum X has the property that *all* maps from X are tight: If X is not scattered, then X maps onto $[0,1]^2$; if we follow that map by the usual projection onto $[0,1]$, we get a map from X onto $[0,1]$ which is not even weakly $\mathfrak{c}$–tight.

The *dissipated* compacta have the property that *unboundedly* many maps onto metric compacta are tight:

Definition 3.1 *Assume that X, Y, Z are compact, $f : X \to Y$, and $g : X \to Z$. Then $f \le g$, or f is* finer than *g, iff there is a $\Gamma \in C(f(X), g(X))$ such that $g = \Gamma \circ f$.*

Lemma 3.2 *Assume that X, Y, Z are compact, $f : X \to Y$, and $g : X \to Z$. Then $f \le g$ iff $\forall x_1, x_2 \in X\, [f(x_1) = f(x_2) \to g(x_1) = g(x_2)]$.*

Proof. For $\leftarrow$, let $\Gamma = \{(f(x), g(x)) : x \in X\} \subseteq f(X) \times g(X)$. ☕

Definition 3.3 *X is κ–*dissipated *iff X is compact and whenever $g : X \to Z$, with Z metric, there is a finer κ–tight $f : X \to Y$ for some metric Y. X is* dissipated *iff X is 2–dissipated. X is* weakly $\mathfrak{c}$–dissipated *iff X is compact and whenever $g : X \to Z$, with Z metric, there is a finer weakly $\mathfrak{c}$–tight $f : X \to Y$ for some metric Y.*

So, the 1–dissipated compacta are the scattered compacta. Metric compacta are trivially dissipated because we can take $Y = X$, with f the identity map. Besides the spaces from [7, 11, 12], an easy example of a dissipated space is given by:

Lemma 3.4 *If X is a compact LOTS, then X is dissipated*

Proof. Fix g, Z as in Definition 3.3. On X, use $[x_1, x_2]$ for the closed interval $[\min(x_1, x_2), \max(x_1, x_2)]$, and define $x_1 \sim x_2$ iff g is constant on $[x_1, x_2]$. Then $\sim$ is a closed equivalence relation, so define $Y = X/{\sim}$ with $f : X \twoheadrightarrow Y$ the natural projection. Then Y is a LOTS and f is order-preserving, so f is tight by Lemma 2.3, and $f \le g$ by Lemma 3.2. To see that Y is metrizable, fix a metric on Z, and then, on Y, define $d(f(x_1), f(x_2)) = \mathrm{diam}(g([x_1, x_2]))$. ☕

By Corollary 2.19, if $w(X) < \mathfrak{c}$ and X is $\mathfrak{c}$–dissipated and not scattered, then X has a Cantor subset, while the double arrow space is an example of an X with $w(X) = \mathfrak{c}$ which is 2–dissipated and has no Cantor subset.

Note that just having *one* tight map g from X onto some metric compactum Z is not sufficient to prove that X is dissipated, since the tightness of g says nothing at all about the complexity of a particular $g^{-1}\{z\}$. Trivial counter-examples are obtained with $|Z| = 1$ and g a constant map. However, if all $g^{-1}\{z\}$ are scattered, then just one tight g is enough:

Lemma 3.5 *Suppose that $g : X \to Z$ is κ–tight and all $g^{-1}\{z\}$ are scattered. Fix $f : X \to Y$ with $f \le g$. Then f is κ–tight. In particular, if Z is also metric, then X is κ–dissipated.*

Proof. Fix $\Gamma \in C(f(X), g(X))$ such that $g = \Gamma \circ f$. Suppose that $\mathcal{P}$ were a loose family for f of size κ; then we have $Q \subseteq f(X)$ with $Q = f(P)$ for all $P \in \mathcal{P}$, and Q is not scattered. But $\Gamma(Q)$ is scattered, since g is κ–tight and $g(P) = \Gamma(f(P)) = \Gamma(Q)$ for all $P \in \mathcal{P}$. It follows that we can fix $z \in Z$ with $Q \cap \Gamma^{-1}\{z\}$ not scattered. But then $f(g^{-1}\{z\}) = \Gamma^{-1}\{z\}$ is not scattered, which is impossible, since $g^{-1}\{z\}$ is scattered.

We next consider the degree of dissipation of products:

Lemma 3.6 *Let $X = A \times B$, where A, B are compact, B is not scattered, and assume that for each $\varphi \in C(A, [0,1]^\omega)$ there is a $z \in [0,1]^\omega$ with $|\varphi^{-1}\{z\}| \geq \kappa$. Then X is not κ–dissipated. If for each $\varphi \in C(A, [0,1]^\omega)$ there is a z such that $\varphi^{-1}\{z\}$ is not scattered, then X is not weakly $\mathfrak{c}$–dissipated.*

Proof. Since B is not scattered, fix $h : B \twoheadrightarrow [0,1]$, and define $g : X \twoheadrightarrow [0,1]$ by $g(a,b) = h(b)$. Now, fix any $f : X \to Y$ with f finer than g and Y metric. We shall show that f is not κ–tight.

Define $\widehat{f} : A \to C(B, Y)$ by $(\widehat{f}(a))(b) = f(a,b)$. Since the range of $\widehat{f}$ is compact and hence embeddable in the Hilbert cube, we can fix $\zeta \in C(B, Y)$ such that $E := \{a : \widehat{f}(a) = \zeta\}$ has size at least κ. Let $Q = \zeta(B)$; $|Q| = \mathfrak{c}$ by $f \leq g$, so Q is not scattered. For $a \in E$, let $P_a = \{a\} \times B$. Then $\{P_a : a \in E\}$ is a loose family of size at least κ.

The second assertion is proved similarly.

Note that A might be scattered; for example, A could be the ordinal $\kappa + 1$ (if κ is uncountable and regular) or the one point compactification of a discrete space of size κ (if κ is uncountable). B may be second countable; for example B can be the Cantor set.

A class of spaces A to which Lemma 3.6 applies is produced by:

Lemma 3.7 *Suppose that $f : \prod_{\alpha<\kappa} X_\alpha \to M$, where M is compact metric and, for each α, X_α is compact and not metrizable. Then there are two-element sets $E_\alpha \subseteq X_\alpha$ for each α such that f is constant on $\prod_{\alpha<\kappa} E_\alpha$.*

Proof. For $p \in \prod_{\alpha<\delta} X_\alpha$, define $\widehat{f_p} : \prod_{\alpha\geq\delta} X_\alpha \to M$ by: $\widehat{f_p}(q) = f(p^\frown q)$. Then inductively choose E_α so that for all $\delta \leq \kappa$, the functions $\widehat{f_p}$ are the same for all $p \in \prod_{\alpha<\delta} E_\alpha$. Say $\delta < \kappa$ and we have chosen E_α for $\alpha < \delta$. Let $g = \widehat{f_p}$ for some (any) $p \in \prod_{\alpha<\delta} E_\alpha$, and define $g^* \in C(X_\delta, C(\prod_{\alpha>\delta} X_\alpha, M))$ by: $(g^*(x))(q) = g(x^\frown q)$. Then g^* maps X_δ into a metric space of functions, so $\mathrm{ran}(g^*)$ is a compact metric space, so g^* cannot be 1-1, so choose E_δ of size 2 with g^* constant on E_δ.

Theorem 3.8 *Assume that each X_k is compact:*

1. *If X_n is not scattered and X_k, for $k < n$, is not metrizable, then $\prod_{k \le n} X_k$ is not 2^n–dissipated.*
2. *If each X_k is not metrizable, then $\prod_{k<\omega} X_k$ is not weakly $\mathfrak{c}$–dissipated.*

Proof. For (1), apply Lemma 3.6 with $A = \prod_{k<n} X_k$ and $B = X_n$. For (2), apply Lemma 3.6 with $A = \prod_{k<\omega} X_{2k}$ and $B = \prod_{k<\omega} X_{2k+1}$.

In (1), if all X_k are scattered, then $\prod_{k \le n} X_k$ is scattered and hence dissipated. As an example of (1) applied to LOTSes, if $S \subseteq (0,1)$ is uncountable, then $(I_S)^2$ is not dissipated (2–dissipated), $(I_S)^3$ is not 4–dissipated, and $(I_S)^4$ is not 8–dissipated. By Theorem 3.9, these three spaces are, respectively, 3–dissipated, 5–dissipated, and 9–dissipated. However, Lemma 3.6 shows that for any κ, we can find a product of two LOTSes which is not κ–dissipated.

The following theorem will often suffice to compute the degree of dissipation of a finite product of separable LOTSes:

Theorem 3.9 *Assume that n is finite and X_i, for $i \le n$, is a compact separable LOTS. Then $\prod_{i \le n} X_i$ is $(2^n + 1)$–dissipated. Furthermore, if all the X_i are not scattered, and at most one of the X_i is second countable, then $\prod_{i \le n} X_i$ is not (2^n)–dissipated.*

Proof. Let $D_i \subseteq X_i$ be countable and dense. Choose $f_i \in C(X_i, [0,1])$ such that f_i is order-preserving and is 1-1 on D_i (such a function f_i exists; see the proof of Lemma 3.6 in [10]). Note that each $|f_i^{-1}\{y\}| \le 2$, and, by Lemma 2.3, each f_i is 2–tight. Applying Lemma 2.14 and induction, $\prod_{i \le n} f_i$ is $(2^n + 1)$–tight. Then $\prod_{i \le n} X_i$ is $(2^n + 1)$–dissipated by Lemma 3.5.

The "furthermore" is by Theorem 3.8.

Next, we note that "dissipated" is a local property:

Definition 3.10 *Let $\mathfrak{K}$ be a class of compact spaces. $\mathfrak{K}$ is* closed-hereditary *iff every closed subspace of a space in $\mathfrak{K}$ is also in $\mathfrak{K}$. $\mathfrak{K}$ is* local *iff $\mathfrak{K}$ is closed-hereditary* and *for every compact X: if X is covered by open sets whose closures lie in $\mathfrak{K}$, then $X \in \mathfrak{K}$.*

Classes of compacta which restrict cardinal functions (first countable, second countable, countable tightness, etc.) are clearly local, whereas the class of compacta which are homeomorphic to a LOTS is closed-hereditary, but not local. To prove that "dissipated" is local, we use as a preliminary lemma:

Lemma 3.11 *Let X be an arbitrary compact space, with $K \subseteq U \subseteq X$, such that U is open, K is closed, and $\overline{U}$ is κ–dissipated. Fix $g : \overline{U} \to Z$, with Z compact metric. Then there is an $f : X \to Y$, with Y compact metric, f κ–tight, and $f \restriction K \le g \restriction K$.*

Proof. Fix $\varphi : X \to [0,1]$ with $\varphi(K) = \{0\}$ and $\varphi(\partial U) = \{1\}$. First get $f_0 : \overline{U} \to Y_0$, with Y_0 compact metric, f_0 κ–tight, $f_0 \le g$, and $f_0 \le \varphi\restriction\overline{U}$ (just let f_0 refine $x \mapsto (g(x), \varphi(x))$). Then $f_0(K) \cap f_0(\partial U) = \emptyset$. Let $Y = Y_0/f_0(\partial U)$, obtained by collapsing $f_0(\partial U)$ to a point, p. Let $f_1 : \overline{U} \to Y$ be the natural map, and extend f_1 to $f : X \to Y$ by letting $f_1(X\backslash U) = \{p\}$. ☕

Lemma 3.12 *For any κ, the class of κ–dissipated compacta is a local class.*

Proof. For closed-hereditary: Assume that X is κ–dissipated and K is closed in X. Fix $g : K \to Z$, with Z metric. Then we may assume that $Z \subseteq I^\omega$, so that g extends to some $\widetilde{g} : X \to I^\omega$. Then there is a κ–tight $\widetilde{f} : X \to Y$ for some metric Y, with $\widetilde{f} \le \widetilde{g}$. If $f = \widetilde{f}\restriction K$, then f is κ–tight and $f \le g$.

For local: Assume that $X = \bigcup_{i<\ell} U_i$, where each U_i is open and $\overline{U_i}$ is κ–dissipated. Fix $g : X \to Z$, with Z metric. Choose closed $K_i \subseteq U_i$ such that $X = \bigcup_{i<\ell} K_i$. Then apply Lemma 3.11 and choose $f_i : X \to Y_i$, with Y_i compact metric, f_i κ–tight, and $f_i\restriction K_i \le g\restriction K_i$. Then the map $x \mapsto (f_0(x), \ldots, f_{\ell-1}(x))$ refines g, and is κ–tight by Lemma 2.15. ☕

Many classes of compacta are closed under continuous images, but this is not true in general of the class of κ–dissipated spaces:

Example 3.13 *There is a continuous image of a 3–dissipated space which is not $\mathfrak{c}$–dissipated.*

Proof. Let $T = (D(\mathfrak{c}) \cup \{\infty\}) \times 2^\omega$, where $D(\mathfrak{c}) \cup \{\infty\}$ is the 1-point compactification of the ordinal $\mathfrak{c}$ with the discrete topology. Then T is not $\mathfrak{c}$–dissipated by Lemma 3.6. Let F_α, for $\alpha < \mathfrak{c}$, be disjoint Cantor subsets of 2^ω such that for some $g : 2^\omega \twoheadrightarrow 2^\omega$, each $g(F_\alpha) = 2^\omega$. Let $X = \{\infty\} \times 2^\omega \ \cup \ \bigcup_{\alpha<\mathfrak{c}}(\{\alpha\} \times F_\alpha) \ \subseteq \ T$. Then X is 3–dissipated by Lemma 3.5 because the natural projection onto 2^ω is 3–tight and all point inverses are scattered (of size ≤ 2). But also, T is a continuous image of X via the map $\mathbb{1} \times g$, $(u, z) \mapsto (u, g(z))$. ☕

Of course, the continuous image of a 1–dissipated (= scattered) compactum is 1–dissipated. We do not know about the dissipated (= 2–dissipated) spaces; perhaps 2 is a special case.

4 LOTS Dimension

We shall apply the results of Section 3 to products of LOTSes. Each I^n has dimension n under any standard notion of topological dimension, so that I^{n+1} is not embeddable into I^n. Now, say we wish to prove such a result replacing I by some totally disconnected LOTS X. Then standard dimension theory gives all X^n dimension 0. Furthermore, the result is false; for example, $X^{n+1} \cong X^n$ if X is the Cantor set. However, if X is the double arrow space, then X^{n+1} is not embeddable into X^n. To study this further, we introduce a notion of LOTS dimension:

Definition 4.1 *If X is any Tychonov space, then* $\mathrm{Ldim}_0(X)$ *is the least κ such that X is embeddable into a product of the form $\prod_{\alpha<\kappa} L_\alpha$, where each L_α is a LOTS. Then* $\mathrm{Ldim}(X)$*, the* LOTS dimension *of X, is the least κ such that every point in X has a neighborhood U such that* $\mathrm{Ldim}_0(\overline{U}) \le \kappa$.

Lemma 4.2 *The class of compacta X such that* $\mathrm{Ldim}(X) \le \kappa$ *is a local class.*

If X is any compact n–manifold, then $\mathrm{Ldim}(X) = n < \mathrm{Ldim}_0(X)$. We follow the usual convention that the empty product $\prod_{\alpha<0} L_\alpha$ is a singleton, so that $\mathrm{Ldim}(X) = 0$ iff X is finite, although $\mathrm{Ldim}_0(X) = 1$ if $1 < |X| < \aleph_0$.

Lemma 4.3 *If X is compact, infinite, and totally disconnected, then $Ldim(X) = Ldim_0(X)$.*

Proof. Use the fact that a disjoint sum of LOTSes is a LOTS.

By Tychonov, $\mathrm{Ldim}(X) \le w(X)$, taking each $L_\alpha = I$. In this section, we focus mainly on spaces whose LOTS dimension is finite, although this cardinal function might be of interest for other spaces. For example, $\mathrm{Ldim}(\beta\mathbb{N}) = 2^{\aleph_0}$; this is easily proved using the theorem of Pospíšil that there are points in $\beta\mathbb{N}$ of character $2^{\aleph_0}$. We shall show (Lemma 4.5) that $\mathrm{Ldim}((I_S)^n) = n$ whenever S is uncountable. When S is countable, this is false if S is dense in I (then $(I_S)^n \cong I_S$ is the Cantor set) and true if S is not dense in I (by standard dimension theory; not by the results of this paper). More generally, we shall prove:

Theorem 4.4 *Let Z_j, for $1 \le j \le s$, be a compact LOTS. Assume that $s = r + m$, where $r, m \ge 0$. For $r + 1 \le j \le s$, assume that Z_j has either has an increasing or decreasing ω_1–sequence. For $1 \le j \le r$, assume that there is a countable $D_j \subseteq Z_j$ such that $\overline{D_j}$ is not scattered, and assume that at most one of the $\overline{D_j}$ is second countable. Then* $\mathrm{Ldim}(\prod_{j=1}^{s} Z_j) = s$.

The following lemma handles the case $r = s, m = 0$ if we replace each Z_j by $L_j = \overline{D_j}$.

Lemma 4.5 *Assume that n is finite and L_j, for $j < n$, is a compact separable LOTS. Also, assume that all the L_j are not scattered, and that at most one of the L_j is second countable. Then* $\mathrm{Ldim}(\prod_{j<n} L_j) = n$.

Proof. This is trivial if $n \le 1$, so assume that $n \ge 2$. Clearly, $\mathrm{Ldim}(\prod_{j<n} L_j) \le \mathrm{Ldim}_0(\prod_{j<n} L_j) \le n$. Also, by Theorem 3.9, $\prod_{j<n} L_j$ is not 2^{n-1}–dissipated.

To see that $\mathrm{Ldim}_0(\prod_{j<n} L_j) \ge n$, assume that we could embed $\prod_{j<n} L_j$ into $\prod_{i<(n-1)} X_i$, where each X_i is a LOTS. Since the continuous image of a compact separable space is compact and separable, we may assume that each X_i is compact and separable, so that $\prod_{i<(n-1)} X_i$ and $\prod_{j<n} L_j$, are $(2^{n-2}+1)$–dissipated by Theorem 3.9, a contradiction since $2^{n-2}+1 \le 2^{n-1}$.

Now, assume that $\mathrm{Ldim}(\prod_{j<n} L_j) < n$. Then we could cover $\prod_{j<n} L_j$ by finitely many open boxes, each of the form $\prod_{j<n} U_j$, with each U_j an open interval in L_j, such that each open box satisfies $\mathrm{Ldim}_0(\prod_{j<n} \overline{U_j}) < n$. But for at least one of these open boxes, the $\overline{U_j}$ would satisfy all the same hypotheses satisfied by the L_j, so that we would again have a contradiction. ☕

In particular, if L is the double arrow space, then L^{n+1} is not embeddable into L^n. Similar results were obtained by Burke and Lutzer [2] and Burke and Moore [3] for the Sorgenfrey line J, which may be viewed as $\{z^+ : z \in (0,1)\} \subseteq L$. We do not see how to derive our results directly from [2, 3], since a map $\varphi : L^{n+1} \to L^n$ need not preserve order, so it does not directly yield a map from J^{n+1} to J^n.

We now extend Lemma 4.5 to include LOTSes which have an increasing or decreasing ω_1–sequence. First some preliminaries:

Definition 4.6 $[A]^{n\uparrow} = \{(\alpha_1, \ldots, \alpha_n) \in A^n : \alpha_1 < \cdots < \alpha_n\}$, *where* $1 \le n < \omega$ *and* $A \subseteq \omega_1$. *We give* $[A]^{n\uparrow}$ *the topology it inherits from* $(\omega_1)^n$. *The* club filter $\mathcal{F}_n$ *on* $[\omega_1]^{n\uparrow}$ *is generated by all the* $[C]^{n\uparrow}$ *such that* C *is club in* ω_1. $\mathcal{I}_n$ *is the dual ideal to* $\mathcal{F}_n$.

Lemma 4.7 *If* $B \subseteq [\omega_1]^{n\uparrow}$ *is a Borel set, then* $B \in \mathcal{F}_n$ *or* $B \in \mathcal{I}_n$.

Proof. Since the $\mathcal{I}_n$ and $\mathcal{F}_n$ are countably complete, it is sufficient to prove this for closed sets K. The case $n = 1$ is obvious, so we proceed by induction. We assume the lemma for n, fix a closed $K \subseteq [\omega_1]^{(n+1)\uparrow}$, and show that $K \in \mathcal{F}_{n+1}$ or $K \in \mathcal{I}_{n+1}$. Applying the lemma for n: For each $\alpha_0 < \omega_1$, choose $\nu(\alpha_0) \in \{0,1\}$ and a club $C_{\alpha_0} \subseteq (\alpha_0, \omega_1)$ such that for all $(\alpha_1, \ldots, \alpha_n) \in [C_{\alpha_0}]^{n\uparrow}$:

$$\nu(\alpha_0) = 0 \;\to\; (\alpha_0, \alpha_1, \ldots, \alpha_n) \notin K \quad ; \quad \nu(\alpha_0) = 1 \;\to\; (\alpha_0, \alpha_1, \ldots, \alpha_n) \in K \qquad (*)$$

Let $C = \{\delta : \delta \in \bigcap\{C_{\alpha_0} : \alpha_0 < \delta\}\}$. Then C is club and $(*)$ holds for all $(\alpha_0, \alpha_1, \ldots, \alpha_n) \in [C]^{(n+1)\uparrow}$. Also, $D := \{\alpha_0 \in C : \nu(\alpha_0) = 1\}$ is closed because K is closed. $[D]^{(n+1)\uparrow} \subseteq K$, so if D is club, then $K \in \mathcal{F}_{n+1}$. If D is bounded, then $C \backslash D$ contains a club, and then $K \in \mathcal{I}_{n+1}$. ☕

Definition 4.8 *If L is a LOTS, $f \in C([\omega_1]^{m\uparrow}, L)$, and $\psi \in C([\omega_1]^{n\uparrow}, L)$, then ψ is* derived from f *iff $n \geq m$ and for some $i_1, \dots i_m$: $1 \leq i_1 < \cdots < i_m \leq n$ and $\psi(\alpha_1, \dots, \alpha_n) = f(\alpha_{i_1}, \dots, \alpha_{i_m})$ for all $(\alpha_1, \dots, \alpha_n) \in [\omega_1]^{n\uparrow}$. Then a set $E \subseteq [\omega_1]^{n\uparrow}$ is* derived from f *iff E is of the form $\{\vec{\alpha} : \psi_1(\vec{\alpha}) < \psi_2(\vec{\alpha})\}$ or $\{\vec{\alpha} : \psi_1(\vec{\alpha}) \leq \psi_2(\vec{\alpha})\}$ or $\{\vec{\alpha} : \psi_1(\vec{\alpha}) = \psi_2(\vec{\alpha})\}$, where ψ_1, ψ_2 are derived from f.*

Lemma 4.9 *Suppose that $f \in C([\omega_1]^{m\uparrow}, L)$, where L is a compact LOTS. Then there is a club C, a continuous $g : C \to L$, and a $j \in \{1, 2, \dots m\}$, such that for all $\vec{\alpha} = (\alpha_1, \dots, \alpha_m) \in [C]^{m\uparrow}$, we have $f(\vec{\alpha}) = g(\alpha_j)$, and g is either strictly increasing or strictly decreasing or constant.*

Proof. Applying Lemma 4.7, and then restricting everything to a club, we may make the following *homogeneity* assumption: for all $n \geq m$ and all $E \subseteq [\omega_1]^{n\uparrow}$ which are derived from f, either $E = \emptyset$ or $E = [\omega_1]^{n\uparrow}$. Then, our club C will be all of ω_1. We first consider the special cases $m = 1$ and $m = 2$.

For $m = 1$, we have $f \in C(\omega_1, L)$. Applying homogeneity to the three derived sets $\{(\alpha, \beta) \in [\omega_1]^{2\uparrow} : f(\alpha) \circledast f(\beta)\}$, where $\circledast$ is one of $<$, $>$, and $=$, we see that f is either strictly increasing or strictly decreasing or constant.

Likewise, for $m > 1$, if we succeed in getting $f(\vec{\alpha}) = g(\alpha_j)$, then g must be either strictly increasing or strictly decreasing or constant.

Next, fix $f \in C([\omega_1]^{2\uparrow}, L)$. If $\alpha < \beta < \gamma \to f(\alpha, \beta) = f(\alpha, \gamma)$, then $f(\alpha, \beta) = g(\alpha)$, and we are done, so WLOG, assume $\alpha < \beta < \gamma \to f(\alpha, \beta) < f(\alpha, \gamma)$. Let $B_\alpha = \{f(\alpha, \beta) : \alpha < \beta < \omega_1\}$, which is a subset of L of order type ω_1. Let $h(\alpha) = \sup(B_\alpha)$. Fix $\alpha < \alpha' < \omega_1$. There are now three cases; Cases II and III will lead to contradictions:

Case I. $h(\alpha) = h(\alpha')$: By continuity of f, there is a club $C \subseteq (\alpha', \omega_1)$ such that $f(\alpha, \beta) = f(\alpha', \beta)$ for all $\beta \in C$. Applying homogeneity, we have $\alpha < \alpha' < \beta \to f(\alpha, \beta) = f(\alpha', \beta)$, so $f(\alpha, \beta) = g(\beta)$.

Case II. $h(\alpha) < h(\alpha')$: Fix β such that $\alpha < \alpha' < \beta$ and $f(\alpha', \beta) > f(\alpha, \gamma)$ for all γ. Then by homogeneity, $\alpha < \alpha' < \beta < \gamma \to f(\alpha, \gamma) < f(\alpha', \beta)$ for *all* $\alpha, \alpha', \beta, \gamma$. Let α' be a limit and consider $\alpha \nearrow \alpha'$: we get, by continuity, $\alpha' < \beta < \gamma \to f(\alpha', \gamma) \leq f(\alpha', \beta)$, contradicting $\alpha < \beta < \gamma \to f(\alpha, \beta) < f(\alpha, \gamma)$.

Case III. $h(\alpha) > h(\alpha')$: Fix β such that $\alpha < \alpha' < \beta$ and $f(\alpha, \beta) > f(\alpha', \gamma)$ for all γ. Then by homogeneity, $\alpha < \alpha' < \beta < \gamma \to f(\alpha', \gamma) < f(\alpha, \beta)$ for *all* $\alpha, \alpha', \beta, \gamma$. Letting $\alpha \nearrow \alpha'$, we get a contradiction as in Case II.

Finally, fix $m \geq 2$ and assume that the lemma holds for m. We shall prove it for $m + 1$, so fix $f \in C([\omega_1]^{(m+1)\uparrow}, L)$. Temporarily fix $(\alpha_1, \dots, \alpha_{m-1}) \in [\omega_1]^{(m-1)\uparrow}$, and let $\widetilde{f}(\alpha_m, \alpha_{m+1}) = f(\alpha_1, \dots, \alpha_{m-1}, \alpha_m, \alpha_{m+1})$; so $\widetilde{f} \in C([\,(\alpha_{m-1}, \omega_1)\,]^{2\uparrow}, L)$. Applying the $m = 2$ case, $\widetilde{f}$ is really just a function of one of its arguments, so that f just depends on an m–tuple (either $(\alpha_1, \dots, \alpha_{m-1}, \alpha_{m+1})$ or $(\alpha_1, \dots, \alpha_{m-1}, \alpha_m)$), so we may now apply the lemma for m. ☕

It is easy to see from this lemma that $\mathrm{Ldim}((\omega_1+1)^m) = m$, but we now want to consider products of $(\omega_1+1)^m$ with separable LOTSes.

Lemma 4.10 *Suppose that $f \in C(X \times [\omega_1]^{m\uparrow}, L)$, where L is a compact LOTS and X is compact, nonempty, first countable, and separable. Then there is a club $C \subseteq \omega_1$, a nonempty open $U \subseteq X$, a $g \in C(\overline{U} \times C, L)$, and a $j \in \{1, 2, \ldots m\}$ such that $f(x, \vec{\alpha}) = g(x, \alpha_j)$ for all $\vec{\alpha} = (\alpha_1, \ldots, \alpha_m) \in [C]^{m\uparrow}$ and all $x \in \overline{U}$, and such that either*

1. *for all $x \in \overline{U}$, the map $\vec{\alpha} \mapsto f(x, \vec{\alpha})$ is constant on $[C]^{m\uparrow}$ or*
2. *for all $x \in \overline{U}$, the map $\xi \mapsto g(x, \xi)$ is strictly increasing on C, or*
3. *for all $x \in \overline{U}$, the map $\xi \mapsto g(x, \xi)$ is strictly decreasing on C.*

Proof. First, let K be the set of all x such that $\vec{\alpha} \mapsto f(x, \vec{\alpha})$ is constant on some set in $\mathcal{F}_m$. Then K is closed, since X is first countable, so, replacing X by some $\overline{U}$, we may assume that $K = X$ or $K = \emptyset$. If $K = X$, then intersecting the clubs for x in a countable dense set, we get one club C such that (1) holds.

Now, assume that $K = \emptyset$. Applying Lemma 4.9, for each $x \in X$ choose a club C_x, a $g_x \in C(C_x, L)$, and $j_x \in \{1, 2, \ldots m\}$ and a $\mu_x \in \{-1, 1\}$ such that for all $\vec{\alpha} = (\alpha_1, \ldots, \alpha_m) \in [C_x]^{m\uparrow}$, we have $f(x, \vec{\alpha}) = g_x(\alpha_{j_x})$, and each g_x is either strictly increasing (when $\mu_x = 1$) or strictly decreasing (when $\mu_x = -1$).

For each j, μ, let $H_j^\mu = \{x : j_x = j \ \&\ \mu_x = \mu\}$. Then the H_j^μ are disjoint, and they are also closed (since $K = \emptyset$). Since $\bigcup_{j,\mu} H_j^\mu = X$, U can be any nonempty H_j^μ.

In situations (2) or (3), we shall apply:

Lemma 4.11 *Suppose that $g \in C(X \times (\omega_1 + 1), L)$, where L is a compact LOTS and X is compact, and suppose that $g(x, \xi) < g(x, \eta)$ for each $x \in X$ and each $\xi < \eta < \omega_1$. Let $h(x) = g(x, \omega_1)$. Then $h(X)$ is finite.*

Proof. Assume that $h(X)$ is infinite. Then, choose $c_n \in X$ for $n \in \omega$ such that the sequence $\langle h(c_n) : n \in \omega \rangle$ is either increasing strictly or decreasing strictly. Let $c \in X$ be any limit point of $\langle c_n : n \in \omega \rangle$, and note that $h(c_n) \to h(c)$. Also note that $h(x) = \sup\{g(x, \xi) : \xi < \omega_1\}$ for every x. Consider the two cases:

Case I. $\langle h(c_n) : n \in \omega \rangle$ is increasing strictly. Then we can fix a large enough countable γ such that $g(c_n, \omega_1) < g(c_{n+1}, \gamma)$ for all n. Then we have the ω–sequence, $g(c_0, \gamma) < g(c_0, \omega_1) < g(c_1, \gamma) < g(c_1, \omega_1) < g(c_2, \gamma) < g(c_2, \omega_1) < \cdots$, whose limit must be $g(c, \gamma) = g(c, \omega_1)$, contradicting $g(c, \gamma) < g(c, \omega_1)$,

Case II. $\langle h(c_n) : n \in \omega \rangle$ is decreasing strictly. Then we can fix a large enough countable γ such that $g(c_n, \gamma) > g(c_{n+1}, \omega_1)$ for all n. Then we have the ω–sequence,

$g(c_0, \omega_1) > g(c_0, \gamma) > g(c_1, \omega_1) > g(c_1, \gamma) > g(c_2, \omega_1) > g(c_2, \gamma) > \cdots$, whose limit must be $g(c, \omega_1) = g(c, \gamma)$, contradicting $g(c, \omega_1) > g(c, \gamma)$, ☕

Now if $h(X)$ is finite, we can always shrink X to a $\overline{U}$ on which h is constant. Then note that if $h(b) = h(c)$ and $\xi \mapsto g(x, \xi)$ is always an increasing function, then there is a club on which $g(b, \xi) = g(c, \xi)$. Putting these last two lemmas together, we get:

Lemma 4.12 *Suppose that $f \in C(X \times (\omega_1 + 1)^m, L)$, where L is a compact LOTS and X is compact, nonempty, first countable, and separable. Then there is a club $C \subseteq \omega_1$ and a nonempty open $U \subseteq X$ such that either:*

1. *For some $j \in \{1, 2, \dots m\}$ and some continuous $g : C \to L$: $f(x, \vec{\alpha}) = g(\alpha_j)$ for all $x \in \overline{U}$ and all $\vec{\alpha} \in [C]^{m\uparrow}$ and g is either strictly increasing or strictly decreasing, or*
2. *For some $h \in C(\overline{U}, L)$: $f(x, \vec{\alpha}) = h(x)$ for all $x \in \overline{U}$ and all $\vec{\alpha} \in [C]^{m\uparrow}$.*

Lemma 4.13 *Assume that X is compact, perfect, first countable, and separable, and* $\mathrm{Ldim}(X \times (\omega_1 + 1)^m) \le n$. *Then $n > m$ and there is a nonempty open $U \subseteq X$ such that* $\mathrm{Ldim}_0(\overline{U}) \le n - m$.

Proof. First, restricting everything to the closure of an open box, we may assume that $\mathrm{Ldim}_0(X \times (\omega_1 + 1)^m) \le n$.

Fix a continuous 1-1 $f : X \times (\omega_1 + 1)^m \to \prod_{r=1}^{n} L_r$, where each L_r is a compact LOTS. Applying Lemma 4.12 to the projections, $f_r : X \times (\omega_1 + 1)^m \to L_r$, and permuting the L_r, we obtain a club C and a $\overline{U}$ such that on $\overline{U} \times [C]^{m\uparrow}$:

$$f(x, \vec{\alpha}) = (g_1(\alpha_{j_1}), \dots, g_p(\alpha_{j_p}), h_1(x), \dots, h_q(x)) \quad ,$$

where $p + q = n$. Then $\{j_1, \dots, j_p\} = \{1, \dots, m\}$, since f is 1-1. Thus, $p \ge m$, so $q \le n - m$, and for any fixed $\vec{\alpha}$, the map $x \mapsto (h_1(x), \dots, h_q(x))$ embeds $\overline{U}$ into $\prod_{i=1}^{q} L_{p+i}$. ☕

Proof of Theorem 4.4. Let $n = \mathrm{Ldim}(\prod_{j=1}^{s} Z_j)$. Clearly $n \le s$. To prove that $n \ge s$, we may replace each Z_j by a closed subset and assume that $Z_j = \omega_1 + 1$ when $r + 1 \le j \le s$, while $Z_j = \overline{D_j}$ when $1 \le j \le r$. We may also assume that whenever $Z_j = \overline{D_j}$ is not second countable, no open interval in Z_j is second countable (since there is always a closed subspace with this property). Let $X = \prod_{j=1}^{r} Z_j$, and apply Lemma 4.13 to obtain $U \subseteq X$ with $\mathrm{Ldim}(\overline{U}) \le n - m$. Since $\mathrm{Ldim}(\overline{U}) = r$ by Lemma 4.5, we have $r \le n - m$, so $s = r + m \le n$. ☕

Note that this theorem does not cover all possible products of LOTSes. For example, one can show by a direct argument that $\mathrm{Ldim}((\omega + 1) \times I_S) = 2$ whenever

S is uncountable, although $(\omega + 1) \times I_S$ is dissipated, so the methods used in the proof of Theorem 4.4 do not apply. Also, Theorem 4.4 says nothing about Aronszajn lines, which have neither an increasing or decreasing ω_1–sequence, nor a countable subset whose closure is not second countable. In particular, it is not clear whether one can have a product of three compact Aronszajn lines which is embeddable into a product of two LOTSes.

In some sense, this "dimension theory" for products of totally disconnected LOTSes is more restrictive, not less restrictive, than the classical dimension theory for I^n, since there is also a limitation on dimension-raising maps. For example, Peano [18] shows how to map I onto I^2, but his map has many changes of direction, so it does not define a map from I_S onto $(I_S)^2$. In fact, this is impossible:

Proposition 4.14 *If S is uncountable, then there is no compact LOTS L such that L maps continuously onto $(I_S)^2$.*

Proof. Say $f : L \twoheadrightarrow (I_S)^2$. Replacing L by a closed subset, we may assume that f is irreducible. Then, L must be separable, since $(I_S)^2$ is separable. It follows (see Lutzer and Bennett [17]) that L is hereditarily separable, which implies (by continuity of f) that $(I_S)^2$ is hereditarily separable, which is false.

We do not know whether, for example, one can map L^2 onto $(I_S)^3$. Again, we may assume that L is separable, so that L^2 is 3–dissipated, while $(I_S)^3$ is not even 4–dissipated. However, as we know from Example 3.13, a continuous image of a 3–dissipated space need not be even $\mathfrak{c}$–dissipated.

5 Measures, L-spaces, and S-spaces

As usual, if X is compact, a *Radon measure* on X is a finite positive regular Borel measure on X, and if $f : X \to Y$ and μ is a measure on X, then μf^{-1} denotes the induced measure ν on Y, defined by $\nu(B) = \mu(f^{-1}(B))$. We shall prove some results relating μ to ν in the case that f is tight, and use this to prove that Radon measures on dissipated spaces are separable. We shall also make some remarks on compact L-spaces and S-spaces which are dissipated.

Definition 5.1 *For any space X, $\mathsf{ro}(X)$ denotes the* regular open algebra *of X. If $\mathcal{B}$ is any boolean algebra and $b \in \mathcal{B}$ with $b \neq \mathbb{0}$, then $b\!\downarrow$ denotes the algebra $\{x \in \mathcal{B} : x \leq b\}$; so $\mathbb{1}_{b\downarrow} = b$. A* Suslin algebra *is an atomless ccc complete boolean algebra which is (ω, ω)–distributive*

So, there is a Suslin tree iff there is a Suslin algebra. We shall prove:

Theorem 5.2 *If X is compact, ccc, not separable, and $\aleph_0$–dissipated, then in* $\mathsf{ro}(X)$ *there is a non-zero b such that $b\downarrow$ is a Suslin algebra.*

Of course, this is well-known in the case where X is a LOTS, and is part of the proof that a Suslin line yields a Suslin tree. Since a Suslin line is a compact L-space and is 2–dissipated (by Lemma 3.4), we have

Corollary 5.3 *There is an $\aleph_0$–dissipated compact L-space iff there is a Suslin line.*

As usual, the *support* of a Radon measure μ is the smallest closed $H \subseteq X$ such that $\mu(H) = \mu(X)$. For this H, $\mathsf{ro}(H)$ cannot be a Suslin algebra, so

Corollary 5.4 *If X is $\aleph_0$–dissipated, then the support of every Radon measure on X is a separable topological space.*

In these two corollaries, the "$\aleph_0$" cannot be replaced by "$\aleph_1$", since the usual compact L-space construction shows the following (see Section 6 for a proof):

Proposition 5.5 CH *implies that there is a compact L-space X which is both $\mathfrak{c}$–dissipated and the support of a Radon measure μ. Furthermore, μ is atomless, and, in X, the ideals of null subsets, meager subsets, and separable subsets all coincide.*

Turning to compact S-spaces, the usual CH construction [14] yields one which is scattered, and hence dissipated. Less trivially, the construction of Fedorčuk [7] shows, under $\diamondsuit$, that there is a dissipated compact S-space with no isolated points and no non-trivial convergent ω–sequences; see Section 6 for further remarks on this construction.

Proof of Theorem 5.2. Since X is ccc, we may replace X by some regular closed set and assume that X is nowhere separable — that is, the closure of every countable subset is nowhere dense. Assume that in $\mathsf{ro}(X)$ no $b\downarrow$ is Suslin, and we shall derive a contradiction.

Since X is ccc, the fact that no $b\downarrow$ is Suslin implies that there are open F_σ sets V_n^j for $n, j \in \omega$ such that for each n, the V_n^j for $j \in \omega$ are disjoint and $\bigcup_j V_n^j$ is dense, and such that for each $\varphi \in \omega^\omega$, $\bigcap_n V_n^{\varphi(n)}$ has empty interior. There is then a compact metric Y and an $f : X \twoheadrightarrow Y$ such that $V_n^j = f^{-1}(f(V_n^j))$ for each n, j. Note that this implies that each $f(V_n^j)$ is open, since $f(V_n^j) = Y \setminus f(X \setminus V_n^j)$.

Replacing f by a finer map, we may also assume that f is $\aleph_0$–tight.

Observe that $f^{-1}\{y\}$ is nowhere dense for each $y \in Y$, since either $f^{-1}\{y\} \subseteq \bigcap_n V_n^{\varphi(n)}$ for some $\varphi \in \omega^\omega$, or $f^{-1}\{y\} \subseteq X \setminus \bigcup_j V_n^j$ for some n.

Now, construct open $U_s \subseteq X$ and closed $K_s \subseteq X$ for $s \in 2^{<\omega}$ as follows: $U_{()} = X$, and each $\overline{U_{s\frown i}} \subseteq U_s \setminus K_s$, with $f(\overline{U_{s\frown 0}}) \cap f(\overline{U_{s\frown 1}}) = \emptyset$. Also, $K_s \subseteq \overline{U_s}$, with $f(K_s) = f(\overline{U_s})$ and $f\restriction K_s \twoheadrightarrow f(K_s)$ irreducible. Note that K_s is separable and $\overline{U_s}$ is nowhere separable, so that the construction can continue. More specifically, to choose $U_{s\frown 0}$ and $U_{s\frown 1}$: First, find $p_0, p_1 \in U_s \backslash K_s$ such that $f(p_0) \neq f(p_1)$; this is possible since otherwise we would have $f(U_s \backslash K_s) \subseteq \{y\}$, contradicting the fact that $f^{-1}\{y\}$ is nowhere dense. Next, find open $W_i \subseteq Y$ with $f(p_i) \in W_i$ and $\overline{W_0} \cap \overline{W_1} = \emptyset$. Then, choose $U_{s\frown i}$ with $U_{s\frown i} \subseteq \overline{U_{s\frown i}} \subseteq (U_s \backslash K_s) \cap f^{-1}(W_i)$.

Let $Q_n = \bigcup\{f(K_s) : s \in 2^n\}$, and let $Q = \bigcap_n Q_n$, which is a non-scattered subset of Y. Let $P_n = f^{-1}(Q) \cap \bigcup\{K_s : s \in 2^n\}$. Then the P_n are disjoint and each $f(P_n) = Q$, contradicting the $\aleph_0$–tightness of f. ☕

To study measures further, we use the following standard definitions:

Definition 5.6 *If μ is any finite measure on X, then* $\mathsf{ma}(\mu)$ *denotes the* measure algebra *of μ — that is, the algebra of measurable sets modulo the null sets. If $f : X \to Y$, μ is a finite measure on X, and $\nu = \mu f^{-1}$, then $f^* : \mathsf{ma}(\nu) \to \mathsf{ma}(\mu)$ is defined by $f^*([A]) = [f^{-1}(A)]$.*

$\mathsf{ma}(\mu)$ is a complete metric space with metric $d([A], [B]) = \mu(A \Delta B)$, where $[A], [B]$ denote the equivalence classes of the sets A, B. Note that we do not require f to be onto here, although $Y \backslash f(X)$ is a ν–null set. f^* is an isometric isomorphism onto some complete subalgebra $f^*(\mathsf{ma}(\nu)) \subseteq \mathsf{ma}(\mu)$.

As usual, a measure μ on X is *separable* iff $L^p(\mu)$ is a separable metric space for some (equivalently, for all) $p \in [1, \infty)$. Also μ is separable iff $\mathsf{ma}(\mu)$ is a separable metric space iff $\mathsf{ma}(\mu)$ is countably generated as a complete boolean algebra. Separability of μ is not related in any simple way to the separability of any topology that X may have. Following [6]:

Definition 5.7 MS *is the class of all compact spaces X such that every Radon measure on X is separable.*

We shall prove:

Theorem 5.8 *If X is a weakly $\mathfrak{c}$–dissipated space then X is in* MS.

In view of Lemma 3.4, Theorem 5.8 generalizes the result from [6] that every compact LOTS is in MS. Note that a space in MS need not be $\mathfrak{c}$–dissipated. For example, MS is closed under countable products (see [6]), but an infinite product of non-metric compacta is never weakly $\mathfrak{c}$–dissipated (see Theorem 3.8).

Theorem 5.8 will be an easy corollary of some general results about measures induced by weakly $\mathfrak{c}$–tight $f : X \twoheadrightarrow Y$, where X, Y are compact. Say μ is a Radon measure on X, with $\nu = \mu f^{-1}$. Even if f is tight (i.e., 2–tight), the separability of ν does not imply the separability of μ; for example, ν may be a point mass concentrating on $\{y\}$, in which case μ can be any measure supported on $f^{-1}\{y\}$ with $\mu(f^{-1}\{y\}) = \nu\{y\}$. However, if ν is atomless, then the form of ν will restrict the form of μ. There are really two kinds of ways that ν might determine μ. We shall denote the stronger way as "X is skinny" and the weaker way as "X is slim". We shall define "skinny" and "slim" also for arbitrary closed subsets of X:

Definition 5.9 *Suppose that X, Y are compact, $f : X \to Y$, μ is a Radon measure on X, and $\nu = \mu f^{-1}$. Then:*

- ☞ *X is* skinny *with respect to f, μ iff for all closed $K \subseteq X$, $\mu(K) = \nu(f(K))$.*
- ☞ *X is* slim *with respect to f, μ iff $f^* : \mathsf{ma}(\nu) \to \mathsf{ma}(\mu)$ maps* onto *$\mathsf{ma}(\mu)$.*

If H is a closed subset of X, then we say that H is skinny *(resp.,* slim*) with respect to f, μ iff H is skinny (resp., slim) with respect to $f{\restriction}H, \mu{\restriction}H$.*

Note that the equation $\mu(K) = \nu(f(K))$ shows that if X is skinny, then ν determines μ; there is no Radon measure $\mu' \neq \mu$ such that $\nu = \mu' f^{-1}$.

Lemma 5.10 *If X is skinny with respect to f, μ, then X is slim.*

Proof. If $K \subseteq X$ is closed, then $\mu(K) = \mu(f^{-1}(f(K)))$ implies that $[K] = [f^{-1}(f(K))] = f^*([f(K)])$ in $\mathsf{ma}(\mu)$. Thus, $[K] \in \mathrm{ran}(f^*)$ for all closed $K \subseteq X$, which implies that f^* is onto.

The converse is false. For example, suppose that H is a closed subset of X such that μ is supported on H and $f{\restriction}H$ is 1-1. Then X is slim, since $\mathsf{ma}(\mu) \cong \mathsf{ma}(\mu{\restriction}H)$, but X need not be skinny, since there may well be closed K disjoint from H with $X = f^{-1}(f(K))$; then $\mu(K) = 0$ but $\nu(f(K)) = \mu(X)$. In this example, H is skinny with respect to f, μ. Some examples of skinny sets on which the function f is not 1-1 are given by:

Lemma 5.11 *Suppose that X, Y are compact, $f : X \to Y$ is tight, μ is a Radon measure on X, and $\nu = \mu f^{-1}$ is atomless. Then X is skinny with respect to f, μ.*

Proof. If X is not skinny, fix a closed $K \subseteq X$ with $\mu(K) < \nu(f(K))$, so that $\mu(f^{-1}(f(K)) \setminus K) > 0$. Then choose a closed $L \subseteq f^{-1}(f(K)) \setminus K$ with $\mu(L) > 0$. Then K, L are disjoint in X and $\nu(f(K) \cap f(L)) = \nu(f(L)) \geq \mu(L) > 0$, so $f(K) \cap f(L)$ cannot be scattered, since ν is atomless, so f is not tight.

One cannot replace "tight" by "3–tight" here. For example, say $X = Y \times \{0, 1\}$, with f the natural projection, which is 2–tight. If ν is any Radon measure on Y, and on X we let $\mu(E_0 \times \{0\} \cup E_1 \times \{1\}) = \frac{1}{2}(\nu(E_0) + \nu(E_1))$, then X is not skinny (or even slim). Here, X is the union of two skinny subsets, and this situation generalized to:

Lemma 5.12 *Suppose that X, Y are compact, $f : X \twoheadrightarrow Y$ is $\aleph_0$–tight and μ is a Radon measure on X with μf^{-1} atomless. Then there is a countable family $\mathcal{H}$ of disjoint skinny subsets of X such that $\mu(X) = \sum\{\mu(H) : H \in \mathcal{H}\}$.*

Proof. If this fails, then the usual exhaustion argument lets us shrink X and assume that $\mu(X) > 0$ and there are no closed skinny $H \subseteq X$ of positive measure. We now build an infinite loose family as follows:

Construct a tree of closed $H_s \subseteq X$ for $s \in 2^{<\omega}$; so $H_{s\frown 0}, H_{s\frown 1}$ will be disjoint closed subsets of H_s, and also $f(H_{s\frown 0}) \cap f(H_{s\frown 1}) = \emptyset$. Each H_s will have positive measure. $H_{()}$ can be X.

Given H_s: Since H_s is not skinny, we can choose a closed $K_s \subset H_s$ with $\mu(H_s \cap (f^{-1}(f(K_s)) \setminus K_s)) > 0$. Then, since μ is regular and μf^{-1} is atomless, we can choose closed $H_{s\frown 0}, H_{s\frown 1} \subseteq H_s \cap (f^{-1}(f(K_s)) \setminus K_s)$ with each $\mu(H_{s\frown i}) > 0$ and $f(H_{s\frown 0}) \cap f(H_{s\frown 1}) = \emptyset$.

Now, let $Q_n = \bigcup\{f(H_s) : s \in 2^n\}$ and let $Q = \bigcap_n Q_n$; so, Q is non-scattered. Let $P_n = f^{-1}(Q) \cap \bigcup\{K_s : s \in 2^n\}$. Then $\{P_n : n \in \omega\}$ is a loose family.

It follows that the measure algebra of μ is a countable sum of measure algebras isomorphic to algebras derived from measures on Y. Note that the K_s in this proof may be null sets, so one cannot split them also to obtain a loose family of size $\mathfrak{c}$, as we did in the proof of Lemma 2.18. In fact, the L-space of Proposition 5.5 shows that one cannot weaken "$\aleph_0$–tight" to "$\aleph_1$–tight" in this lemma. To see this, note that μ is a separable measure on X by Theorem 5.8, so one can get an $f : X \twoheadrightarrow Y$ such that Y is compact metric, $\nu = \mu f^{-1}$ atomless, and $f^*(\mathsf{ma}(\nu)) = \mathsf{ma}(\mu)$. Since X is $\aleph_1$–dissipated, one can refine f and assume also that f is $\aleph_1$–tight. Now, if H is skinny, let K be a closed subset of H such that $f(K) = f(H)$ and $f \restriction K : K \twoheadrightarrow f(H)$ is irreducible. Then K is separable and hence null (by the properties of X), and $\mu(H) = \mu(K)$ (since H is skinny), so $\mu(H) = 0$. Thus, there cannot be a family $\mathcal{H}$ as in Lemma 5.12.

However, the analogous result with "slim" (Theorem 5.14) just uses $\mathfrak{c}$–tightness.

Definition 5.13 *Suppose that X, Y are compact, $f : X \to Y$, and μ is a Radon measure on X. Then X is* simple *with respect to f, μ iff there is a countable disjoint family $\mathcal{H}$ of slim subsets of X such that $\sum\{\mu(H) : H \in \mathcal{H}\} = \mu(X)$.*

We shall prove:

Theorem 5.14 *Suppose that X, Y are compact, $f : X \to Y$, and μ is a Radon measure on X, with $\nu = \mu f^{-1}$, and suppose that X is not simple with respect to f, μ. Then there is a $\varphi : \mathrm{dom}(\varphi) \to 2^\omega$, where $\mathrm{dom}(\varphi)$ is closed in X, such that for some closed $Q \subseteq Y$, $\nu(Q) > 0$ and $\varphi(f^{-1}\{y\}) = 2^\omega$ for all $y \in Q$. In particular, if ν is atomless, then f is not weakly $\mathfrak{c}$–tight.*

In proving this, the notion of *conditional expectation* (see [9], §48) will be useful in comparing the induced measure $(\mu\restriction S)f^{-1}$ to ν for various $S \subseteq X$:

Definition 5.15 *Suppose that $f : X \to Y$, with X, Y compact, μ is a measure on X and $\nu = \mu f^{-1}$. If S is a measurable subset of X, then the* conditional expectation*, $\mathbb{E}(S|f) = \mathbb{E}_\mu(S|f)$, is the measurable $\varphi : Y \to [0,1]$ defined so that $\int_A \varphi(y)\, d\nu(y) = \mu(f^{-1}(A) \cap S)$ for all measurable $A \subseteq Y$.*

Of course, φ is only defined up to equivalence in $L^\infty(\nu)$. Conditional expectations are usually defined for probability measures, but they make sense in general for finite measures; actually, $\mathbb{E}_\mu(S|f) = \mathbb{E}_{c\mu}(S|f)$ for any non-zero c. Note that $\int_A \varphi(y)\, d\nu(y) = \int_{f^{-1}(A)} \varphi(f(x))\, d\mu(x)$. We may also characterize $\varphi = \mathbb{E}_\mu(S|f)$ by the equation:

$$\int_S g(f(x))\, d\mu(x) = \int_X \varphi(f(x))\, g(f(x))\, d\mu(x) = \int_Y \varphi(y)\, g(y)\, d\nu(y) \quad .$$

for $g \in L^1(Y, \nu)$. φ is obtained either by the Radon-Nikodym Theorem, or, equivalently, by identifying $(L^1(Y,\nu))^*$ with $L^\infty(Y,\nu)$, since $\Gamma(g) := \int_S g(f(x))\, dx$ defines $\Gamma \in (L^1(Y,\nu))^*$, with $\|\Gamma\| \le 1$.

Now, given μ on X and $f : X \to Y$, we shall consider various closed subsets $H \subseteq X$ while studying the tightness properties of f. When $S \subseteq H \subseteq X$, one must be careful to distinguish $\mathbb{E}_\mu(S|f)$ (computed using μ and $f : X \to Y$) from $\mathbb{E}_{\mu\restriction H}(S \mid f\restriction H)$ (computed using $\mu\restriction H$ and $f\restriction H : H \to Y$). These are related by:

Lemma 5.16 *Suppose that $f : X \to Y$, with X, Y compact, H is a closed subset of X, and μ is a Radon measure on X. Let S be a measurable subset of H. Then $\mathbb{E}_\mu(S|f) = \mathbb{E}_\mu(H|f) \cdot \mathbb{E}_{\mu\restriction H}(S \mid f\restriction H)$.*

Proof. Let $\varphi = \mathbb{E}_\mu(S|f)$, $\psi = \mathbb{E}_\mu(H|f)$, and $\gamma = \mathbb{E}_{\mu\restriction H}(S \mid f\restriction H)$. We may take these to be bounded Borel-measurable functions from Y to $\mathbb{R}$. For any bounded Borel-measurable $g : Y \to \mathbb{R}$, we have

$$\int_S g(f(x))\, d\mu(x) = \int_X \varphi(f(x))\, g(f(x))\, d\mu(x)$$
$$\int_H g(f(x))\, d\mu(x) = \int_X \psi(f(x))\, g(f(x))\, d\mu(x)$$
$$\int_S g(f(x))\, d\mu(x) = \int_H \gamma(f(x))\, g(f(x))\, d\mu(x) = \int_X \psi(f(x))\, \gamma(f(x))\, g(f(x))\, d\mu(x),$$

which yields $\varphi = \psi\gamma$. ☕

We now relate conditional expectations to slimness:

Lemma 5.17 *Suppose that X, Y are compact, $f : X \to Y$, and μ is a measure on X, with $\nu = \mu f^{-1}$. Let $S \subseteq X$ be measurable. Then $[S] \in \mathrm{ran}(f^*)$ iff $[\mathbb{E}(S|f)] = [\chi_T]$ for some measurable $T \subseteq Y$, in which case $[S] = f^*([T])$.*

Proof. For $\to$: If $[S] = f^*([T])$ then $\mu(S \Delta f^{-1}(T)) = 0$, which implies $\mathbb{E}(S|f) = \mathbb{E}(f^{-1}(T)|f) = \chi_T$.

For $\leftarrow$: If $[\mathbb{E}(S|f)] = [\chi_T]$ then $\mu(f^{-1}(A) \cap S) = \nu(A \cap T)$ for all measurable $A \subseteq Y$. Setting $A = Y \backslash T$, we get $\mu(S \backslash f^{-1}(T)) = 0$, so $[S] \leq [f^{-1}(T)]$. Setting $A = T$, we get $\mu(S \cap f^{-1}(T)) = \nu(T) = \mu(f^{-1}(T))$, so $[S] \geq [f^{-1}(T)]$.

In particular, X is slim with respect to f, μ iff every $\mathbb{E}(S|f)$ is the characteristic function of a set; this remark will be useful when applied also to $\mu \restriction H$ for various $H \subseteq X$.

Lemma 5.18 *Suppose that X, Y are compact, $f : X \to Y$, and μ is a measure on X, with $\nu = \mu f^{-1}$, and suppose that X is not slim with respect to f, μ. Then there are disjoint closed $H_0, H_1 \subseteq X$ with $f(H_0) = f(H_1) = K$, such that $\nu(K) > 0$ and, for $i = 0, 1$, $0 < \mathbb{E}(H_i|f)(y) < 1$ for a.e. $y \in K$.*

Proof. First, let $\widetilde{H}_0 \subseteq X$ be closed with $[H_0] \notin \mathrm{ran}(f^*)$. We can then, by Lemma 5.17, get a closed $\widetilde{K} \subseteq f(\widetilde{H}_0)$ with $\nu(\widetilde{K}) > 0$ and $\mathbb{E}(\widetilde{H}_0|f)(y) \in (0, 1)$ for a.e. $y \in \widetilde{K}$. Then, choose a closed $\widetilde{H}_1 \subseteq f^{-1}(\widetilde{K}) \backslash \widetilde{H}_0$ with $\mu(\widetilde{H}_1) > 0$. Then, choose a closed $K \subseteq \widetilde{f}(\widetilde{H}_1)$ with $\nu(K) > 0$ and $\mathbb{E}(\widetilde{H}_1|f)(y) > 0$ for a.e. $y \in K$, and let $H_i = \widetilde{H}_i \cap f^{-1}(K)$.

We now consider the opposite of slim:

Definition 5.19 *X is* nowhere slim *with respect to f, μ iff there is no closed $H \subseteq X$ with $\mu(H) > 0$ such that H is slim with respect to f, μ.*

Lemma 5.20 *Suppose that X, Y are compact, $f : X \to Y$, and μ is a measure on X, with $\nu = \mu f^{-1}$, and suppose that X is nowhere slim with respect to f, μ. Fix $\varepsilon > 0$. Then there are disjoint closed $H_0, H_1 \subseteq X$ with $f(H_0) = f(H_1) = K$, such that $\nu(Y \backslash K) < \varepsilon$ and, for $i = 0, 1$, $0 < \mathbb{E}(H_i|f)(y) < 1$ for a.e. $y \in K$.*

Proof. Fix $\mathcal{K}$ such that

1. $\mathcal{K}$ is a disjoint family of non-null closed subsets of Y.
2. For $K \in \mathcal{K}$, there are disjoint closed $H_0^K, H_1^K \subseteq X$ with $f(H_0^K) = f(H_1^K) = K$, and, for $i = 0, 1$, $0 < \mathbb{E}(H_i^K|f)(y) < 1$ for a.e. $y \in K$.
3. $\mathcal{K}$ is maximal with respect to (1)(2).

Then $\mathcal{K}$ is countable. If $\nu(Y \backslash \bigcup \mathcal{K}) = 0$, choose a finite $\mathcal{K}' \subseteq \mathcal{K}$ such that $\nu(Y \backslash \bigcup \mathcal{K}') < \varepsilon$, set $K = \bigcup \mathcal{K}'$, and set $H_i = \bigcup \{H_i^K : K \in \mathcal{K}'\}$. If $\nu(Y \backslash \bigcup \mathcal{K}) \neq 0$, choose a closed $E \subseteq Y \backslash \bigcup \mathcal{K}$ with $\nu(E) > 0$, and use Lemma 5.18 to derive a contradiction from maximality of $\mathcal{K}$ and the fact that $f^{-1}(E)$ is not slim. ☕

We can now use a tree argument to prove Theorem 5.14:

Proof of Theorem 5.14. Since f is not simple, there must be a closed $H \subseteq X$ such that H is nowhere slim with respect to $\mu \restriction H, f \restriction H$. Restricting everything to H, we may assume that X itself is nowhere slim. Also, WLOG $\mu(X) = \nu(Y) = 1$ and $f(X) = Y$. Now, get $P_s \subseteq X$ for $s \in 2^{<\omega}$ and $Q_n \subseteq Y$ for $n \in \omega$ so that:

1. $P_{()} = X$ and $Q_0 = Y$.
2. P_s is closed in X and Q_n is closed in Y.
3. $Q_n = \bigcap \{f(P_s) : \mathrm{lh}(s) = n\}$.
4. $P_{s \frown 0}$ and $P_{s \frown 1}$ are disjoint subsets of P_s.
5. $\nu(f(P_s) \setminus f(P_{s \frown i})) \leq 6^{-n-1}$ when $\mathrm{lh}(s) = n$ and $i = 0, 1$.
6. $Q_{n+1} \subseteq Q_n$ and $\nu(Q_n \backslash Q_{n+1}) \leq 2^{n+1} \cdot 6^{-n-1} = 3^{-n-1}$.
7. $\mathbb{E}_\mu(P_s | f)(y) > 0$ for ν–a.e. $y \in f(P_s)$.

Assuming that this can be done, let $Q = \bigcap_n Q_n$. $Q \subseteq f(P_s)$ for all $s \in 2^{<\omega}$, so for $t \in 2^\omega$, let $P_t = f^{-1}(Q) \cap \bigcap_n P_{t \restriction n}$. Then the P_t are disjoint and $f(P_t) = Q$ for all t. Also, $\mu(Q) \geq 1 - 1/3 - 1/9 - 1/27 - \cdots = 1/2$. Let $\mathrm{dom}(\varphi) = \bigcup_t P_t$, with $\varphi(x) = t$ for $x \in P_t$.

Now, to do the construction, note first that (6) follows from (3)(4)(5). We proceed by induction on $\mathrm{lh}(s)$, using (7) to accomplish the splitting. For $\mathrm{lh}(s) = 0$, (1)(2)(3)(7) are trivial, since $\mathbb{E}(X|f)(y) = 1$ for a.e. $y \in Y$. Now fix s with $\mathrm{lh}(s) = n$. We obtain $P_{s \frown 0}$ and $P_{s \frown 1}$ by applying Lemma 5.20, with the X, Y there replaced by $P_s, f(P_s)$; but then we must replace ν by $\lambda := (\mu \restriction P_s)\,(f \restriction P_s)^{-1}$ on $f(P_s)$. Let $\varphi = \mathbb{E}_\mu(P_s | f)$; then, by (7) for P_s, $\varphi(y) > 0$ for ν–a.e. $y \in f(P_s)$; also $\varphi(y) = 0$ for a.e. $y \notin f(P_s)$, and $\int_A \varphi(y)\, d\nu(y) = \mu(f^{-1}(A) \cap P_s) = \lambda(A)$ for all measurable $A \subseteq f(P_s)$. Fix $\delta > 0$ such that $\nu(\{y \in f(P_s) : \varphi(y) < \delta\}) \leq 6^{-n-1}/2$. Now apply Lemma 5.20 to get closed $P_{s \frown 0}, P_{s \frown 1}$ satisfying (4) with $K_s := f(P_{s \frown 0}) = f(P_{s \frown 1})$ so that, for $i = 0, 1$, $\mathbb{E}_{\mu \restriction P_s}(P_{s \frown i} \mid f \restriction P_s)(y) > 0$ for λ–a.e. $y \in K_s$, and $\lambda(f(P_s) \backslash K_s) < \delta \cdot 6^{-n-1}/2$. Now, by Lemma 5.16, $\mathbb{E}_\mu(P_{s \frown i} | f) = \varphi \cdot \mathbb{E}_{\mu \restriction P_s}(P_{s \frown i} \mid f \restriction P_s)$, which yields (7) for $P_{s \frown i}$. To obtain (5), let $A = f(P_s) \setminus K_s$. we need $\nu(A) \leq 6^{-n-1}$, and we have $\int_A \varphi(y)\, d\nu(y) = \lambda(A) < \delta \cdot 6^{-n-1}/2$. Let $A = A' \cup A''$, where $\varphi < \delta$ on A' and $\varphi \geq \delta$ on A''. Then $\nu(A') \leq 6^{-n-1}/2$ and $\nu(A'') \leq (1/\delta) \int_{A''} \varphi(y)\, d\nu(y) \leq 6^{-n-1}/2$, so $\nu(A) \leq 6^{-n-1}$. ☕

Corollary 5.21 *Suppose that X, Y are compact, $f : X \twoheadrightarrow Y$ is weakly $\mathfrak{c}$–tight, and μ is a Radon measure on X, with $\nu = \mu f^{-1}$ atomless and separable. Then μ is separable.*

Proof. X is simple with respect to f, μ, by Theorem 5.14, which implies that $\mathsf{ma}(\mu)$ is a countable disjoint sum of separable measure algebras. ☕

Proof of Theorem 5.8. Assume that μ is a non-separable Radon measure on X; we shall derive a contradiction. By subtracting the point masses, we may assume that μ is atomless.

First, fix a compact metric Z and a $g : X \twoheadrightarrow Z$ such that μg^{-1} is atomless. This is easily done by an elementary submodel argument. More concretely, one can assume that $X \subseteq [0,1]^\kappa$; then $g = \pi^\kappa_d$ for a suitably chosen countable $d \subseteq \kappa$. We construct d as $\bigcup_i d_i$, where the d_i are finite and non-empty and $d_0 \subseteq d_1 \subseteq \cdots$. Given d_i, we have the space $Z_i = \pi^\kappa_{d_i}(X)$, with measure $\nu_i = \mu(\pi^\kappa_{d_i})^{-1}$. Let $\{F^\ell_i : \ell \in \omega\}$ be a family of closed non-null subsets of Z_i which is dense in the measure algebra, and make sure that for each ℓ, there is some $j > i$ such that Z_j contains a closed set $K \subseteq (\pi^{d_j}_{d_i})^{-1}(F^\ell_i)$ with $\nu_j(K)/\mu_i(F^\ell_i) \in (1/3, 2/3)$.

Let $f : X \twoheadrightarrow Y$ be weakly $\mathfrak{c}$–tight, where Y is metric and f is finer than g. We then have $\Gamma \in C(Y, Z)$ such that $g = \Gamma \circ f$, so $\mu g^{-1} = (\mu f^{-1})\Gamma^{-1}$, so μf^{-1} is atomless. Also, μf^{-1} is separable because Y is metric, contradicting Corollary 5.21. ☕

6 Inverse Limits

Some compacta built as inverse limits in ω_1 steps are dissipated. We avoid explicit use of the inverse limit by viewing X as a subspace of some M^{ω_1}, so the bonding maps in the inverse limit will be the projection maps.

Definition 6.1 *For any space M and ordinals $\alpha \leq \beta$: $\pi^\beta_\alpha : M^\beta \twoheadrightarrow M^\alpha$ denotes the natural projection.*

Theorem 6.2 *Let M be compact metric, and suppose that X is a closed subset of M^{ω_1}. Let $X_\alpha = \pi^{\omega_1}_\alpha(X)$. Assume that for each $\alpha < \omega_1$, the map $\pi^{\alpha+1}_\alpha \restriction X_{\alpha+1} : X_{\alpha+1} \twoheadrightarrow X_\alpha$ is tight. Then*

1. *For each $\alpha < \beta \leq \omega_1$, the map $\pi^\beta_\alpha \restriction X_\beta : X_\beta \twoheadrightarrow X_\alpha$ is tight.*
2. *X is dissipated.*

Proof. For (1), fix α and induct on β. For successor stages, use Lemma 2.13. For limit $\beta > \alpha$, use the fact that if P_0, P_1 are disjoint closed subsets of X_β, then there is a δ with $\alpha < \delta < \beta$ and $\pi^\beta_\delta(P_0) \cap \pi^\beta_\delta(P_1) = \emptyset$.

For (2), observe that given $g : X \twoheadrightarrow Z$, with Z metric, there is an $\alpha < \omega_1$ with $\pi^{\omega_1}_\alpha \restriction X$ finer than g. Now, use the fact that all $\pi^{\omega_1}_\beta \restriction X$ are tight.

The proof of (2) did not actually require all $\pi^{\omega_1}_\beta \restriction X$ to be tight; we only needed unboundedly many. More generally, the definition of "dissipated" requires the family of tight maps to be unbounded, but it does not necessarily contain a club, although it does contain a club in the "natural" examples of dissipated spaces. We first point out an example where the tight maps do not contain a club. Then we shall formulate precisely what "contains a club" means.

Example 6.3 *There is a closed $X \subseteq 2^{\omega_1}$ such that, setting $X_\alpha = \pi^{\omega_1}_\alpha(X)$:*

a. X is dissipated

b. For all $\alpha < \omega_1$, $\pi^{\omega_1}_\alpha \restriction X : X \twoheadrightarrow X_\alpha$ is tight iff α is not a limit ordinal.

Proof. First note that $(b) \to (a)$ because whenever $g : X \to Z$, with Z metric, there is always an $\alpha < \omega_1$ with $\pi^{\omega_1}_\alpha \restriction X \le g$. Then $\pi^{\omega_1}_{\alpha+1} \restriction X \le \pi^{\omega_1}_\alpha \restriction X \le g$ and $\pi^{\omega_1}_{\alpha+1} \restriction X$ is tight.

To prove (b), we use a standard inverse limit construction, building X_α by induction on α. We shall have:

1. X_α is a closed subset of 2^α for all $\alpha \le \omega_1$, and $X = X_{\omega_1}$.
2. $X_\alpha = \pi^\beta_\alpha(X_\beta)$ whenever $\alpha \le \beta \le \omega_1$.
3. $X_\alpha = 2^\alpha$ for $\alpha \le \omega$.
4. For $\alpha < \omega_1$: $X_{\alpha+1} = X_\alpha \times \{0\} \cup F_\alpha \times \{1\}$, where F_α is a closed subset of X_α.
5. F_γ is a perfect set for all limit $\gamma < \omega_1$.
6. $\pi^\alpha_\delta(F_\alpha)$ is finite whenever $\delta < \alpha < \omega_1$.
7. Whenever $\delta < \alpha < \omega_1$ and δ is a successor ordinal, there is an n with $0 < n < \omega$ such that $\pi^{\alpha+n}_{\delta+1}(F_{\alpha+n}) = F_\delta \times \{0, 1\}$.

Conditions (1)(2) imply that X_γ, for limit γ, is determined by the X_α for $\alpha < \gamma$; then, by (4), the whole construction is determined by the choice of the $F_\alpha \subseteq X_\alpha$; as usual, in stating (4), we are identifying $2^{\alpha+1}$ with $2^\alpha \times \{0, 1\}$. By (3), $F_\alpha = X_\alpha$ when $\alpha < \omega$. By (6), F_α is finite for successor α. Conditions (1)–(6) are sufficient to verify (b) of the theorem, but (7) was added to ensure that the construction can be carried out. Using (7), it is easy to construct F_γ for limit γ to satisfy (5)(6), and (7) itself is easy to ensure by a standard enumeration argument, since there are no further restrictions on the finite sets $F_{\alpha+n} \subseteq X_{\alpha+n}$ when $n > 0$.

To verify (b): If $\alpha < \omega_1$ is a limit ordinal, then (4)(5) guarantee that $\pi^{\omega_1}_{\alpha} \restriction X : X \twoheadrightarrow X_\alpha$ is not tight. Now, fix a successor $\alpha < \omega$. We prove by induction that $\pi^{\beta}_{\alpha} \restriction X_\beta : X_\beta \twoheadrightarrow X_\alpha$ is tight whenever $\alpha \le \beta \le \omega_1$. This is trivial when $\beta = \alpha$. If $\beta > \alpha$ is a limit ordinal and $\pi^{\beta}_{\alpha} \restriction X_\beta$ fails to be tight, then we have disjoint closed $P_0, P_1 \subset X_\beta$ with $Q = \pi^{\beta}_{\alpha}(P_0) = \pi^{\beta}_{\alpha}(P_1)$ and Q not scattered; but then there is a δ with $\beta > \delta > \alpha$ such that $\pi^{\beta}_{\delta}(P_0) \cap \pi^{\beta}_{\delta}(P_1) = \emptyset$, and then the $\pi^{\beta}_{\delta}(P_i)$ refute the tightness of π^{δ}_{α}.

Finally, assume that $\alpha \le \beta < \omega_1$ and that $\pi^{\beta}_{\alpha} \restriction X_\beta$ is tight. We shall prove that $\pi^{\beta+1}_{\alpha} \restriction X_{\beta+1}$ is tight. If β is a successor, we note that $\pi^{\beta+1}_{\beta} \restriction X_{\beta+1}$ is tight because F_β is finite, so that $\pi^{\beta+1}_{\alpha} \restriction X_{\beta+1} = \pi^{\beta}_{\alpha} \restriction X_\beta \circ \pi^{\beta+1}_{\beta} \restriction X_{\beta+1}$ is tight by Lemma 2.13. Now, assume that β is a limit (so $\alpha < \beta$) and that $\pi^{\beta+1}_{\alpha} \restriction X_{\beta+1}$ is not tight. Fix disjoint closed $P_0, P_1 \subset X_{\beta+1}$ with $Q = \pi^{\beta+1}_{\alpha}(P_0) = \pi^{\beta+1}_{\alpha}(P_1)$ and Q not scattered. Since $\pi^{\beta}_{\alpha}(F_\beta)$ is finite, we may shrink Q and the P_i and assume that $Q \cap \pi^{\beta}_{\alpha}(F_\beta) = \emptyset$. Then $\pi^{\beta+1}_{\beta}(P_i) \cap F_\beta = \emptyset$, so that $\pi^{\beta+1}_{\beta}(P_0) \cap \pi^{\beta+1}_{\beta}(P_1) = \emptyset$, and the $\pi^{\beta+1}_{\beta}(P_i)$ contradict the tightness of $\pi^{\beta}_{\alpha} \restriction X_\beta$.

There are various equivalent ways to formulate "contains a club"; the following is probably the quickest to state:

Definition 6.4 *The compact X is* wasted *iff whenever θ is a suitably large regular cardinal and $M \prec H(\theta)$ is countable and contains X and its topology, the natural evaluation map $\pi_M : X \to [0,1]^{C(X,[0,1]) \cap M}$ is tight.*

For the X of Example 6.3, no π_M is tight, since π_M is equivalent to $\pi^{\omega_1}_{\gamma}$, where $\gamma = \omega_1 \cap M$. The X of Theorem 6.2 is wasted, as is every compact LOTS. A notion intermediate between "dissipated" and "wasted" is obtained by requiring π_M to be tight for a stationary set of $M \prec H(\theta)$.

In Theorem 6.2: since $X_{\alpha+1}$ and X_α are compact metric, the assumption that $\pi^{\alpha+1}_{\alpha}$ is tight is equivalent to saying that $\{y \in X_\alpha : |(\pi^{\alpha+1}_{\alpha})^{-1}\{y\} \cap X_{\alpha+1}| > 1\}$ is countable (see Theorem 2.7). In the constructions of [7, 11, 12], this set is actually a singleton. In some cases, the spaces are also *minimally generated* in the sense Koppelberg [15] and Dow [4]:

Definition 6.5 *Let X, Y be compact. Then $f : X \twoheadrightarrow Y$ is* minimal *iff $|f^{-1}\{y\}| = 1$ for all $y \in Y$ except for one y_0, for which $|f^{-1}\{y_0\}| = 2$.*

We remark that this is the same as minimality in the sense that if $f = g \circ h$, where $h : X \twoheadrightarrow Z$ and $g : Z \twoheadrightarrow Y$, then either g or h is a bijection. Clearly, every minimal map is tight.

Definition 6.6 *X is* minimally generated *iff X is a closed subspace of some 2^ρ, where, setting $X_\alpha = \pi^{\rho}_{\alpha}(X)$, all the maps $\pi^{\alpha+1}_{\alpha} \restriction X_{\alpha+1} : X_{\alpha+1} \twoheadrightarrow X_\alpha$, for $\alpha < \rho$, are minimal.*

Examples of such spaces are the Fedorčuk S-space [7], obtained under $\diamondsuit$ (here, $\rho = \omega_1$), and the Efimov spaces obtained by Fedorčuk [8] and Dow [4], where $\rho > \omega_1$.

Clearly, if $\rho = \omega_1$, then X must be dissipated by Theorem 6.2, but this need not be true for $\rho > \omega_1$. For example, if $A(\aleph_1)$ is the 1-point compactification of a discrete space of size $\aleph_1$, and $X = A(\aleph_1) \times 2^\omega$, then X is not $\aleph_1$–dissipated by Lemma 3.6, but X is minimally generated, with $\rho = \omega_1 + \omega$.

Note that if we weaken "tight" to "3–tight" in Theorem 6.2, we get nothing of any interest in general. In fact, if $M = 2 = \{0, 1\}$ and each $X_\alpha = M^\alpha$, then all $\pi^{\alpha+1}_\alpha \restriction X_{\alpha+1}$ are 3–tight, but X is not weakly $\mathfrak{c}$–dissipated by Theorem 3.8. However, one can in some cases use an inverse limit construction build a space which is $\aleph_0$–dissipated:

Proof of Proposition 5.5. We modify the standard construction of a compact L-space under CH, following specifically the details in [16]; similar constructions are in Haydon [13] and Talagrand [19]. So, X will be a closed subset of 2^{ω_1}.

We inductively define $X_\alpha \subseteq 2^\alpha$, for $\omega \le \alpha \le \omega_1$, along with an atomless Radon probability measure μ_α on X_α such that the support of μ_α is all of X_α. Let $X_\omega = 2^\omega$ with μ_ω the usual product measure. The measures will all cohere, in the sense that $\mu_\alpha = \mu_\beta\,(\pi^\beta_\alpha)^{-1}$ whenever $\alpha < \beta$. Along with the measures, we choose a countable family $\mathcal{F}_\alpha$ of closed μ_α–null subsets of X_α and a specific closed nowhere dense non-null $K_\alpha \subseteq X_\alpha$. When $\alpha < \beta < \omega_1$, $\mathcal{F}_\beta$ will contain $(\pi^\beta_\alpha)^{-1}(F)$ for all $F \in \mathcal{F}_\beta$, along with some additional sets. Since $\mathcal{F}_\alpha$ is countable, we can choose a perfect $C_\alpha \subseteq K_\alpha$ such that $\mu_\alpha(C_\alpha) > 0$, C_α is the support of $\mu_\alpha \restriction C_\alpha$, and $C_\alpha \cap F = \emptyset$ for all $F \in \mathcal{F}_\alpha$. Then we let $X_{\alpha+1} = X_\alpha \times \{0\} \cup C_\alpha \times \{1\}$. In the construction of [16], $\mu_{\alpha+1}$ can be chosen arbitrarily to satisfy $\mu_\alpha = \mu_{\alpha+1}\,(\pi^{\alpha+1}_\alpha)^{-1}$, as long as all non-empty open subsets of $C_\alpha \times \{1\}$ have positive measure; there is some flexibility here in distributing the measure on C_α among its copies $C_\alpha \times \{0\}$ and $C_\alpha \times \{1\}$. In particular, depending on the choices made, the final measure $\mu = \mu_{\omega_1}$ on $X = X_{\omega_1}$ may be separable or non-separable. In any case, [16] shows that, assuming CH, one may choose the $\mathcal{F}_\alpha$ and K_α appropriately to guarantee X is an L-space and that the ideals of null subsets, meager subsets, and separable subsets all coincide.

Now, always choose $\mu_{\alpha+1}$ such that $\mu_{\alpha+1}(C_\alpha \times \{0\}) = 0$. This will guarantee that μ on X is separable, with $\mathsf{ma}(\mu)$ isomorphic to $\mathsf{ma}(\mu_\omega)$ via $(\pi^{\omega_1}_\omega)^*$. Also, put the set $C_\alpha \times \{0\}$ into $\mathcal{F}_{\alpha+1}$. Then, for all $x \in X_\omega$, $(\pi^{\omega_1}_\omega)^{-1}\{x\}$ is scattered (as is easy to verify), and hence countable (since X is HL). But then $\pi^{\omega_1}_\omega \restriction X : X \twoheadrightarrow X_\omega$ is $\aleph_1$–tight, so that X is $\aleph_1$–dissipated by Lemma 3.5. ☕

We remark that by Theorem 5.8, we know that the μ of Proposition 5.5 must be separable, so it was natural to make $\mathsf{ma}(\mu)$ isomorphic to $\mathsf{ma}(\mu_\omega)$ in the construction.

7 Absoluteness

We shall prove here that tightness is absolute. This can then be applied in forcing arguments, but the absoluteness itself has nothing at all to do with forcing; it is just a fact about transitive models of ZFC, and is related to the absoluteness of Π^1_1 statements. Since we never need absoluteness of Π^1_2 (Shoenfield's Theorem), we do not need the models to contain all the ordinals. So, we consider arbitrary transitive models M, N of ZFC with $M \subseteq N$. If in M, we have compacta X, Y and $f : X \to Y$, we want to show that f is tight in M iff f is tight in N.

To make this discussion precise, we must, in N, replace X, Y by the corresponding compact spaces $\widetilde{X}, \widetilde{Y}$. This concept was described by Bandlow [1] (and later in [5, 6, 12]), and is defined as follows:

Definition 7.1 *Let $M \subseteq N$ be transitive models of ZFC. In M, assume that X is compact. Then $\widetilde{X}$ denotes the compactum in N characterized by:*

1. *X is dense in $\widetilde{X}$.*
2. *Every $\varphi \in C(X, [0,1]) \cap M$ extends to a $\widetilde{\varphi} \in C(\widetilde{X}, [0,1])$ in N.*
3. *The functions $\widetilde{\varphi}$ (for $\varphi \in M$) separate the points of $\widetilde{X}$.*

If, in M, X, Y are compact and $f \in C(X, Y)$, then in N, $\widetilde{f} \in C(\widetilde{X}, \widetilde{Y})$ denotes the (unique) continuous extension of f.

In forcing, $\overset{\diamond}{X}$ denotes the $\widetilde{X}$ of $V[G]$, while $\check{X}$ denotes the X of $V[G]$.

Theorem 7.2 *Let $M \subseteq N$ be transitive models of ZFC. In M, assume that X, Y are compact, K is compact metric, and $f : X \to Y$. Then the following are equivalent:*

1. *In M: There is a K–loose function for f.*
2. *In N: There is a $\widetilde{K}$–loose function for $\widetilde{f}$.*

Proof. For (1) → (2), just observe that if in M, we have φ, Q satisfying Definition 2.4 (of K–loose), then $\widetilde{\varphi}, \widetilde{Q}$ satisfy Definition 2.4 in N.

For ¬(1) → ¬(2), we shall define a partial order $\mathbb{T}$ in M. We shall then prove that ¬(1) implies the well-founded of $\mathbb{T}$ in M, while the well-founded of $\mathbb{T}$ in N implies ¬(2). The result then follows by the absoluteness of well-foundedness.

As in the proof of Theorem 2.10, let $H = [0,1]^\omega$, and assume that $K \subseteq H$. Then the existence of a K–loose function is equivalent to the existence of a $\varphi \in C(X, H)$ such that for some non-scattered $Q \subseteq Y$ we have $\psi(f^{-1}\{y\}) \supseteq K$ for all $y \in Q$.

$\mathbb{T}$ is a tree of finite sequences, ordered by extension. $\mathbb{T}$ contains the empty sequence and all non-empty sequences

$$\big\langle (\mathcal{E}_0, \psi_0), (\mathcal{E}_1, \psi_1), \ldots, (\mathcal{E}_{n-1}, \psi_{n-1}) \big\rangle$$

satisfying:

a. Each $\psi_i \in C(X,H)$.
b. Each $\mathcal{E}_i$ is a disjoint family of 2^i non-empty closed subsets of Y.
c. Whenever $y \in E \in \mathcal{E}_i$ and $z \in K$: $d(z, \psi_i(f^{-1}\{y\})) \le 2^{-i}$.
d. When $i+1 < n$: $d(\psi_i, \psi_{i+1}) \le 2^{i-1}$, and each $E \in \mathcal{E}_i$ has exactly two subsets in $\mathcal{E}_{i+1}$.

In M, if $\mathbb{T}$ is not well-founded and $\langle (\mathcal{E}_0, \psi_0), (\mathcal{E}_1, \psi_1), \ldots \rangle$ is an infinite path through $\mathbb{T}$, then we get $\varphi = \lim_i \psi_i \in C(X,H)$ using (a)(d) and $Q = \bigcap_i \bigcup \mathcal{E}_i$, which is a non-scattered subset of Y using (b)(d), and (c)(d) implies that $\varphi(f^{-1}\{y\}) \supseteq K$ for all $y \in Q$, so (1) holds.

Now, suppose, in N, that we have Q, φ for which (2) holds; then we construct a path through $\mathbb{T}$. To obtain the ψ_i (all of which must be in M), use the fact that $\{\widetilde{\psi} : \psi \in C(X,H)^M\}$ is dense in $C(\widetilde{X}, \widetilde{H})$. Likewise each $E \in \mathcal{E}_i$ will be a closed set in M such that $\widetilde{E} \cap Q$ is not scattered.

Note that Theorem 7.2 says that the existence of the φ and Q described in the proof Theorem 2.10 is absolute. The corresponding "absoluteness version" of Theorem 2.9 is false. For example, suppose that in V, we have $X = Y \times K$, where X, Y, K are compact and non-scattered, and in addition, K has no non-trivial convergent ω–sequences. Then clearly in V, there can be no perfect $Q \subseteq Y$ and 1-1 map $i : Q \times (\omega + 1) \to X$ such that $f(i(q,u)) = q$ for all $(q, y) \in Q \times (\omega + 1)$, whereas if $V[G]$ collapses enough cardinals, it will contain such a Q, i.

An application of the absoluteness result in Theorem 7.2 is:

Proof of Theorem 2.5. Assume that in the universe, V: X and Y are compact, $f : X \to Y$, and we have an infinite loose family $\{P_i : i \in \omega\}$. Let $V[G]$ be any forcing extension of V which makes the weights of X and Y countable, so that in $V[G]$, we still have $f : \widetilde{X} \to \widetilde{Y}$ and a loose family $\{\widetilde{P_i} : i \in \omega\}$, but $\widetilde{X}$ and $\widetilde{Y}$ are now compact metric, so that Theorem 2.10 gives us an $(\omega+1)$–loose function in $V[G]$. Hence, by absoluteness, there is one in V.

A direct proof of this can be given without forcing, but it seems quite a bit more complicated, since one must embed into the proof the method of Suslin used in proving Lemma 2.8; one cannot just quote Suslin's theorem, since the spaces are not Polish. Theorem 2.5 is needed for the $\kappa = \omega$ part of:

Corollary 7.3 *Fix $\kappa \le \omega$. Let M, N be transitive models of ZFC, with $M \subseteq N$. Assume that in M we have X, Y, f with X, Y compact and $f : X \to Y$. Then $M \models$ "$f : X \to Y$ is κ–tight" iff $N \models$ "$\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ is κ–tight".*

Of course, the $\leftarrow$ direction is trivial, and holds for all κ if we rephrase Definition 2.1 appropriately so that κ is not required to be a cardinal (since "cardinal" is not absolute). That is, if in M, we have a loose family $\{P_\alpha : \alpha < \kappa\}$, then $\{\widetilde{P_\alpha} : \alpha < \kappa\}$ is loose in N. For a version of Corollary 7.3 for $\kappa = \mathfrak{c}$, we use the notion of "weakly $\mathfrak{c}$–tight" from Definition 2.6.

Corollary 7.4 *Fix $\kappa \le \omega$. Let M, N be transitive models of ZFC, with $M \subseteq N$. Assume that in M we have X, Y, f with X, Y compact and $f : X \to Y$. Then $M \models$ "$f : X \to Y$ is weakly $\mathfrak{c}$–tight" iff $N \models$ "$\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ is weakly $\mathfrak{c}$–tight".*

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