

Bohr Compactifications of Non-Abelian Groups ^{*}

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May 23, 2002

Abstract

We investigate properties of the Bohr compactification, $\mathbf{b}G$, and the Bohr topology, $G^\#$, for discrete non-abelian groups G . These properties depend on the algebraic structure of G , and can be analyzed with the aid of the compact space of unitary representations of G . Using our analysis, we show that a compact Hausdorff space is embeddable in some $G^\#$ iff it is Eberlein compact.

1 Introduction

For any discrete group G , its Bohr compactification $\mathbf{b}G$ is its maximal compactification, and $G^\#$ denotes G given the topology induced by its embedding into $\mathbf{b}G$. For abelian groups, the cardinal functions on $\mathbf{b}G$ and $G^\#$ have been known for sixty years; for example, their size and weight are as large as possible. In the non-abelian case, these properties are sensitive to the group-theoretic properties of G .

In this paper, we discuss properties of G which help describe $\mathbf{b}G$ and $G^\#$. Most of our results use the standard theory of representations of compact groups (see [6, 10, 11, 22]). Following von Neumann [18], homomorphisms from G into unitary groups are used to define the *maxap* (maximally almost periodic) and *minap* (minimally almost periodic) groups. We also classify

^{*}2000 Mathematics Subject Classification: Primary 54H11, 22D10; Secondary 54D35.
Key Words and Phrases: Bohr compactification, representation, character.
The authors would like to thank the referee of an original draft of this paper for a number of very helpful comments.

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[§]Author partially supported by NSF Grant DMS-0097881.

groups into *small*, *medium*, and *large* according to the structure of the space of these homomorphisms. In addition, we include results on self-bohrifying groups (Definition 1.13), on compact subspaces of $G^\#$, and on the cardinality and weight of $\mathbf{b}G$.

Before describing our results in more detail, we review some basic terminology.

Definition 1.1 *If G, H are groups, then $\text{Hom}(G, H)$ is the set of all homomorphisms from G to H . If G, H are topological groups, then $\text{Hom}_c(G, H)$ is the set of continuous elements of $\text{Hom}(G, H)$.*

Definition 1.2 *A compactification of the group G is a pair (X, φ) , such that X is a compact group, $\varphi \in \text{Hom}(G, X)$, and $\text{ran}(\varphi)$ is dense in X .*

As usual, “compact group” means “compact *Hausdorff* topological group”. Note that this definition ignores any topology G might have; equivalently, we treat G as a discrete group. One can define a more general notion, where G is an arbitrary topological group, by considering only $\varphi \in \text{Hom}_c(G, X)$, but we do not pursue that here. We make essential use of the fact that $\text{Hom}(G, X)$ is closed in X^G (with the product topology), and hence is compact when X is compact; this is not true in general for $\text{Hom}_c(G, X)$ for non-discrete groups G .

Definition 1.3 *If $(X, \varphi), (Y, \psi)$ are two compactifications of G , then $(X, \varphi) \leq (Y, \psi)$ iff $\varphi = \Gamma \circ \psi$ for some $\Gamma \in \text{Hom}_c(Y, X)$. (X, φ) and (Y, ψ) are equivalent iff $(X, \varphi) \leq (Y, \psi) \leq (X, \varphi)$.*

Observe that the Γ in Definition 1.3 is unique and onto (since the range of a compactification is dense). In the case that (X, φ) and (Y, ψ) are equivalent, Γ is a continuous isomorphism.

Definition 1.4 *The Bohr compactification of G , $(\mathbf{b}G, \Phi_G)$, is the unique (up to equivalence) largest compactification of G in the order \leq . $G^\#$ denotes the group G with the Bohr topology: U is open in $G^\#$ iff $U = (\Phi_G)^{-1}(V)$ for some open $V \subseteq \mathbf{b}G$.*

Note that $G^\#$ is Hausdorff iff Φ_G is 1-1. Also, U is open in $G^\#$ iff $U = \varphi^{-1}(V)$ for some compactification (X, φ) of G and some open $V \subseteq X$.

$\mathbf{b}G$ was first described by Weil [24]. By Holm [12], the existence of a largest compactification holds for arbitrary algebraic structures, not just groups; see [8] for further discussion and references. In the case of groups, one may “compute” $\mathbf{b}G$ explicitly by using (finite dimensional unitary) representations (see Lemma 1.8). As usual, $U(n)$ denotes the group of all $n \times n$ unitary complex matrices.

Definition 1.5 Let G be a topological group. φ is a representation of G iff $\varphi \in \text{Hom}_c(G, U(n))$ for some finite n , called the degree of φ . A linear subspace $E \subseteq \mathbb{C}^n$ is an invariant subspace for $\mathcal{S} \subseteq U(n)$ if $ME \subseteq E$ for all $M \in \mathcal{S}$. If there is a subspace E with $\{0\} \subsetneq E \subsetneq \mathbb{C}^n$ which is invariant for \mathcal{S} , then \mathcal{S} is called reducible; otherwise, \mathcal{S} is irreducible. An irrep (irreducible representation) of G is a $\varphi \in \text{Hom}_c(G, U(n))$ such that $\text{ran}(\varphi)$ is irreducible. Two representations, φ, ψ , are equivalent ($\varphi \sim \psi$) iff there is an $M \in U(n)$ such that $\psi(x) = M^{-1}\varphi(x)M$ for all $x \in G$.

Because \mathbb{C}^n is finite dimensional, every representation decomposes into a sum of irreducible representations.

We plan to use Definition 1.5 in only two cases: If G is a compact group, the Peter-Weyl Theorem (Lemma 1.7) implies that the irreps of G yield a complete analysis of G . If G is a discrete group, the Peter-Weyl Theorem applied to compactifications of G implies that the irreps of G can be used to obtain $\text{b}G$ (Lemma 1.8).

If φ is a representation of the discrete group G , then $(\text{cl}(\text{ran}(\varphi)), \varphi)$ is a compactification of G . Equivalence of φ, ψ as representations implies equivalence as compactifications, but not conversely; for example, for finite G , any two faithful representations of G are equivalent as compactifications. When discussing equivalence of representations, we shall always mean *as* representations, not compactifications.

As in von Neumann [18] (Definition 11), one must make an arbitrary choice of a representation in each equivalence class. We shall index these choices by a (von Neumann) cardinal θ :

Definition 1.6 For any topological group G , let $\theta = \theta_G$ be the number of inequivalent irreps of G . Then, let $\{\varphi_\alpha^G : \alpha < \theta\}$ list one irrep for each equivalence class, and let n_α^G be the degree of φ_α^G . Let $\theta^n = \theta_G^n = |\{\alpha < \theta : n_\alpha^G = n\}|$. Define the evaluation map $\mathcal{E} = \mathcal{E}^G : G \rightarrow \prod_{\alpha < \theta} U(n_\alpha)$ by $(\mathcal{E}(x))_\alpha = \varphi_\alpha(x)$.

We drop the superscript or subscript G when G is clear from context. What we need from the Peter-Weyl Theorem ([6, 11, 22]) is stated by Lemma 1.7:

Lemma 1.7 Let X be a compact group. Then \mathcal{E} is a continuous isomorphism from X into $\prod_\alpha U(n_\alpha)$, and the linear span of the projections of the φ_α onto their matrix elements is uniformly dense in $C(X)$.

Lemma 1.8 Let G be a discrete group. Let $Y = \text{cl}(\text{ran}(\mathcal{E}^G))$. Then (Y, \mathcal{E}^G) is the Bohr compactification of G , and $\theta_Y = \theta_G$, with the irreps of Y being the projections from Y onto the coordinates in $\prod_\alpha U(n_\alpha)$.

Proof. If (X, φ) is any compactification of G , we prove $(X, \varphi) \leq (Y, \mathcal{E}^G)$ by applying the Peter-Weyl Theorem to X . ☺

$w(X)$, the *weight* of X , is the least cardinality of a base for the topology of X . If X is any infinite compact group, then $w(X) = \theta_X$ (by Lemma 1.7), and $|X| = 2^{w(X)}$ (see [2], §3). Hence, by Lemma 1.8:

Lemma 1.9 *Whenever $\mathfrak{b}G$ is infinite and $\theta = \theta_G$: $w(\mathfrak{b}G) = \theta$ and $|\mathfrak{b}G| = 2^\theta$.*

If G is finite, then $\mathfrak{b}G = G$ and θ_G is the number of conjugacy classes of G . Note that $\mathfrak{b}G$ may be finite even when G is infinite.

Definition 1.10 *G is maxap (maximally almost periodic) iff $\ker(\Phi_G) = \{1\}$. G is minap (minimally almost periodic) iff $\ker(\Phi_G) = G$.*

These notions were introduced by von Neumann [18], and $\ker(\Phi_G)$ is called the *von Neumann kernel*; see also Rothman [20]. G is minap iff $|\mathfrak{b}G| = 1$ iff G has only the trivial homomorphism into any compact group – equivalently, into any $U(n)$. So, every simple group of order greater than 2^{\aleph_0} is clearly minap. Less trivially, $SL(k, F)$ and $PSL(k, F)$ are minap whenever F is an infinite field and $k \geq 2$; see Corollaries 5.12 and 5.15.

G is maxap iff G has a 1-1 homomorphism into some compact group. Every abelian group is maxap, as is every group which already has some compact group topology.

Lemma 1.11 *For any G : $G/\ker(\Phi_G)$ is maxap and its Bohr compactification is $(\mathfrak{b}G, \Phi_G/\ker(\Phi_G))$, where $\Phi_G/\ker(\Phi_G) : G/\ker(\Phi_G) \rightarrow \mathfrak{b}G$ is the natural quotient map.*

Thus, to study how $\Phi(G)$ lies within $\mathfrak{b}G$, it is enough to consider maxap groups. When G is maxap, Φ is 1-1, and we may simplify the notation by identifying G with $\Phi(G)$ and regarding G as a dense subgroup of $\mathfrak{b}G$, thereby dropping explicit mention of Φ . The inclusion $G \hookrightarrow \mathfrak{b}G$ is then characterized by the property that each $\varphi \in \text{Hom}(G, Y)$, for any compact Y , extends uniquely to a $\tilde{\varphi} \in \text{Hom}_c(\mathfrak{b}G, Y)$.

One important result relating $\Phi(G)$ with $\mathfrak{b}G$ is:

Theorem 1.12 (Moran [17]) *If for some n , θ_G^n is infinite (see Definition 1.6), then $\Phi(G)$ is a Haar null set in $\mathfrak{b}G$.*

The converse of this result is false. For example, if G is countable and maxap then $\Phi(G)$ is a null set in $\mathbf{b}G$. It is easy to find such G with θ_G^n finite for every n ; in particular, there are examples of the form $G = \sum_{k \in \omega} G_k$, where the G_k are finite groups; see [17] and Section 3. However, by Theorem 1.15, the converse is true when $G = X_d$ for some compact group X .

Definition 1.13 *If X is a topological group, then X_d denotes the same group with the discrete topology. A compact group X is self-bohrifying iff $\mathbf{b}(X_d)$ is the identity map: $X_d \rightarrow X$.*

Actually, by the definitions in this paper, $\mathbf{b}(X_d)$ and $\mathbf{b}X$ have the same meaning, since $\mathbf{b}X$ ignores any topology X might have, but we write $\mathbf{b}(X_d)$ to avoid confusion with terminology elsewhere in the literature. Observe:

Lemma 1.14 *Let X be a compact group. Then the following are equivalent:*

- ✦ X is self-bohrifying.
- ✦ $\text{Hom}(X, Y) = \text{Hom}_c(X, Y)$ for every compact group Y .
- ✦ $\text{Hom}(X, U(n)) = \text{Hom}_c(X, U(n))$ for each n .

A compact abelian group is self-bohrifying iff it is finite, but there are many examples of infinite self-bohrifying non-abelian groups. For example, by van der Waerden ([23]; see also [11], Theorem 5.64), all compact connected semisimple Lie groups are self-bohrifying. A specific example is $SO(3)$, for which there is an easier direct proof which does not use Lie theory (Comfort and Robertson [4]). Further properties of self-bohrifying groups are discussed in this paper; in summary:

Theorem 1.15 *Let X be a compact group. Then the following are equivalent:*

1. X is not a null set in $\mathbf{b}(X_d)$.
2. X is self-bohrifying.
3. $\theta_{X_d}^n < 2^{\aleph_1}$ for each n .
4. $\theta_{X_d}^n$ is finite for each n – that is, X_d has only finitely many inequivalent irreps of each degree.

Moreover, (4) implies:

5. X has only finitely many inequivalent continuous irreps of each degree.

Also, (5) \rightarrow (4) holds when X is a compact Lie group, but not (in general) for profinite groups.

Of course, (2) \rightarrow (1) and (4) \rightarrow (3) are obvious, and (1) \rightarrow (4) is clear by Theorem 1.12. (3) \rightarrow (2) follows from Theorem 3.15. We do not know of any criterion for a general compact group X to be self-bohrifying which is expressed just in terms of the *continuous* representations of X , although (5) provides such a criterion for Lie groups.

The counter-example (Example 7.2) to (5) \rightarrow (4) will also have X' dense in X but not all of X (where X' is the subgroup generated by the commutators $[x, y] = x^{-1}y^{-1}xy$), so that X will have only the trivial continuous representation of degree 1, but infinitely many discontinuous ones. For compact Lie groups, X' is always closed (see [11], Theorem 6.11), and this will allow us (see Lemma 7.1) to derive (5) \rightarrow (4) from van der Waerden's theorem.

Further information about the structure of $\mathbf{b}G$ may be obtained by passing from group representations to characters, using the trace function $\text{tr} : U(n) \rightarrow \mathbb{C}$. A *character* of degree n is of the form $\text{tr} \circ \varphi$, where φ is a representation of degree n .

Definition 1.16 $C_n(G) = \{\text{tr} \circ \varphi : \varphi \in \text{Hom}(G, U(n))\}$ is the space of all characters of G of degree n .

Note that $\text{Hom}(G, U(n))$ is a closed subset of $U(n)^G$, and is hence a compact Hausdorff space. Then, $C_n(G) \subseteq \mathbb{C}^G$ is the image of $\text{Hom}(G, U(n))$ by the continuous trace map, so $C_n(G)$ is also compact Hausdorff. By the following lemma, we may identify $C_n(G)$ with the quotient space $\text{Hom}(G, U(n))/\sim$, where \sim is as in Definition 1.5.

Lemma 1.17 If $\varphi, \psi \in \text{Hom}(G, U(n))$, then $\text{tr} \circ \varphi = \text{tr} \circ \psi$ iff $\varphi \sim \psi$.

Proof. Apply in $\mathbf{b}G$ the fact that for compact groups, two continuous representations are equivalent iff their characters are equal. $\odot\odot$

Lemma 1.18 If $\mathbf{b}G$ is infinite, then $w(\mathbf{b}G) = \sup_n |C_n(G)|$.

Proof. Apply Lemmas 1.9 and 1.17, plus the fact that every character is a sum of irreducible characters. $\odot\odot$

Some of the representation-counting arguments of this paper use the following well-known theorem of Jordan:

Theorem 1.19 (Jordan) There is a function $f : \omega \rightarrow \omega$ such that for all finite $H < U(n)$, there is an abelian $A \triangleleft H$ with $|H : A| \leq f(n)$.

For a proof and a discussion of the growth rate of $f(n)$, see §36 of Curtis and Reiner [5].

Definition 1.20 For a finite group G , $\text{md}(G)$ is the minimal degree of a non-trivial irrep of G .

So, for $|G| \geq 2$, G is *perfect* ($G = G'$) iff $\text{md}(G) \geq 2$. By Jordan's Theorem, $\text{md}(G) > n$ whenever G is finite nonabelian and simple and $|G| > f(n)$.

Since counting characters is the same as counting equivalence classes of representations, we classify groups G in terms of their character spaces:

Definition 1.21 A (discrete) group G is:

- = small iff $C_n(G)$ is finite for all n .
- = medium iff $C_n(G)$ is scattered for all n , and infinite for some n .
- = large iff $C_n(G)$ is non-scattered for some n .

The small groups include all compact self-bohrifying groups (by Moran's Theorem 1.12). No compact group can be medium (Theorem 3.15).

If $G = \sum_n G_n$, where the G_n are finite perfect groups, then G is small if $\liminf_n \text{md}(G_n) = \infty$, and medium otherwise (see Theorem 3.9).

The large groups include all compact non-self-bohrifying G , and all groups G such that $|G : G'|$ is infinite.

In Section 6, we use our structure results to show that a compact Hausdorff space Z can be embedded in some $G^\#$ iff Z is Eberlein compact. By Glicksberg [7], every compact subspace of an abelian $G^\#$ is finite.

In spite of the seemingly set-theoretic nature of some of our definitions, the notions "small", "medium", "large", "minap", and "maxap" are *absolute* algebraic properties of a group G , in that they do not vary with the model of set theory in which G sits. We prove this in Section 8.

2 Subgroups and Quotients

The functorial properties of the Bohr compactification are expressed by:

Lemma 2.1 If $f \in \text{Hom}(G, H)$, then there is unique $\text{bf} \in \text{Hom}_c(\text{b}G, \text{b}H)$ such that the following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{\Phi_G} & \text{b}G \\ f \downarrow & & \downarrow \text{bf} \\ H & \xrightarrow{\Phi_H} & \text{b}H \end{array}$$

Proof. $\mathbf{b}f$ exists because $\Phi_H \circ f$ is a compactification of G and Φ_G is the largest compactification. \odot

Lemma 2.2 *If $f \in \text{Hom}(G, H)$ is onto H , then the $\mathbf{b}f$ in Lemma 2.1 is onto $\mathbf{b}H$ and $\ker \mathbf{b}f = \text{cl}(\Phi_G(\ker f))$.*

Proof. $\mathbf{b}f$ is onto because $\text{ran}(\mathbf{b}f)$ is closed and contains $\Phi_H(\text{ran } f) = \text{ran } \Phi_H$, which is dense.

Let $K = \text{cl}(\Phi_G(\ker f))$. Then $x \in \ker f \Rightarrow \mathbf{b}f(\Phi_G(x)) = 1 \Rightarrow \Phi_G(x) \in \ker \mathbf{b}f$, so $\Phi_G(\ker f) \subseteq \ker \mathbf{b}f$, so $K \subseteq \ker \mathbf{b}f$. Note that $K \triangleleft \mathbf{b}G$. To prove $K = \ker \mathbf{b}f$, consider the following diagram:

$$\begin{array}{ccccc}
 G & \xrightarrow{\Phi_G} & \mathbf{b}G & & \\
 f \downarrow & & \mathbf{b}f \swarrow & & \searrow \pi \\
 H & \xrightarrow{\Phi_H} & \mathbf{b}H & \xleftarrow{\tau} & (\mathbf{b}G)/K \\
 & \searrow \sigma & & &
 \end{array}$$

Here, π is the natural quotient map, as is τ (since $K \leq \ker \mathbf{b}f$). σ exists so that $\sigma \circ f = \pi \circ \Phi_G$ because $\ker f \subseteq \ker(\pi \circ \Phi_G)$. Note that the diagram commutes. Also, $\text{ran } \sigma = \pi(\text{ran } \Phi_G)$, which is dense, so $((\mathbf{b}G)/K, \sigma)$ is a compactification of H and $((\mathbf{b}G)/K, \sigma) \geq (\mathbf{b}H, \Phi_H)$ (because of τ), so that $((\mathbf{b}G)/K, \sigma)$ and $(\mathbf{b}H, \Phi_H)$ are equivalent (because $\mathbf{b}H$ is the largest compactification), so that τ is a bijection, whence $\ker \mathbf{b}f = \ker \pi = K$. \odot

Thus, the Bohr compactification of quotients of G can be computed from the Bohr compactification of G . Lemma 1.11 is a special case of this, where $H = G/\ker(\Phi_G)$.

Now, consider what happens when the f of Lemma 2.1 is 1-1. Here, we can simplify notation by considering G to be a subgroup of H , and replacing f by inclusion $i : G \hookrightarrow H$. One cannot assert that $\mathbf{b}i$ is 1-1. For example, H may be minap and G an abelian subgroup, so that $|\mathbf{b}G| > 1 = |\mathbf{b}H|$. If $\mathbf{b}i$ is 1-1, then we can compute $\mathbf{b}G$ from $\mathbf{b}H$. Since this happens in a number of cases, we give it a name:

Definition 2.3 *G is a \mathbf{b} -faithful subgroup of H iff $\mathbf{b}i : \mathbf{b}G \rightarrow \mathbf{b}H$ is 1-1, where $i : G \hookrightarrow H$ is inclusion.*

Lemma 2.4 *If $G \leq H$ then G is a \mathbf{b} -faithful subgroup iff $(\text{cl}(\Phi_H(G)), \Phi_H \upharpoonright G)$ is (up to equivalence) the Bohr compactification of G .*

The following lemma (a variant of [8], Lemma 2.7.1) restates “b-faithful” in terms of extensions of homomorphisms.

Lemma 2.5 *If $G \leq H$, then G is a b-faithful subgroup iff for every compactification (X, φ) of G , there are compactifications (X_1, φ_1) of G and (Y, ψ) of H such that $(X, \varphi) \leq (X_1, \varphi_1)$, X_1 is a closed subgroup of Y , and ψ extends φ_1 .*

Proof.

\Rightarrow : Let $(X_1, \varphi_1) = (\mathbf{b}G, \Phi_G)$.

\Leftarrow : Let $(X, \varphi) = (\mathbf{b}G, \Phi_G)$, the largest compactification, so we may assume that $(X_1, \varphi_1) = (\mathbf{b}G, \Phi_G)$. Now, $\mathbf{b}i$ is 1-1 by the following diagram:

$$\begin{array}{ccc}
 G & \xrightarrow{\Phi_G} & \mathbf{b}G \\
 \downarrow i & & \downarrow \mathbf{b}i \\
 H & \xrightarrow{\psi} & Y \\
 & \searrow \Phi_H & \vdots \\
 & & \mathbf{b}H
 \end{array}$$



For example, if H is abelian, then, as is well-known, every subgroup is b-faithful. To prove this using Lemma 2.5: X is compact abelian, so we may assume that $X \subseteq \mathbb{T}^\kappa$ for some κ (where $\mathbb{T} = U(1)$). Take $(X_1, \varphi_1) = (X, \varphi)$; then ψ exists because \mathbb{T}^κ is divisible.

One cannot always take $(X_1, \varphi_1) = (X, \varphi)$, even for finite groups. For example, let H be simple and G a proper non-simple subgroup, with φ non-trivial and not 1-1.

A direct sum is a b-faithful subgroup of a direct product. We use the following notation for sums and products:

Definition 2.6 *If $h \in \prod_{\alpha < \kappa} K_\alpha$, then $\text{supt}(h) = \{\alpha : h(\alpha) \neq 1\}$, and $\sum_{\alpha < \kappa} K_\alpha = \{h \in \prod_{\alpha < \kappa} K_\alpha : |\text{supt}(h)| < \aleph_0\}$.*

Lemma 2.7 *Suppose that $G = \sum_{\alpha < \kappa} K_\alpha$, and $H = \prod_{\alpha < \kappa} K_\alpha$. Then for every compactification (X, φ) of G , there is a compactification (X, ψ) of H such that ψ extends φ . Hence, G is a b-faithful subgroup of H .*

Proof. If $s \in [\kappa]^{<\omega}$ and $h \in H$, define $h|s \in G$ so that $(h|s)(\alpha)$ is $h(\alpha)$ for $\alpha \in s$ and 1 for $\alpha \notin s$. Let \mathcal{V} be an ultrafilter on the set $[\kappa]^{<\omega}$ such that $\{s : \alpha \in s\} \in \mathcal{V}$ for all $\alpha < \kappa$. Then, let $\psi(h) = \mathcal{V}\text{-lim}_s \varphi(h|s)$. ☺

Also, by Theorem 2.7.3 of [8], G is a \mathbf{b} -faithful subgroup whenever G is algebraically closed in H . This implies that every group has a countable \mathbf{b} -faithful subgroup. It can also be used to derive Lemma 2.7, since the direct sum is algebraically closed in the direct product.

Lemma 2.8 *Assume that $G \leq H$ and $|H : G|$ is finite. Then G is a \mathbf{b} -faithful subgroup of H , $\text{cl}(\Phi_H(G))$ is clopen in $\mathbf{b}H$, and $|\mathbf{b}H : \text{cl}(\Phi_H(G))| = |H : G|$.*

Proof. Let $n = |H : G|$, and list the right cosets of G as Ga_1, \dots, Ga_n , where $a_1 = 1$. Define $\theta : H \rightarrow S_n$ so that $(i)(\theta(h))$ is the j such that $Ga_i h = Ga_j$. As usual, for $\sigma, \tau \in S_n$, $\sigma\tau$ means “do σ , then τ ”, so that $\theta \in \text{Hom}(H, S_n)$. Let N be the core, $N = \ker \theta$. Then $N \triangleleft H$ and $N \triangleleft G$. Let $R = \text{ran } \theta$; then (R, θ) is a finite compactification of H . Let $\Gamma \in \text{Hom}_c(\mathbf{b}H, R)$ with $\Gamma \circ \Phi_H = \theta$. Then the $\Gamma^{-1}\{\sigma\}$ for $\sigma \in R$ partition $\mathbf{b}H$ into clopen sets. Since $\Phi_H(H)$ is dense in $\mathbf{b}H$, $\Phi_H(N)$ is dense in $\Gamma^{-1}\{1\}$. From this, it is easily seen that $\text{cl}(\Phi_H(G)) = \Gamma^{-1}(\theta(G))$ is clopen in $\mathbf{b}H$, and $|\mathbf{b}H : \text{cl}(\Phi_H(G))| = |R : \theta(G)| = |H : G|$.

To prove that G is a \mathbf{b} -faithful subgroup, we apply Lemma 2.5, so fix a compactification (X, φ) of G , and we obtain (X_1, φ_1) and (Y, ψ) by the method of induced representations. Consider X to be contained in the group ring $\mathbb{Z}X$, and let $M_n(\mathbb{Z}X)$ be the ring of all $n \times n$ matrices of elements of $\mathbb{Z}X$. Let $P_n(X) \subseteq M_n(\mathbb{Z}X)$ be the group of all $n \times n$ “permutation matrices” over X ; $B \in P_n(X)$ iff every row and every column of B has all entries equal 0, except for one entry, which is a member of X . For example, $\begin{pmatrix} 0 & x & 0 \\ y & 0 & 0 \\ 0 & 0 & z \end{pmatrix} \in P_3(X)$ when $x, y, z \in X$. Note that P_n is a compact group homeomorphic to a sum of $n!$ copies of X^n .

Define $\psi : H \rightarrow P_n(X)$ so that $(\psi(h))_{ij} = 0$ unless $Ga_i h = Ga_j$, in which case $(\psi(h))_{ij} = \varphi(a_i h a_j^{-1})$. Let $Y = \text{cl}(\text{ran } \psi)$. Let $\varphi_1 = \psi|_G$ and $X_1 = \text{cl}(\text{ran}(\varphi_1))$, so clearly ψ extends φ_1 . Also, $(X, \varphi) \leq (X_1, \varphi_1)$ via the map $\Delta : X_1 \rightarrow X$ defined by $\Delta(B) = b_{11}$; to see that $\Delta(\psi(g)) = \varphi(g)$ when $g \in G$, recall that $a_1 = 1$, so that $(\psi(g))_{11} = \varphi(g)$. $\odot\odot$

In the usual theory of induced representations (see Chapter 6 of [6] or §12.5 of [21]), $X \leq U(k)$ for some k , in which case $P_n(X)$ may be identified with a subgroup of $U(nk)$, so we get a representation of H of degree nk . Lemma 2.8 can fail when $|H : G|$ is infinite – even when G, H are compact groups with $G \triangleleft H$. For example, let $H = \prod_n SL(2, p_n)$, where the p_n are distinct odd primes, and let $G = Z(H)$, which is an infinite product of 2-element groups. Then H is self-bohrifying (by Lemma 5.20), so G is not a \mathbf{b} -faithful subgroup of H ; in fact, $|\mathbf{b}H| = |H| = \mathfrak{c} = 2^{\aleph_0}$, and $|\mathbf{b}G| = 2^{2^{\mathfrak{c}}}$.

Finally, each factor in a finite product is \mathbf{b} -faithful:

Lemma 2.9 $\mathbf{b}(G \times H) = \mathbf{b}G \times \mathbf{b}H$, and $\Phi_{G \times H}$ is the product map, $\Phi_G \times \Phi_H$.

This seems to be first due to Holm [12]; see also §2.9 of [8] for another proof and some historical remarks.

3 The Size of $\mathbf{b}G$

Lemma 1.9 easily yields the following bounds for $\mathbf{b}G$:

Lemma 3.1 *Suppose that $|G| = \kappa \geq \aleph_0$ and G is maxap. Then $w(\mathbf{b}G) \leq 2^\kappa$ and $|G| \leq |\mathbf{b}G| \leq 2^{2^\kappa}$.*

When G is abelian, $w(\mathbf{b}G)$ and $|\mathbf{b}G|$ have their largest possible values:

Theorem 3.2 (Kakutani [15]) *If $|G| = \kappa \geq \aleph_0$ and G is abelian, then $w(\mathbf{b}G) = 2^\kappa$ and $|\mathbf{b}G| = 2^{2^\kappa}$.*

Lemma 3.3 *If G is countable and maxap, then $w(\mathbf{b}G)$ is either \aleph_0 or 2^{\aleph_0} .*

Proof. Apply Lemma 1.18. Each $C_n(G)$ is compact and second countable, so it is either countable or of size 2^{\aleph_0} . ☺☺

This lemma is also Corollary 2.10.20 of [8], which gives a longer proof which is valid for Bohr compactifications of general structures. Since a compact second countable space is scattered iff it is countable, a countable G will be large when $w(\mathbf{b}G) = 2^{\aleph_0}$, and either medium or small when $w(\mathbf{b}G) = \aleph_0$.

For countable maxap G , we do not know a simple criterion for deciding whether $w(\mathbf{b}G)$ is \aleph_0 or 2^{\aleph_0} , but the following lemmas will provide some partial information. First, we may apply Theorem 3.2 to the abelian group G/G' . Since representations of G/G' yield representations of G , we get:

Lemma 3.4 *If G is countable and $|G : G'|$ is infinite, then $w(\mathbf{b}G) = 2^{\aleph_0}$.*

The converse is false, since $w(\mathbf{b}G)$ might be large due to irreps of degree larger than 1:

Example 3.5 *There is a countable H with $H' = H$ and $w(\mathbf{b}H) = 2^{\aleph_0}$.*

Proof. Suppose that $H = G \rtimes K$, where K is a finite group and G is a countably infinite abelian group. Then G and H are maxap and $w(\mathfrak{b}H) = w(\mathfrak{b}G) = 2^{\aleph_0}$ (applying Lemma 2.8).

To make $H' = H$: Let $K = A_5$. Let $G = \{a \subset \omega \times \{1, 2, 3, 4, 5\} : |a| < \aleph_0 \ \& \ \forall n[a_n \text{ is even}]\}$. Here, $a_n = \{i : (n, i) \in a\}$, the group operation is symmetric difference (so G is a boolean group), and the action of K on G is defined by: $a^\sigma = \{(n, i\sigma) : (n, i) \in a\}$. View K, G as subgroups of H , so $H = GK$. H' contains K because $K' = K$. H' contains G because H' contains all elements of G of the form $\{(n, i), (n, j)\} = [(k, j, i), \{(n, i), (n, k)\}]$, and these elements generate G . Thus, $H' = H$. ☺

The converse of Lemma 3.4 is true (Theorem 3.9) in the case that G is a direct sum of finite groups. First, observe, by a compactness argument:

Lemma 3.6 *If $\mathcal{S} \subseteq U(n)$ is irreducible, then some finite $\mathcal{S}_0 \subseteq \mathcal{S}$ is irreducible.*

Hence, every representation of a direct sum of perfect groups has finite support:

Lemma 3.7 *Assume that $K = \sum_{\alpha < \kappa} H_\alpha$, where $H_\alpha = (H_\alpha)'$ for all but finitely many α . Fix $\varphi \in \text{Hom}(K, U(n))$. Then there is a finite $F \subseteq \kappa$ such that $\varphi(x) = \varphi(y)$ for all $x, y \in K$ such that $x \upharpoonright F = y \upharpoonright F$.*

Proof. Since φ can be decomposed into irreducible representations, we may (and shall) assume that φ itself is irreducible. Then, by Lemma 3.6, we can fix a finite F such that if $K_F = \sum_{\alpha \in F} H_\alpha$, then $\varphi \upharpoonright K_F$ is irreducible. We may also assume that $H_\alpha = (H_\alpha)'$ for all $\alpha \notin F$.

Now $K \cong K_F \times K^F$, where $K^F = \sum_{\alpha \notin F} H_\alpha$; so the factors K_F and K^F correspond to subgroups of K . Irreducibility of $\varphi \upharpoonright K_F$, plus the fact that $\varphi(x)\varphi(y) = \varphi(y)\varphi(x)$ whenever $x \in K_F$ and $y \in K^F$, implies that $\varphi(y) = \lambda_y I$ for all $y \in K^F$, where $\lambda_y \in \mathbb{C}$ (see [10], Theorem 21.30). Then $(y \mapsto \lambda_y)$ is a homomorphism from K^F to $U(1)$; but $(K^F)' = K^F$, so $\lambda_y = 1$ for all $y \in K^F$.

☺

Lemma 3.8 *Let Φ_Σ , Φ_Π be the natural maps from $\sum_{\alpha < \kappa} G_\alpha$, $\prod_{\alpha < \kappa} G_\alpha$, respectively, into the compact group $\prod_{\alpha < \kappa} \mathfrak{b}G_\alpha$. Then*

1. Φ_Σ is the Bohr compactification iff $G'_\alpha = G_\alpha$ for all but finitely many α
2. Φ_Π is the Bohr compactification iff Φ_Σ is the Bohr compactification and $\mathfrak{b}(\prod_{\alpha < \kappa} G_\alpha / \sum_{\alpha < \kappa} G_\alpha) = \{1\}$.

Proof. (1) \rightarrow : Choose $\varphi_\alpha \in \text{Hom}(G_\alpha, U(1))$ such that φ_α is non-trivial whenever $G'_\alpha \neq G_\alpha$. Define $\varphi \in \text{Hom}(\sum_\alpha G_\alpha, U(1))$ so that $\varphi(x) = \prod_\alpha \varphi_\alpha(x_\alpha)$. Then φ cannot be lifted continuously to $\prod_{\alpha < \kappa} \mathbf{b}G_\alpha$ unless $\varphi_\alpha = 1$ for all but finitely many α .

(1) \leftarrow : By Lemmas 3.7 and 2.9.

(2) \rightarrow : Φ_Σ is the Bohr compactification because $\sum_\alpha G_\alpha$ is \mathbf{b} -faithful in $\prod_\alpha G_\alpha$ (Lemma 2.7). If $\pi : \prod_\alpha G_\alpha \rightarrow \prod_\alpha G_\alpha / \sum_\alpha G_\alpha$ is the quotient map, then $\mathbf{b}\pi : \mathbf{b}\prod_\alpha G_\alpha = \prod_\alpha \mathbf{b}G_\alpha \rightarrow \mathbf{b}(\prod_\alpha G_\alpha / \sum_\alpha G_\alpha)$ and $\ker(\mathbf{b}\pi) = \prod_\alpha \mathbf{b}G_\alpha$ by Lemma 2.2, so that $\mathbf{b}(\prod_\alpha G_\alpha / \sum_\alpha G_\alpha) = \{1\}$.

(2) \leftarrow : Consider any $\varphi \in \text{Hom}(\prod_\alpha G_\alpha, U(n))$. Applying (1) \leftarrow and Lemma 3.7, we get a finite $F \subseteq \kappa$ such that $\varphi(x) = 1$ whenever $\text{supt}(x) \cap F = \emptyset$ and $x \in \sum_\alpha G_\alpha$. But then also $\varphi(x) = 1$ for all $x \in \prod_\alpha G_\alpha$ such that $\text{supt}(x) \cap F = \emptyset$; otherwise, φ would define a non-trivial homomorphism from $\prod_{\alpha \in (\kappa \setminus F)} G_\alpha / \sum_{\alpha \in (\kappa \setminus F)} G_\alpha = \prod_{\alpha \in \kappa} G_\alpha / \sum_{\alpha \in \kappa} G_\alpha$ into $U(n)$. Now, proceed as in (1) \leftarrow . \odot

The proof of (2) \leftarrow is a modification of an argument in Moran [17].

Theorem 3.9 *Suppose that $G = \sum_{n \in \omega} G_n$, where each G_n is finite. Then G is maxap and the following are equivalent:*

1. $\mathbf{b}G = \prod_n G_n$ (with the natural inclusion).
2. $w(\mathbf{b}G) = \aleph_0$.
3. $|G : G'|$ is finite.
4. $(G_n)' = G_n$ for all but finitely many n .

Furthermore, if these conditions hold, then G is small iff $\liminf_n \text{md}(G_n) = \infty$ (see Definition 1.20), and medium otherwise.

Proof. (1) \rightarrow (2) is obvious, (2) \rightarrow (3) is by Lemma 3.4, (3) \leftrightarrow (4) is easy, and (4) \rightarrow (1) is by Lemma 3.8. \odot

Now, we turn to the question of computing $w(\mathbf{b}(X_d))$ when X is an arbitrary compact metric group. When X is self-bohrifying, $w(\mathbf{b}(X_d)) = \aleph_0$. If not, we shall show (Theorem 3.15) that $w(\mathbf{b}(X_d)) \geq 2^{\aleph_1}$.

Note that if X is not self-bohrifying, then it has a discontinuous representation, and hence a discontinuous character by:

Lemma 3.10 *If X is a topological group, $\varphi \in \text{Hom}(X_d, U(n))$, and $\chi = \text{tr} \circ \varphi$ is continuous, then φ is continuous.*

Proof. Since φ is a homomorphism, it is sufficient to prove continuity at 1. Thus, we fix an open $W \subseteq U(n)$ with $I \in W$, and we must find an open $V \subseteq X$ with $1 \in V$ and $\varphi(V) \subseteq W$. Since I is the unique element of $U(n)$ whose trace is n , we can, by compactness of $U(n)$, fix $\varepsilon > 0$ such that $A \in W$ for all $A \in U(n)$ with $|\text{tr}(A) - n| < \varepsilon$. Then, let $V = \{x \in X : |\chi(x) - n| < \varepsilon\}$; V is open because χ is continuous. \odot

Lemma 3.11 *Suppose that G is any group and $\varphi \in \text{Hom}(G, U(n))$. Then there is a countable $A \subseteq G$ such that for all $\psi \in \text{Hom}(G, U(n))$, if $\psi \upharpoonright A = \varphi \upharpoonright A$ and $\psi \sim \varphi$, then $\psi = \varphi$.*

Proof. For $x \in G$, let $W_x = \{M \in U(n) : M^{-1}\varphi(x)M \neq \varphi(x)\}$. Since W_x is open and $U(n)$ is second countable, we may choose a countable $A \subseteq G$ such that $\bigcup_{x \in A} W_x = \bigcup_{x \in G} W_x$. \odot

For many topological groups, all measurable homomorphisms are continuous; for example, Theorem 22.18 of Hewitt and Ross [10] implies:

Lemma 3.12 *If X, Y are compact groups, and $\varphi \in \text{Hom}(X, Y)$ is such that $\varphi^{-1}(V)$ is Haar measurable for all open $V \subseteq Y$, then φ is continuous.*

Lemma 3.13 *Assume that X is a compact metric group, $\chi \in C_n(X_d)$ is not continuous, and A is a countable subset of X . Then there is an $\eta \in C_n(X_d)$ such that $\eta \upharpoonright A = \chi \upharpoonright A$ but $\eta \neq \chi$.*

Proof. Let $\chi = \text{tr} \circ \varphi$; then φ is not continuous. By Lemma 3.11 (and expanding A), we may assume that for all $\psi \in \text{Hom}(X, U(n))$, if $\psi \upharpoonright A = \varphi \upharpoonright A$ and $\psi \neq \varphi$, then $\psi \not\sim \varphi$ so that $\text{tr} \circ \psi \neq \text{tr} \circ \varphi$ by Lemma 1.17. It is now sufficient to find a $\psi \in \text{Hom}(X, U(n))$ such that $\psi \upharpoonright A = \varphi \upharpoonright A$ and $\psi \neq \varphi$. So, assume that there is no such ψ . We show that (the graph of) φ is a G_δ set in $X \times U(n)$, which, by Lemma 3.12, implies that φ is continuous, yielding a contradiction.

We approximate $\varphi \subseteq X \times U(n)$ by open sets W_k as follows: List A as $\{a_\ell : \ell \in \omega\}$. \mathbb{C}^n has the usual Hilbert space norm, and $\|\cdot\|$ denotes the operator norm on $M_n(\mathbb{C}) \cong \text{Hom}(\mathbb{C}^n, \mathbb{C}^n)$. For any $k \geq 1$, let

$$\begin{aligned} V_k = \{ & (x_0, \dots, x_k, M_0, \dots, M_k) \in X^{k+1} \times U(n)^{k+1} : \forall i, j, \ell \leq k : \\ & x_i = x_j \rightarrow \|M_i - M_j\| < 2^{-k} \quad \& \\ & x_\ell = x_i x_j^{-1} \rightarrow \|M_\ell - M_i M_j^{-1}\| < 2^{-k} \quad \& \\ & x_i = a_j \rightarrow \|M_i - \varphi(a_j)\| < 2^{-k} \} \quad . \end{aligned}$$

Then V_k is open. Let

$$W_k = \{(x_0, M_0) \in X \times U(n) : \\ \forall x_1 \cdots x_k \exists M_1 \cdots M_k (x_0, \dots, x_k, M_0, \dots, M_k) \in V_k\} .$$

Then W_k is open because projection from a compact factor is both an open and a closed map. Thus, the lemma will follow if we show that $\varphi = \bigcap_k W_k$.

To see that $\varphi \subseteq \bigcap_k W_k$, note that each $(x_0, \dots, x_k, \varphi(x_0), \dots, \varphi(x_k)) \in V_k$, so that $(x_0, \varphi(x_0)) \in W_k$.

To see that $\varphi \supseteq \bigcap_k W_k$. Fix $(x_0, M_0) \in \bigcap_k W_k$. We construct a $\psi \in \text{Hom}(X, U(n))$ such that $\psi(x_0) = M_0$ and $\psi \upharpoonright A = \varphi \upharpoonright A$. Then, $\psi = \varphi$ by our assumed uniqueness of φ , so that $(x_0, M_0) \in \varphi$.

Let $S = \bigcup \{X^k : 1 \leq k < \omega\}$. For $s = (x_1, \dots, x_k) \in S$, we use $\text{ran}(s)$ to denote the set $\{x_1, \dots, x_k\}$, and define $s \leq t$ iff $\text{ran}(s) \subseteq \text{ran}(t)$. Let \mathcal{D} be an ultrafilter on S such that $\{t : t \geq s\} \in \mathcal{D}$ for all $s \in S$.

For $s = (x_1, \dots, x_k) \in S$, choose $(M_1, \dots, M_k) = (M_1^s, \dots, M_k^s)$ such that $(x_0, x_1, \dots, x_k, M_0, M_1, \dots, M_k) \in V_k$. For $s = (x_1, \dots, x_k) \in S$ and $z \in \text{ran}(s)$, choose $m = m_{s,z}$ such that $x_m = z$, and then let $f(s, z) = M_{m_{s,z}}^s$.

Now, let $\psi(z) = \mathcal{D}\text{-lim}_s f(s, z)$. The three clauses in the definition of V_k are used, respectively, to verify that $\psi(x_0) = M_0$, ψ is a homomorphism, and $\psi \upharpoonright A = \varphi \upharpoonright A$. \odot

Lemma 3.14 *If X is compact and not second countable, then there is a closed $N \triangleleft X$ such that X/N is second countable and not self-bohrifying.*

Proof. Fix n such that there are uncountably many inequivalent continuous irreps of degree n , and then choose an ω -sequence of these, $\{\varphi_i : i < \omega\}$. Let $N = \bigcap_i \ker(\varphi_i)$. X/N is second countable because it is embedded in $U(n)^\omega$, and X/N is not self-bohrifying by Moran's Theorem 1.12. \odot

Theorem 3.15 *If X is compact and not self-bohrifying, then $w(\mathbf{b}(X_d)) \geq 2^{\aleph_1}$ and X is large.*

Proof. By Lemma 3.14, we may assume that X is second countable (if not, work with a suitable quotient). By Lemma 3.10, fix a discontinuous character, η . Since X is second countable, we can fix a countable B such that $\eta \upharpoonright B$ is not continuous. Let $Z = \{\chi \in C_n(X) : \chi \upharpoonright B = \eta \upharpoonright B\}$. Then Z is compact Hausdorff and non-empty. By Lemma 3.13, applied to arbitrary countable $A \supseteq B$, we see that no point in Z has a countable neighborhood base. Thus, by the Čech - Pospíšil Theorem, $|Z| \geq 2^{\aleph_1}$, and hence $w(\mathbf{b}(X_d)) \geq 2^{\aleph_1}$ by Lemma 1.18. Also, Z shows that $C_n(X)$ is not scattered, so X is large. \odot

4 Self-bohrifying groups

We collect here some consequences of the previous results for the self-bohrifying groups.

Lemma 4.1 *If X is self-bohrifying and $N \triangleleft X$ is closed, then X/N is self-bohrifying.*

Lemma 4.2 *If X is self-bohrifying, then every subgroup of finite index is clopen.*

Lemma 4.3 *If $Y \leq X$ are compact groups with $|X : Y|$ finite, then Y is self-bohrifying iff X is.*

Proof. By Lemma 2.8. ☺

Applied to X' , this yields \Leftarrow of the following lemma:

Lemma 4.4 *X is self-bohrifying iff X' is clopen in X , $|X : X'| < \aleph_0$, and X' is self-bohrifying.*

Proof. For \Rightarrow : $X' = \bigcap \{\ker \varphi : \varphi \in \text{Hom}(X, U(1))\} = \bigcap \{\ker \varphi : \varphi \in \text{Hom}_c(X, U(1))\}$, which is closed. If $|X : X'|$ were infinite, then $X/(X')$ would be an infinite compact abelian group, which cannot be self-bohrifying, contradicting Lemma 4.1. Since X' is closed and of finite index, it is also open. Finally, X' is self-bohrifying by Lemma 4.3. ☺

Thus, if X is self-bohrifying, the derived series, $X \geq X' \geq X'' \geq X''' \dots$, yields a descending sequence of clopen self-bohrifying subgroups. We do not know if this series can be infinite. By Lemma 4.5, it cannot be infinite when X is self-bohrifying and a product of finite groups. Every finite length is possible, even for finite groups.

Lemma 4.5 *Let X_n , for $n \in \omega$ be compact groups and $X = \prod_{n \in \omega} X_n$. Then X is self-bohrifying iff all of the following hold:*

- ☞ *Each X_n is self-bohrifying.*
- ☞ *$X'_n = X_n$ for all but finitely many n .*
- ☞ *$\prod_n X_n / \sum_n X_n$ is minap.*

This is immediate from Lemma 3.8. In particular, if an X of this form is self-bohrifying with all X_n finite, then X must be of the form $X = Y \times G$, where G is finite and $Y' = Y$, so that the derived length of X is finite (and equal to the derived length of G).

By this lemma, the main difficulty in proving such a product to be self-bohrifying is in checking that the quotient is minap, which we take up next.

5 Some Minimally Almost Periodic Groups

We describe some general methods for proving that a group is minap. For many groups, such as $SL(n, F)$ and $PSL(n, F)$, where F is any infinite field, infinite “bad” subsets kill off all nontrivial homomorphisms into the unitary groups. To see that certain groups of the form $\prod_n G_n / \sum_n G_n$ are minap, we employ a stronger notion, “ugly”, which has the advantage that it is productive.

To determine whether $T \subseteq G$ is “bad” for G , we consider $\mathcal{N}_\omega(T)$, the normal subgroup T generates:

Definition 5.1 *If T is a subset of the group G , then $\mathcal{N}_k(T)$ is the set of all elements of G of the form $(a_1^{-1}z_1a_1)(a_2^{-1}z_2a_2)\cdots(a_k^{-1}z_ka_k)$, where each $a_1, \dots, a_k \in G$ and each $z_i \in T \cup T^{-1} \cup \{1\}$. Let $\mathcal{N}_\omega(T) = \bigcup_{k \in \omega} \mathcal{N}_k(T)$. $\mathcal{N}_\omega(x) = \mathcal{N}_\omega(\{x\})$ and $\mathcal{N}_k(x) = \mathcal{N}_k(\{x\})$ for $x \in G$.*

So $\mathcal{N}_\omega(T)$ is the least normal subgroup containing T . Generating elements in a more uniform way still produces a normal subgroup of G , and also makes working with products of groups easier:

Definition 5.2 *If T is a subset of the group G , then $\mathcal{N}_2^u(T)$ is the set of all elements of the form $(a_1^{-1}z_1a_1)(a_2^{-1}z_2^{-1}a_2)$ such that a_1, a_2 in G and $z_1, z_2 \in T$; i.e., $\mathcal{N}_2^u(T) = \bigcup_{a_1, a_2 \in G} (a_1^{-1}Ta_1)(a_2^{-1}T^{-1}a_2)$. Let $\mathcal{N}_0^u(T) = \{1\}$ and let $\mathcal{N}_{2k}^u(T)$ be the set of all products of exactly k elements of $\mathcal{N}_2^u(T)$. Let $\mathcal{N}_\omega^u(T) = \bigcup_{k \in \omega} \mathcal{N}_{2k}^u(T)$. $\mathcal{N}_\omega^u(x) = \mathcal{N}_\omega^u(\{x\})$ and $\mathcal{N}_{2k}^u(x) = \mathcal{N}_{2k}^u(\{x\})$ for $x \in G$.*

Observe that each $\mathcal{N}_{2k}^u(T)$ is closed under inverses and conjugations, and that $\mathcal{N}_0^u(T) \subseteq \mathcal{N}_2^u(T) \subseteq \mathcal{N}_4^u(T) \cdots$. It follows that $\mathcal{N}_\omega^u(T)$ is a normal subgroup of G , although it need not contain any elements of T ; in abelian groups, $\mathcal{N}_\omega^u(x) = \{1\}$, while $\mathcal{N}_\omega(x) = \langle x \rangle$. Also note that $\mathcal{N}_{2k}^u(T) \subseteq \mathcal{N}_{2k}(T)$. We remark that putting the “ $(a_2^{-1}z_2^{-1}a_2)$ ” before the “ $(a_1^{-1}z_1a_1)$ ” in the definition of $\mathcal{N}_2^u(T)$ results in an equivalent notion because $(a_2^{-1}z_2^{-1}a_2)(a_1^{-1}z_1a_1) = (a_1^{-1}z_1a_1)(b_2^{-1}z_2^{-1}b_2)$, where $b_2 = a_2a_1^{-1}z_1a_1$.

Definition 5.3 *$S \subseteq G$ is a k -bad set for G iff $G = \mathcal{N}_k(x^{-1}y)$ whenever x, y are distinct elements of S . G is (λ, k) -bad iff there is a k -bad set of size $\geq \lambda$. $S \subseteq G$ is a $2k$ -ugly set for G iff $G = \mathcal{N}_{2k}^u(x^{-1}y)$ whenever x, y are distinct elements of S . G is $(\lambda, 2k)$ -ugly iff there is a $2k$ -ugly set of size $\geq \lambda$.*

Lemma 5.4 *If G is $(\lambda, 2k)$ -ugly, then G is $(\lambda, 2k)$ -bad.*

First, we use badness to see that some groups are minap:

Lemma 5.5 *If $A \in U(n)$ and $A \neq I$, then $\|A^\ell - I\| \geq \sqrt{3}$ for some ℓ .*

Proof. Diagonalize A . ☺

As before, we are using the operator norm on $U(n)$.

Lemma 5.6 *There is a function $f : \omega \times \omega \rightarrow \omega$ such that for all groups G and all k : If G is $(f(n, k), k)$ -bad, then $\text{Hom}(G, U(n)) = \{1\}$.*

Proof. Choose $f(n, k)$ so that whenever $M_i \in U(n)$ for $i = 1, 2, \dots, f(n, k)$, there are distinct i, j such that $\|M_i - M_j\| \leq 1/k$. Now, fix $\varphi \in \text{Hom}(G, U(n))$; we show that φ is trivial. Let S be a k -bad set of size $f(n, k)$. Then we may fix distinct $x, y \in S$ such that $\|\varphi(y) - \varphi(x)\| \leq 1/k$. Hence, $\|\varphi(x^{-1}y) - I\| \leq 1/k$, so that $\|\varphi((a_1^{-1}z_1a_1)(a_2^{-1}z_2a_2) \cdots (a_k^{-1}z_ka_k)) - I\| \leq 1$, whenever $a_1, \dots, a_k \in G$ and $z_1, \dots, z_k \in \{1, x^{-1}y, y^{-1}x\}$. Thus, $\|M - I\| \leq 1$ for all $M \in \varphi(G)$, which implies that $\varphi(G) = \{I\}$ by Lemma 5.5. ☺

Corollary 5.7 *If G is (\aleph_0, k) -bad for some finite k , then $\text{b}G = \{1\}$.*

This will apply directly to $SL(n, F)$ and $PSL(n, F)$, but there is a problem with applying it to products such as $\prod_n G_n / \sum_n G_n$ because the notion of “bad” is not productive. For example, in the symmetric group S_n , the set $\{I, (1\ 2)\}$ is $(n-1)$ -bad, since every element of S_n is a product of $(n-1)$ or fewer conjugates of the transposition $(1\ 2)$. However, $S_n \times S_n$ is not $(2, k)$ -bad for any k because every element of $S_n \times S_n$ lies in a proper normal subgroup – either $A_n \times S_n$ or $S_n \times A_n$ or $\{(x, y) : xy \in A_n\} = \mathcal{N}_\omega(((1\ 2), (1\ 2)))$.

On the other hand, it is easily seen that any product of $(\lambda, 2k)$ -ugly groups is $(\lambda, 2k)$ -ugly. Moreover, we can turn finite λ s into infinite ones as follows:

Lemma 5.8 *Suppose that G_n (for $n \in \omega$) are groups such that for some fixed k : For each finite λ , all but finitely many of the groups G_n are $(\lambda, 2k)$ -ugly. Then $\prod_n G_n / \sum_n G_n$ is $(\aleph_0, 2k)$ -ugly.*

Proof. For each n , let λ_n be the largest $\lambda \leq n$ such that G_n is $(\lambda, 2k)$ -ugly. Let $\{x_n^i : i < \lambda_n\}$ be a $2k$ -ugly set of size λ_n , and let $x_n^i = 1$ for $i \geq \lambda_n$. Let x^i be the equivalence class of the sequence $\langle x_n^i : n < \omega \rangle$ in $\prod_n G_n / \sum_n G_n$. Then $\{x^i : i < \omega\}$ is $2k$ -ugly. ☺

Question 5.9 *Is there a fixed k such that for each finite λ , all but finitely many of the finite nonabelian simple groups are (λ, k) -ugly?*

A “yes” answer would imply that whenever the G_n are distinct finite nonabelian simple groups, $\mathfrak{b}(\prod_n G_n / \sum_n G_n) = \{1\}$, and hence, by Lemma 4.5, $\prod_n G_n$ is self-bohrifying, answering a question of Comfort and Remus [3].

A “yes” answer also would imply (by Lemma 5.6) that for each n , only finitely many finite nonabelian simple groups embed into $U(n)$, but this is true anyway by Jordan’s Theorem (Theorem 1.19).

We shall derive a “yes” answer below for the alternating groups, and for the simple groups of type $PSL(n, F)$ for each fixed n (but the k we derive gets bigger as n gets bigger). The PSL groups are handled easily by:

Lemma 5.10 *Suppose that \mathcal{C} is a class of nonabelian simple groups and every ultraproduct of groups in \mathcal{C} is in \mathcal{C} . Then for some fixed finite k , every group $G \in \mathcal{C}$ is $(|G|, 2k)$ -ugly, with G itself a $2k$ -ugly set.*

Proof. We must prove that $\exists k \forall G \in \mathcal{C} \forall x \in G \setminus \{1\} [\mathcal{N}_{2k}^u(x) = G]$. If not, then for each k , choose $G_k \in \mathcal{C}$ such that $\exists x \in G_k \setminus \{1\} [\mathcal{N}_{2k}^u(x) \neq G_k]$. Let $G = \prod_k G_k / \mathcal{D}$, where \mathcal{D} is some non-principal ultrafilter on ω . Then $G \in \mathcal{C}$ and $\mathcal{N}_\omega^u(x) \neq G$ for some $x \in G \setminus \{1\}$. Since G is simple, $\mathcal{N}_\omega^u(x) = \{1\}$, so $x \in Z(G) = \{1\}$ (since G is nonabelian), a contradiction. $\odot\odot$

In particular, consider groups of the form $SL(n, F)$ (all $n \times n$ matrices of determinant 1 over the field F) and $PSL(n, F) = SL(n, F) / Z(SL(n, F))$. $PSL(n, F)$ is simple except when $|F| \leq 3$ and $n = 2$ (see Scott [21], Theorem 10.8.1). Applying Lemma 5.10 and the fact that $\prod_i PSL(n, F_i) / \mathcal{D} = PSL(n, \prod_i F_i / \mathcal{D})$, we get

Lemma 5.11 *For each n , there is a fixed $k = k_n$ such that every simple group G of the form $PSL(n, F)$ is $(|G|, 2k)$ -ugly, with $2k$ -ugly set G itself.*

One may also prove this lemma and get a bound on k_n by examining the proofs in [21]. Note that $k_n \nearrow \infty$ as $n \nearrow \infty$. Applying 5.11, 5.4, and 5.7:

Corollary 5.12 $\mathfrak{b}(PSL(n, F)) = \{1\}$ whenever F is an infinite field.

We shall see below that the same results hold for groups of the form $SL(n, F)$. The fact that $SL(2, \mathbb{R})$ is minap was proved by von Neumann and Wigner [18, 19] by different methods. The proof in [19] is easily seen to generalize to other fields of characteristic 0; see also Moran [17] and §22.22 of Hewitt and Ross [10]. Actually, Corollary 5.12 follows immediately from its $n = 2$ case, since $PSL(2, F) \leq PSL(n, F)$ and $PSL(n, F)$ is simple. But we need our general “badness” analysis of $PSL(n, F)$ anyway to handle products of finite groups (see Lemma 5.20).

Now, to handle $SL(n, F)$, we need a lemma which lets us derive the badness of G from the badness of G/N when N is finite. This cannot work in general. For example, suppose that $G = N \times H$. Then G cannot be minap because it has a homomorphism onto the compact (finite) group N . So, $H = G/N$ might be (\aleph_0, k) -bad (hence minap), but we cannot conclude that G is (\aleph_0, ℓ) -bad for any finite ℓ . The following lemma postulates the opposite extreme from such a product. We state the “ugly” version, there is also a “bad” version, but the “ugly” one is needed when applying Lemma 5.8.

Lemma 5.13 *Suppose that $N \triangleleft G$, G/N is $(\lambda, 2k)$ -ugly, and whenever $T \subseteq G$ meets every coset of N , $N \subseteq \mathcal{N}_j(T)$. Then G is $(\lambda, 2jk + 2k)$ -ugly.*

Proof. Apply Definition 5.3 to G/N and choose representatives of cosets to get an $S \subseteq G$ such that $|S| = \lambda$ and such that whenever x, y are distinct elements of S , $T := \mathcal{N}_{2k}^u(x^{-1}y)$ contains at least one element from each coset. Now, $N \subseteq \mathcal{N}_j(T) \subseteq \mathcal{N}_{2jk}^u(x^{-1}y)$ and $G = NT$, so $\mathcal{N}_{2jk+2k}^u(x^{-1}y) = G$. ☺

Note that this lemma required only $N \subseteq \mathcal{N}_j(T)$, not $N \subseteq \mathcal{N}_j^u(T)$.

Lemma 5.14 *For each $n \geq 2$, there is a fixed k (depending on n) such that every group G of the form $SL(n, F)$ is $(|G|/n, k)$ -ugly, except for $n = 2$, $|F| \leq 3$.*

Proof. $PSL(n, F) = SL(n, F)/N$, where N is the group of all uI such that $u \in F$ and $u^n = 1$; thus, $|N| \leq n$. So, we may apply Lemmas 5.13 and 5.11 if we can find a j , depending on n but independent of F , so that whenever $T \subseteq G$ meets every coset, $N \subseteq \mathcal{N}_j(T)$.

If $n = 2$, then $N = \{I, -I\}$. Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then $A^2 = -I$ and T contains A or $-A$, so that $N \subseteq \mathcal{N}_2(T)$.

If $n \geq 3$, let $m = |N|$. Then $m \mid n$, and $N = \{I, vI, \dots, v^{m-1}I\}$, where v is a primitive m^{th} root of 1. Write $\Delta(a_1, \dots, a_n)$ for the $n \times n$ diagonal matrix with entries a_1, \dots, a_n . Observe that if π is any permutation of $\{1, 2, \dots, n\}$ then $\Delta(a_1, \dots, a_n)$ is conjugate to $\Delta(a_{1\pi}, \dots, a_{n\pi})$ in $SL(n, F)$, since $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$. Fix r such that $v^r \Delta(v, v^2, v^{-3}, 1, 1, \dots, 1) \in T$. Then $v^r \Delta(v^2, v, v^{-3}, 1, 1, \dots, 1) \in \mathcal{N}_1(T)$. Dividing, and then multiplying and conjugating, we get

$$\begin{aligned} \Delta(v, v^{-1}, 1, 1, 1, \dots, 1, 1) &\in \mathcal{N}_2(T) \\ \Delta(1, v^2, v^{-2}, 1, 1, \dots, 1, 1) &\in \mathcal{N}_4(T) \\ \Delta(1, 1, v^3, v^{-3}, 1, \dots, 1, 1) &\in \mathcal{N}_6(T) \\ &\dots\dots\dots \\ \Delta(1, 1, 1, 1, 1, \dots, v^{n-1}, v^{1-n}) &\in \mathcal{N}_{2n-2}(T) \end{aligned}$$

Multiplying all of these and using $v^n = 1$, we get $vI \in \mathcal{N}_{n(n-1)}(T)$, so that all $v^\ell I$ are in $\mathcal{N}_{nm(n-1)}(T)$. ☺

Corollary 5.15 $\mathfrak{b}(SL(n, F)) = \{1\}$ whenever F is an infinite field.

We now turn to the alternating groups. For the infinite A_κ , the notion of “bad” is irrelevant, but A_κ is easily seen to be minap by a different argument. In general, if for each $x \in G$, there are infinitely many distinct finite nonabelian simple groups H such that $x \in H \leq G$, then G is minap by Jordan’s Theorem 1.19. This clearly applies to the A_κ . However, to handle products of the finite A_n , we need a more detailed analysis:

Lemma 5.16 *Let M be any set of size 2^m . Then the boolean group $(\mathcal{P}(M), \Delta)$ (of order 2^{2^m}) has a subgroup H of order 2^m such that every element of H has size 0 or 2^{m-1} .*

Proof. We may assume that M is the boolean group $\{0, 1\}^m$. Then $\widehat{M} = \text{Hom}(M, \{0, 1\}) \cong M$. Define $F : \widehat{M} \rightarrow \mathcal{P}(M)$ by: $F(\gamma) = \gamma^{-1}\{1\}$, and note that F is a homomorphism and 1-1. ☺

Of course, a similar argument works for infinite compact boolean groups; now each non-empty $F(\gamma)$ has Haar measure $1/2$.

Lemma 5.17 *If $\pi \in S_n$, then there are $\sigma, \tau \in S_n$ with $\pi = \sigma\tau$ and $\sigma^2 = \tau^2 = 1$.*

Proof. Since every permutation is a product of disjoint cycles, we may assume that π is a cycle on some k elements. For k even, use the fact that

$$[(1, 2)(3, 4) \cdots (2\ell - 1, 2\ell)] \cdot [(2, 3)(4, 5) \cdots (2\ell - 2, 2\ell - 1)]$$

is a cycle on $\{1, 2, \dots, 2\ell\}$. For k odd, use the fact that

$$[(1, 2)(3, 4) \cdots (2\ell - 1, 2\ell)] \cdot [(2, 3)(4, 5) \cdots (2\ell, 2\ell + 1)]$$

is a cycle on $\{1, 2, \dots, 2\ell + 1\}$. ☺

Lemma 5.18 *If $\pi \in A_n$, with $n \geq 5$, then there are $\sigma, \lambda, \tau \in A_n$ with $\pi = \sigma\lambda\tau$ and $\sigma^2 = \lambda^2 = \tau^2 = 1$.*

Proof. Let $\pi = \sigma_0 \tau_0$, where $\sigma_0^2 = \tau_0^2 = 1$ and $\sigma_0, \tau_0 \in S_n$. We assume that σ_0 and τ_0 are both odd, since otherwise we are done (taking $\lambda = 1$). If σ_0 and τ_0 are both single transpositions, use the observation that $(a, b)(a, c) = (a, b)(x, y) \cdot (x, y)(a, c)$ (we may again take $\lambda = 1$). Now, suppose that τ_0 is a product of 3 or more disjoint transpositions. Then we can write $\sigma_0 = \sigma \cdot (a, b)$ and $\tau_0 = (c, d) \cdot \tau$, where the four objects a, b, c, d are all distinct, and then let $\lambda = (a, b)(c, d)$. ☺

This lemma is best possible, since in A_7 , $(1, 2, 3, 4, 5, 6, 7)$ is not the product of 2 involutions. Probably the bounds in the next lemma could be improved, but it is good enough for our purposes.

Lemma 5.19 *Suppose that $2^r \leq n < 2^{r+1}$ and $r \geq 3$. Then A_n is $(2^{r-1}, 12)$ -ugly.*

Proof. Let $m = r - 1$, and, in S_n , let M be a set of 2^m disjoint transpositions, and apply Lemma 5.16 to get H ; then $B \in H$ is a subset of M , and let ΠB be the product of the 2^{r-2} transpositions in B . $\Pi B \in A_n$ since $r \geq 3$. Let $\tilde{H} = \{\Pi B : B \in H\}$; then \tilde{H} is a boolean subgroup of A_n isomorphic to H and $|\tilde{H}| = 2^{r-1}$. We show that \tilde{H} is a 12-ugly set.

If x, y are distinct elements of \tilde{H} , then $x^{-1}y \in \tilde{H}$ is a product of 2^{r-2} disjoint transpositions. Thus, the conjugates of $x^{-1}y$ contain every element of A_n which is a product of 2^{r-2} disjoint transpositions. $x^{-1}y$ has order 2, so $\mathcal{N}_2^u(x^{-1}y)$ contains every element of A_n which is a product of 2^{r-1} or fewer disjoint transpositions, since every such element is of the form

$$\pi_1 \pi_2 \cdots \pi_{2s} = (\pi_1 \pi_2 \cdots \pi_s \tau_1 \tau_2 \cdots \tau_{2^{r-2}-s}) \cdot (\pi_{s+1} \pi_{s+2} \cdots \pi_{2s} \tau_1 \tau_2 \cdots \tau_{2^{r-2}-s})$$

where $1 \leq s \leq 2^{r-2}$ and $\pi_1, \pi_2, \dots, \pi_{2s}, \tau_1, \tau_2, \dots, \tau_{2^{r-2}-s}$ are disjoint transpositions. Since $n/2 \leq 2^r$, every element of A_n of order 2 is in $\mathcal{N}_4^u(x^{-1}y)$. Hence, by Lemma 5.18, $\mathcal{N}_{12}(x^{-1}y) = A_n$. ☺

Applying the above with Lemma 4.5:

Lemma 5.20 *If $\langle G_k : k \in \omega \rangle$ is a list of finite groups, then $\prod_k G_k / \sum_k G_k$ is minap whenever all three of the following hold:*

1. No group is listed infinitely often.
2. G_k is either A_{k_n} or $PSL(j_k, q_k)$ or $SL(j_k, q_k)$.
3. $\sup_k j_k < \infty$.

In this case, $\prod_k G_k$ is self-bohrifying.

We do not know the extent to which the hypotheses can be weakened. Clearly, (1) is necessary. We do not know if (3) can be dropped; possibly it can be by modifying the argument for the $SO(n)$ (below). Comfort and Remus [3] conjecture that (2)(3) can be replaced by the assumption that all the G_k are simple and nonabelian.

Of course, the SL groups are not simple when they differ from the PSL groups, so it is reasonable to ask for a condition which also works for general finite groups. It is certainly necessary that $\liminf_k \text{md}(G_k) = \infty$ (see Definition 1.20). Unfortunately, by Example 7.2, this is not sufficient.

Lemma 5.21 *Suppose that $2^r \leq n < 2^{r+1}$ and $r \geq 2$. Then $SO(n)$ is $(2^r, 4)$ -ugly.*

Proof. If $2k \leq n$, let $R(\theta_0, \dots, \theta_{k-1})$ be the rotation matrix $A \in SO(n)$ which rotates θ_i in the $(2i+1, 2i+2)$ coordinate plane. Thus,

$$\begin{pmatrix} a_{2i+1,2i+1} & a_{2i+1,2i+2} \\ a_{2i+2,2i+1} & a_{2i+2,2i+2} \end{pmatrix} = \begin{pmatrix} \cos \theta_i & \sin \theta_i \\ -\sin \theta_i & \cos \theta_i \end{pmatrix}$$

for $i = 0, \dots, k-1$ and $a_{i,i} = 1$ for $2k+1 \leq i \leq n$, and the other components of A are 0. Note that if n is $2k$ or $2k+1$, then every element of $SO(n)$ is conjugate to some $R(\theta_0, \dots, \theta_{k-1})$ (see [22], Proposition VII.5.3).

As in the proof of Lemma 5.19, there is a boolean subgroup \tilde{H} of $SO(n)$ of order 2^r such that every non-identity element of \tilde{H} is a diagonal matrix with exactly 2^{r-1} entries equal -1 and the rest equal 1.

Now, in $SO(4)$, if D is the diagonal matrix with entries $(1, -1, 1, -1)$, then

$$R(\theta/2, \varphi/2) D R(-\theta/2, -\varphi/2) D = R(\theta, \varphi) .$$

Hence, $\mathcal{N}_2^u(D) = SO(4)$.

In $SO(n)$, if X, Y are distinct elements of \tilde{H} , then there is a diagonal matrix conjugate to $X^{-1}Y$ whose first 2^r entries are $(1, -1, 1, -1, \dots)$ and the other $n - 2^r$ entries are 1. Hence, $\mathcal{N}_2^u(X^{-1}Y)$ contains all matrices of the form $R(\theta_0, \dots, \theta_{2^r-1-1})$. Since n is $2k$ or $2k+1$ for some $k \leq 2 \cdot 2^{r-1}$, $\mathcal{N}_4^u(X^{-1}Y) = SO(n)$. \odot

Corollary 5.22 $\prod_n SO(n) / \sum_n SO(n)$ is minap and $\prod_{3 \leq n < \omega} SO(n)$ is self-bohrifying.

6 Compacta in $G^\#$

When G is abelian, every compact subset of $G^\#$ is finite (Glicksberg [7]). This is clearly false for non-abelian groups, since many compact Lie groups are self-bohrifying. Moreover, by Corollary 5.22, every compact second countable space is a subspace of some self-bohrifying group. Every self-bohrifying group is second countable by Moran's Theorem 1.12. However, a compact subset of a $G^\#$ need not be second countable, and in fact can be any Eberlein compactum.

Many equivalent definitions of the Eberlein compacta are discussed in Arkhangel'skii [1]; for example, Z is Eberlein compact iff Z is homeomorphic to a weakly compact subspace of some Banach space. The Amir-Lindenstrauss Theorem ([1], Theorem IV.4.12) yields an equivalent in terms of Σ_* products:

Definition 6.1 *Let H be a topological group. Then $\Sigma(H, \kappa) = \{x \in H^\kappa : |\text{supt}(x)| < \aleph_0\}$. $\Sigma_*(H, \kappa)$ is the set of all $x \in H^\kappa$ such that for all neighborhoods U of 1, $\{\alpha < \kappa : x(\alpha) \notin U\}$ is finite. Both $\Sigma(H, \kappa)$ and $\Sigma_*(H, \kappa)$ are given the topology inherited from the usual product topology on H^κ .*

So, $\Sigma(H, \kappa) \subseteq \Sigma_*(H, \kappa) \subseteq H^\kappa$, and $\Sigma(H, \kappa)$ is just the direct sum of κ copies of H , as in Definition 2.6.

Definition 6.2 *A space Z is Eberlein compact iff for some κ , Z is homeomorphic to a compact subspace of $\Sigma_*(\mathbb{R}, \kappa)$.*

Of course, in Definition 6.1, the "1" refers to the identity element of the group, which is 0 in \mathbb{R} . For example, $\Sigma_*(\mathbb{R}, \omega)$ contains a copy of the Hilbert cube, $\prod_n [0, 2^{-n}]$; hence, every compact metric space is Eberlein compact. The 1-point compactification of a discrete space of size κ is Eberlein compact, realized as the set of $y \in \{0, 1\}^\kappa$ such that $y(\alpha) = 0$ for all but at most one α .

Theorem 6.3 *If Z is compact Hausdorff, then Z is Eberlein compact iff Z is homeomorphic to a subspace of $G^\#$ for some group G .*

The \leftarrow direction is proved in [9], Theorem 3.13. The rest of this section is devoted to proving \rightarrow .

Lemma 6.4 *If Z is Eberlein compact and H is a compact Lie group of dimension ≥ 1 , then Z is homeomorphic to a subset of $\Sigma_*(H, \kappa)$ for some κ .*

Theorem 6.3 will follow immediately when we show that whenever $G = H^\kappa$ and H is any compact connected semisimple Lie group, then the usual product topology agrees with the topology of $G^\#$ on $\Sigma_*(H, \kappa)$. For such H , van der Waerden's theorem (see [23], or [11] Theorem 5.64) says that H is self-bohrifying; that is, $\text{Hom}(H, Y) = \text{Hom}_c(H, Y)$ whenever Y is a compact group. Now, H^κ cannot be self-bohrifying when κ is infinite (see Proposition 1.6 of [9]), but an analog of "self-bohrifying" holds if we replace the product topology by the uniform topology:

Definition 6.5 *If H is a compact group, then $W \subseteq H^\kappa$ is open in the uniform topology on H^κ iff for each $a \in W$, there is an open neighborhood U of 1 in H such that $aU^\kappa \subseteq W$.*

Note that we can always choose U such that $x^{-1}Ux = U$ for all $x \in H$, so that we can replace aU^κ by $U^\kappa a$. U^κ itself will be a neighborhood of 1 in this topology, but will not in general be open; $\{x \in H^\kappa : \text{cl}\{x_\alpha : \alpha < \kappa\} \subseteq U\}$ is an open neighborhood of 1, and these sets form a basis at 1 in H^κ .

If H is a compact metric group, then it has a biinvariant metric ($d(x, y) = d(xz, yz) = d(zx, zy)$; see [11], Corollary A4.19), and then the uniform topology on H^κ is induced by the biinvariant metric d_{sup} defined by $d_{\text{sup}}(x, y) = \sup_\alpha d(x_\alpha, y_\alpha)$.

Definition 6.6 *For finite m , a topological group G has property $\ast(m)$ iff there are $\gamma_1, \dots, \gamma_m \in \text{Hom}_c(\mathbb{R}, G)$ such that whenever $r_1, \dots, r_m \in (0, 1)$:*

$$1 \in \text{int}\{[\gamma_1(r_1), x_1][\gamma_2(r_2), x_2] \cdots [\gamma_m(r_m), x_m] : x_1, x_2, \dots, x_m \in G\} \quad .$$

Applying the γ_i coordinatewise in a product, we get:

Lemma 6.7 *If H is a compact group with property $\ast(m)$, then H^κ , with the uniform topology, also has property $\ast(m)$.*

Now, following [11], van der Waerden's theorem is immediate from the next two lemmas. We remark that in the case of $SO(3)$, one may replace this argument by the proof of Comfort and Robertson [4], yielding a proof of Theorem 6.3 which does not use any Lie theory.

Lemma 6.8 *If H is a compact connected semisimple Lie group, then H has property $\ast(m)$ for some m .*

In the proof, m is the dimension of H and the γ_i are obtained via the exponential map from the Lie algebra.

Lemma 6.9 *If H is a topological group with property $\ast(m)$ for some finite m , then $\text{Hom}(H, Y) = \text{Hom}_c(H, Y)$ for every compact group Y .*

Proof. Given $\varphi \in \text{Hom}(H, Y)$ and a neighborhood U of 1 in Y , use compactness to find a neighborhood V of 1 such that for all $s_1, \dots, s_m \in V$ and all $z_1, \dots, z_m \in Y$, $[s_1, z_1][s_2, z_2] \cdots [s_m, z_m] \in U$. Then, use compactness again to fix $r_1, \dots, r_m \in (0, 1)$ such that each $\varphi(\gamma_i(r_i)) \in V$. It follows that $\varphi([\gamma_1(r_1), x_1][\gamma_2(r_2), x_2] \cdots [\gamma_m(r_m), x_m]) \in U$ for all $x_1, x_2, \dots, x_m \in H$, so that $1 \in \text{int}(\varphi^{-1}(U))$. \odot

Lemma 6.10 *Let H be a compact connected semisimple Lie group, and let $G = H^\kappa$. Then the Bohr topology on G and the product topology on G agree on $\Sigma_*(H, \kappa)$.*

Proof. Since the Bohr topology is finer, it is sufficient to fix $\varphi \in \text{Hom}(G, U(n))$ and prove that its restriction to $\Sigma_*(H, \kappa)$ is continuous with respect to the product topology. Let $K = \Sigma(H, \kappa)$. Since $H = H'$, apply Lemma 3.7 to $\varphi|_K$ to get a finite $F \subseteq \kappa$ such that $\varphi(x) = \varphi(y)$ for all $x, y \in K$ such that $x|F = y|F$. Define $\psi \in \text{Hom}(G, U(n))$ so that $\psi(x) = \varphi(y)$ whenever $x|F = y|F$ and $\text{supt}(y) \subseteq F$. Then ψ is continuous because H^F is self-bohrifying, and ψ agrees with φ on K .

We are now done if we show that ψ and φ agree on $\Sigma_*(H, \kappa)$. If not, fix $x \in \Sigma_*(H, \kappa)$ such that $\|\psi(x) - \varphi(x)\| = c > 0$. Since φ and ψ are continuous with respect to the uniform topology (by Lemmas 6.7, 6.8, 6.9), there is an $\varepsilon > 0$ such that $\|\varphi(x) - \varphi(w)\| \leq c/3$ and $\|\psi(x) - \psi(w)\| \leq c/3$ whenever $d_{\text{sup}}(x, w) \leq \varepsilon$. Now, let $\tilde{x} \in K$ be such that $d_{\text{sup}}(\tilde{x}, x) \leq \varepsilon$. Then $\psi(\tilde{x}) \neq \varphi(\tilde{x})$, a contradiction. \odot

Proof of Theorem 6.3. The \rightarrow direction is immediate from Lemmas 6.4 and 6.10.

7 Small Compact Groups

Let X be a compact group. If X is self-bohrifying, then it is small (Definition 1.21) by Moran's Theorem 1.12. If X is not self-bohrifying, then it is large by Theorem 3.15. But “small” and “large” were defined by counting the irreps of X_d , not the *continuous* irreps of X , which is the more natural object of study in the theory of compact groups. It would be nice to have a characterization of “self-bohrifying”, such as the following, stated in terms of the continuous irreps:

Lemma 7.1 *Let X be a compact Lie group. Then X is self-bohrifying iff for each n , X has only finitely many inequivalent continuous irreps of degree n .*

Proof. Of course, one direction is again by Theorem 1.12, so assume throughout that X is not self-bohrifying, and we show that X has infinitely many inequivalent continuous irreps of some degree.

If X is connected: X' is compact and semisimple (see [11], Theorem 6.18). If $X = X'$, then X would be self-bohrifying by van der Waerden theorem. Thus, X' is a Lie group of smaller dimension, so X/X' is an infinite compact abelian group, yielding infinitely many continuous irreps of degree 1.

In general, let $X_0 \triangleleft X$ be the component of 1. Since X is not self-bohrifying, neither is X_0 (by Lemma 4.3), so that X_0 has infinitely many continuous irreps of degree 1. Then, since $|X : X_0|$ is finite, the method of induced representations yields an infinite set of inequivalent continuous irreps of X of some degree $\leq |X : X_0|$. \odot

Unfortunately, this lemma does not hold for profinite (totally disconnected) compact groups.

Example 7.2 *There is a compact profinite group $X = \prod_{n \in \omega} G_n$, where the G_n are finite, such that X is not self-bohrifying but X has only finitely many inequivalent continuous irreps of each degree.*

Proof. For a finite group G , let $c(G)$ be the least c such that every element of G is a product of c commutators. X will be $\prod_n G_n$ where:

1. G_n is finite and $(G_n)' = G_n$.
2. $\liminf_n c(G_n) = \infty$.
3. $\liminf_n \text{md}(G_n) = \infty$ (see Definition 1.20).

Then (1,2) imply that X' is a proper dense subgroup of X , so that X has discontinuous homomorphisms into $U(1)$, and (3) implies that X has only finitely many inequivalent continuous irreps of each degree.

There are many examples in the literature satisfying (1) and (2). In particular, consider the one in Lemma 2.1.10 of Holt and Plesken [13]. In this example, one chooses a sequence of primes $5 \leq p_0 < p_1 < p_2 < \dots$, and each $G_n = P_n \rtimes SL(2, p_n)$, where P_n is a p_n -group. (1) and (2) are verified in [13], and we now verify (3).

Note that one composition series for G_n is of the form $\dots \triangleleft H_n^1 \triangleleft H_n^0 = G_n$, where $H_n^0/H_n^1 \cong PSL(2, p_n)$ and H_n^1 is solvable, so that the other factors are all cyclic. Let $f : \omega \rightarrow \omega$ be as in Jordan's Theorem 1.19. We show

that for each k , $\text{Hom}(G_n, U(k))$ is trivial unless $|PSL(2, p_n)| \leq f(k)$. So, fix $\varphi \in \text{Hom}(G_n, U(k))$, and assume that $|PSL(2, p_n)| > f(k)$ and φ is non-trivial. Fix an abelian $A \triangleleft \text{ran}(\varphi)$ with $|\text{ran}(\varphi) : A| \leq f(k)$. By (1), $\text{ran}(\varphi)$ is non-abelian, so $A \neq \text{ran}(\varphi)$. Thus, G_n has a proper normal subgroup of index $\leq f(k)$, so there is another composition series $\cdots \triangleleft K_n^1 \triangleleft K_n^0 = G_n$, where, $|K_n^0/K_n^1| \leq f(k)$. Hence, by the Jordan-Hölder Theorem, K_n^0/K_n^1 is cyclic, contradicting (1). ☺

8 Absoluteness

At first sight, the properties maxap, minap, large, medium, and small for an infinite group G might seem to be sensitive to the axioms of set theory. However, the existence of homomorphisms into $U(n) = U(n, \mathbb{C})$ can be rephrased in terms of “approximate homomorphisms” from finite subsets of G into $U(n, \mathbb{A})$, where \mathbb{A} is the field of complex algebraic numbers. Thus, maxap, minap, etc. are all absolute (see Theorem 8.6 for a precise statement); so, for example, they do not change if one does some forcing argument and passes to a different model of set theory. Note that the countable sets \mathbb{A} and $U(n, \mathbb{A})$, unlike \mathbb{C} and $U(n, \mathbb{C})$, are the same in all transitive models of set theory.

Definition 8.1 *An approximate representation of the group G of degree n is a function f such that $\text{dom}(f)$ is a subset of G of some finite size k , $\text{ran}(f) \subset U(n, \mathbb{A})$, and for all $x, y, z \in \text{dom}(f)$: if $z = xy^{-1}$ then $\|f(z) - f(x)f(y)^{-1}\| < 2^{-k}$.*

As in the proof of Lemma 3.13, $\|\cdot\|$ denotes the operator norm on $M_n(\mathbb{C})$.

Lemma 8.2 *For any n , any $a_1, \dots, a_r \in G$, any matrices $B_1, \dots, B_r \in U(n, \mathbb{A})$, and any $\varepsilon > 0$, the following are equivalent:*

1. *For some $\varphi \in \text{Hom}(G, U(n))$, each $\|\varphi(a_\ell) - B_\ell\| < \varepsilon$.*
2. *For some ε' with $0 < \varepsilon' < \varepsilon$: For every finite s with $\{a_1, \dots, a_r\} \subseteq s \subseteq G$, there is an approximate representation f of degree n such that $\text{dom}(f) = s$ and each $\|f(a_\ell) - B_\ell\| < \varepsilon'$.*

Proof. (1) \rightarrow (2) holds because $U(n, \mathbb{A})$ is dense in $U(n) = U(n, \mathbb{C})$. Now, assume (2). For each finite s with $\{a_1, \dots, a_r\} \subseteq s \subseteq G$, choose an approximate representation f_s such that $\text{dom}(f_s) = s$ and each $\|f_s(a_\ell) - B_\ell\| < \varepsilon'$. Let \mathcal{D} be an ultrafilter on $[G]^{<\omega}$ such that for each $x \in G$ the set $\{s \in [G]^{<\omega} : x \in s\}$

is in \mathcal{D} . Define $\varphi : G \rightarrow U(n)$ by $\varphi(x) = \mathcal{D}\text{-lim}_s f_s(x)$; this is defined because \mathcal{D} -a.e. s contains all the elements a_1, \dots, a_r, x . Our choice of f_s guarantees that each $\|\varphi(a_\ell) - B_\ell\| \leq \varepsilon' < \varepsilon$, and the definition of “approximate representation” guarantees that $\|\varphi(z) - \varphi(x)\varphi(y)^{-1}\| \leq 2^{-k}$ for each k and each $x, y, z \in G$, so that $\varphi \in \text{Hom}(G, U(n))$. ☺

As usual in set theory (see, e.g., [14, 16]), if M is a transitive model of ZFC and \mathfrak{A} is a formula or a defined object, then $[\mathfrak{A}]^M$ is that formula or object as viewed within M . Note that $\mathbb{A} = [\mathbb{A}]^M \subset [\mathbb{C}]^M = \mathbb{C} \cap M \subseteq \mathbb{C}$. If we assert that “minap” is *absolute* for M , we mean that if $G \in M$, then $[G \text{ is minap}]^M$ iff G is really minap in the universe (V) .

Lemma 8.3 *Let G be a group, and let M be a transitive model of ZFC with $G \in M$. Let $E = [\text{Hom}(G, U(n))]^M$. Then $\text{Hom}(G, U(n)) = \text{cl}(E)$, where the closure is taken in $U(n)^G$.*

Proof. Since $E = \text{Hom}(G, U(n)) \cap M$ is a subset of the closed $\text{Hom}(G, U(n))$, we have $\text{cl}(E) \subseteq \text{Hom}(G, U(n))$. Now suppose $\psi \in \text{Hom}(G, U(n)) \setminus \text{cl}(E)$. Then there is a basic open set $W \subseteq U(n)^G$ such that $\psi \in W$ and $W \cap E = \emptyset$. We may assume that $W = \{\varphi : \forall \ell \leq r \|\varphi(a_\ell) - B_\ell\| < \varepsilon\}$, where ε is a positive rational, $a_1, \dots, a_r \in G$, and $B_1, \dots, B_r \in U(n, \mathbb{A})$. Since $\psi \in W$, condition (2) of Lemma 8.2 holds, and this condition involves just finite subsets of G and \mathbb{A} , so that $[(2)]^M$ holds as well. But then, applying Lemma 8.2 within M , we produce a $\varphi \in W \cap E$, contradicting the choice of W . ☺

Corollary 8.4 *Let G be a group, and let M be a transitive model of ZFC with $G \in M$. Let $Y = [C_n(G)]^M$. Then $C_n(G) = \text{cl}(Y)$, where the closure is taken in \mathbb{C}^G .*

Proof. Define $\text{TR} : U(n)^G \rightarrow \mathbb{C}^G$ by: $\text{TR}(\varphi) = \text{tr} \circ \varphi$. Then $C_n(G) = \text{TR}(\text{Hom}(G, U(n)))$, and, applying this within M : $Y = \text{TR}(E)$, where E is as in Lemma 8.3. Hence, $C_n(G) = \text{TR}(\text{cl}(E)) = \text{cl}(\text{TR}(E)) = \text{cl}(Y)$, since TR is continuous, and hence, by compactness of $U(n)^G$, a closed map. ☺

In particular, if $[G \text{ is small}]^M$, then each $C_n(G) = [C_n(G)]^M$ is finite, so that G is really small in V . To get the same argument for “medium”, apply the following well-known fact about scattered spaces, which is easily proved by induction on the Cantor-Bendixon rank:

Lemma 8.5 *Let Y be a set, with topology \mathcal{T} , and let M be any transitive model of ZFC with $Y, \mathcal{T} \in M$ such that $[\mathcal{T} \text{ is a Hausdorff topology on } Y]^M$ and $[(Y, \mathcal{T}) \text{ is compact scattered}]^M$. Then (in V) \mathcal{T} is a base for a compact scattered Hausdorff topology on Y .*

Theorem 8.6 *The properties “maxap”, “minap”, “small”, “medium”, “large” are all absolute for transitive models of ZFC.*

Proof. Fix $G \in M$, a transitive model of ZFC.

By Lemma 8.3, $[\ker(\Phi_G)]^M = \ker(\Phi_G)$, so “maxap” ($\ker(\Phi_G) = \{1\}$) and “minap” ($\ker(\Phi_G) = G$) are absolute.

By Corollary 8.4, “small” is clearly absolute.

If $[G \text{ is medium}]^M$, then by Lemma 8.5, each $[C_n(G)]^M$ remains compact scattered in V , and hence, by Corollary 8.4, $C_n(G) = [C_n(G)]^M$. Hence, G is really medium in V .

If G is medium in V , then each $C_n(G)$ is scattered, and hence $[C_n(G)]^M$ is scattered (in V), since every subspace of a scattered space is scattered. But the Cantor-Bendixon sequence is absolute, so $[C_n(G) \text{ is scattered}]^M$ as well, so that $[G \text{ is medium}]^M$.

Hence, “medium” is absolute, and thus also “large” (= “neither small nor medium”) is absolute as well. ☺

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