

Limits in Compact Abelian Groups*

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Abstract

For X a compact abelian group and B an infinite subset of its dual \widehat{X} , let \mathcal{C}_B be the set of all $x \in X$ such that $\langle \varphi(x) : \varphi \in B \rangle$ converges to 1. If \mathcal{F} is a free filter on \widehat{X} , let $\mathcal{D}_{\mathcal{F}} = \bigcup \{ \mathcal{C}_B : B \in \mathcal{F} \}$. The sets \mathcal{C}_B and $\mathcal{D}_{\mathcal{F}}$ are subgroups of X . \mathcal{C}_B always has Haar measure 0, while the measure of $\mathcal{D}_{\mathcal{F}}$ depends on \mathcal{F} . We show that there is a filter \mathcal{F} such that $\mathcal{D}_{\mathcal{F}}$ has measure 0 but is not contained in any \mathcal{C}_B . This generalizes previous results for the special case where X is the circle group.

1 Introduction

In this paper we study the pointwise convergence of sequences of characters of compact abelian groups and its relation to Bohr topologies. We begin with some abstract definitions. All spaces considered here are assumed to be Hausdorff.

Definition 1.1 *If X, Y are topological spaces, then $C(X, Y)$ is the set of continuous functions from X to Y , and $C_p(X, Y)$ denotes $C(X, Y)$ given the topology of pointwise convergence (i.e., regarding $C_p(X, Y)$ as a subset of Y^X with the usual product topology). If Y contains a distinguished point 1, then $\mathbf{1}$ denotes the constant function $x \mapsto 1$ in $C(X, Y)$.*

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See Arkhangel'skii [1] for a discussion of such function spaces.

Suppose X is a compact abelian group and $Y = \mathbb{T} \subset \mathbb{C}$, where \mathbb{T} is the circle group. As usual (see [6, 9, 13]), \widehat{X} denotes the dual group of X ; that is, the group of characters, or continuous homomorphisms into \mathbb{T} ; then $\mathbf{1}$ is the identity element of \widehat{X} . If $G = \widehat{X}$ and we view G as a discrete abelian group, then $X \cong \widehat{G}$ by the Pontrjagin Duality Theorem. However, if we consider $G \subseteq C_p(X, \mathbb{T})$, then its inherited topology is the *Bohr topology* on G , and the closure of G in \mathbb{T}^X is the *Bohr compactification*, $\mathbf{b}G$, of G . $G^\#$ denotes G with its Bohr topology. Since the compact group $\mathbf{b}G$ is dense in itself, and $G^\#$ is dense in $\mathbf{b}G$, we have:

Lemma 1.2 *If X is an infinite compact abelian group, then \widehat{X} is dense in itself in the topology inherited from $C_p(X, \mathbb{T})$.*

However, \widehat{X} has no pointwise convergent sequences. To study pointwise convergence, we use the following notation:

Definition 1.3 *If X, Y are topological spaces, $y \in Y$, and $B \subseteq C(X, Y)$ is infinite, then $\mathcal{C}_B(y)$ is the set of all $x \in X$ such that the sequence $\langle \varphi(x) : \varphi \in B \rangle$ converges to y (that is, every neighborhood of y contains $\varphi(x)$ for all but finitely many $\varphi \in B$). $\widetilde{\mathcal{C}}_B = \bigcup_{y \in Y} \mathcal{C}_B(y)$. If Y is a topological group with identity 1, then \mathcal{C}_B denotes $\mathcal{C}_B(1)$.*

If X and Y are topological groups and B is a family of homomorphisms, then \mathcal{C}_B and $\widetilde{\mathcal{C}}_B$ are subgroups of X . Clearly, $\mathcal{C}_B \subseteq \widetilde{\mathcal{C}}_B$. The sequence $\langle \varphi : \varphi \in B \rangle$ converges pointwise (i.e., in $C_p(X, Y)$) iff $\widetilde{\mathcal{C}}_B = X$. So when X is compact abelian and $B \subseteq \widehat{X}$, $\widetilde{\mathcal{C}}_B$ can never equal X , but it can be non-trivial. In §2 we prove the following, which gives some results involving the sizes of \mathcal{C}_B and $\widetilde{\mathcal{C}}_B$:

Theorem 1.4 *Let X be an infinite compact abelian group with $G = \widehat{X}$. Then:*

1. $\widetilde{\mathcal{C}}_B$ is a Haar null set for each infinite $B \subseteq G$.
2. For any countable $Q \subseteq X$, there is an infinite $B \subseteq \widehat{X}$ such that $Q \subseteq \mathcal{C}_B$, \mathcal{C}_B contains a perfect subset, and \mathcal{C}_B is dense in X .
3. $\lambda(\overline{B}) \leq 1/|\widetilde{\mathcal{C}}_B|$ for all infinite $B \subseteq G$. Here, \overline{B} is the closure of B in $\mathbf{b}G$, λ is the Haar probability measure on $\mathbf{b}G$, and $1/|\widetilde{\mathcal{C}}_B| = 0$ when $|\widetilde{\mathcal{C}}_B|$ is infinite.

So, every $\widetilde{\mathcal{C}}_B$ is small in the sense of measure, but by (2), for some B even the smaller \mathcal{C}_B is big in other senses. And (3) implies that whenever $\widetilde{\mathcal{C}}_B$ is infinite, B itself is small in the sense that \overline{B} is a Haar null set in $\mathbf{b}G$.

When $X = \mathbb{T}$, the fact that \mathcal{C}_B is null is pointed out in [2, 4].

Note that both \mathcal{C}_B and $\tilde{\mathcal{C}}_B$ get bigger as B gets smaller, so that the detailed arguments in this paper will only involve countable B . For example, it is sufficient to prove (1) for countable B , and the B produced in the proof of (2) will be countable.

If $X = \mathbb{T}$, then $\widehat{\mathbb{T}} \subset C(\mathbb{T}, \mathbb{T})$ is the set of functions $z \mapsto z^n$ for $n \in \mathbb{Z}$; we identify $\widehat{\mathbb{T}}$ with \mathbb{Z} . As an illustration of (3), let $B = \{kn : n \in \mathbb{Z}\}$. Then $\mathcal{C}_B = \tilde{\mathcal{C}}_B = \{z \in \mathbb{T} : z^k = 1\}$, and $\lambda(\overline{B}) = 1/k = 1/|\tilde{\mathcal{C}}_B| = 1/|\mathcal{C}_B|$.

For $X = \mathbb{T}$, Barbieri, Dikranjan, Milan, and Weber [2] showed that assuming Martin's Axiom, there is a Haar null subgroup D of \mathbb{T} which is not contained in any \mathcal{C}_B . In [7] we showed that this holds in ZFC; in fact, we gave an explicit definition of such a D which is a Borel set in \mathbb{T} .

There are two natural generalizations of these results about $C(\mathbb{T}, \mathbb{T})$. First, one may study the maps $(z \mapsto z^n) \in C(X, X)$ for any compact group X ; this was done in [7]. In this paper, we consider the second generalization. For an arbitrary compact abelian group X , we have $B \subseteq \widehat{X} \subset C(X, \mathbb{T})$. We shall produce (Theorem 1.9) a Haar null subgroup D of X such that D is not contained in any countable union of the form $\bigcup_\ell \tilde{\mathcal{C}}_{B_\ell}$. As in [7], it is convenient to define the null group D from a filter:

Definition 1.5 *Suppose that X, Y are topological spaces, $y \in Y$, and \mathcal{F} is a free filter on the set $C(X, Y)$. Then $\mathcal{D}_{\mathcal{F}}(y) = \bigcup\{\mathcal{C}_B(y) : B \in \mathcal{F}\}$, and $\tilde{\mathcal{D}}_{\mathcal{F}} = \bigcup\{\tilde{\mathcal{C}}_B : B \in \mathcal{F}\}$. If Y is a topological group with identity 1, then $\mathcal{D}_{\mathcal{F}}$ denotes $\mathcal{D}_{\mathcal{F}}(1)$.*

As usual, a filter \mathcal{F} is *free* iff it contains the complements of finite sets. As in [7], our null group D will be $\mathcal{D}_{\mathcal{F}}$, where \mathcal{F} is a filter of sets of asymptotic density one:

Definition 1.6 *For $E \subseteq \omega$, let $\underline{d}(E)$ and $\overline{d}(E)$ denote the lower and upper asymptotic density:*

$$\underline{d}(E) = \liminf_{n \rightarrow \infty} \frac{|E \cap n|}{n} \leq \limsup_{n \rightarrow \infty} \frac{|E \cap n|}{n} = \overline{d}(E) \quad .$$

If equality holds, let $d(E) = \underline{d}(E) = \overline{d}(E)$ denote the asymptotic density of E .

Definition 1.7 *Let X be a compact abelian group and let $\varphi = \langle \varphi_n : n \in \omega \rangle$ be a sequence of distinct elements of \widehat{X} . Then \mathcal{F}_{φ} is the filter \mathcal{F} generated by all sets of the form $\{\varphi_n : n \in E\}$ such that $d(E) = 1$.*

Proposition 1.8 *For \mathcal{F}_φ defined as in 1.7, $\widetilde{\mathcal{D}}_{\mathcal{F}_\varphi}$ is a Haar null subgroup of X .*

Note that $\widetilde{\mathcal{D}}_{\mathcal{F}_\varphi}$ is clearly a subgroup. We prove that it is null in §2. The group $\widetilde{\mathcal{D}}_{\mathcal{F}_\varphi}$ could be trivial; for example, if $X = \mathbb{T}$ and $\varphi_n(z) = z^n$, then $\widetilde{\mathcal{D}}_{\mathcal{F}_\varphi} = \{1\}$. In [7], our null subgroup of \mathbb{T} was of the form $\widetilde{\mathcal{D}}_{\mathcal{F}_\varphi}$, where $\varphi_n(z) = z^{n!}$.

The null group $\widetilde{\mathcal{D}}_{\mathcal{F}_\varphi}$ contains $\mathcal{D}_{\mathcal{F}_\varphi}$. Nevertheless, Theorem 1.9 shows that for suitable φ , even $\mathcal{D}_{\mathcal{F}_\varphi}$ is not contained in any countable union of $\widetilde{\mathcal{C}}_B$ sets.

Theorem 1.9 *For any infinite compact abelian group X , there is a D such that:*

1. D is a Haar null subgroup of X ;
2. D is dense in X ;
3. D is not a subset of any countable union of the form $\bigcup_\ell \widetilde{\mathcal{C}}_{B_\ell}$, where each B_ℓ is an infinite subset of \widehat{X} ;
4. $D = \mathcal{D}_{\mathcal{F}_\varphi}$ for some sequence of distinct characters $\varphi = \langle \varphi_n : n \in \omega \rangle$.

The proof of Theorem 1.9 has two parts. In §4, we prove the theorem when X is one of four types of “stock” compact groups. And in §3, we show that it is sufficient to prove the theorem for those stock groups. This argument applies the structure theory for abelian groups to \widehat{X} , and is similar to the analysis used in constructing I_0 sets (Hartman and Ryll-Nardzewski [8], Thm. 5; see also [12]).

The stock groups are all second countable (that is, their \widehat{X} are countable). The $|\widehat{X}|$ in Proposition 1.8 and Theorem 1.9 can be an arbitrary infinite cardinal. However, since $\widetilde{\mathcal{C}}_B$ gets bigger as B gets smaller, it is sufficient to prove Theorem 1.9 in the case that all the B_ℓ are countable. For countable B , \mathcal{C}_B and $\widetilde{\mathcal{C}}_B$ are Borel (in fact, $F_{\sigma\delta}$) sets; likewise, $\mathcal{D}_{\mathcal{F}_\varphi}$ and $\widetilde{\mathcal{D}}_{\mathcal{F}_\varphi}$ are $F_{\sigma\delta}$ sets (see Proposition 5.3).

Our results are related to the notions of \mathfrak{g} -closure and \mathfrak{g} -density described by Dikranjan, Milan, and Tonolo [5]. These notions may be expressed in terms of an intersection involving our \mathcal{C}_B :

Definition 1.10 *Let X be a compact abelian group, and $J \leq X$, with \overline{J} its (usual topological) closure. Then $\mathfrak{g}_X(J) = \overline{J} \cap \bigcap \{\mathcal{C}_B : B \in [\widehat{X}]^{\aleph_0} \text{ \& } J \subseteq \mathcal{C}_B\}$.*

They call $\mathfrak{g}_X(J)$ the \mathfrak{g} -closure of J and say that J is \mathfrak{g} -dense iff $\mathfrak{g}_X(J) = X$. Barbieri, Dikranjan, Milan, and Weber [3] ask (see Question 5.7) whether for every infinite compact abelian group, there is a \mathfrak{g} -dense subgroup which is a Haar null set, and they provide an affirmative answer under Martin’s Axiom in some cases. Our D from Theorem 1.9 provides an affirmative answer in all cases in ZFC.

2 Elementary Facts

Proposition 1.8 is easily proved using Cesàro limits:

Definition 2.1 Given $r_n \in \mathbb{C}$ for $n \in \omega$ and $s \in \mathbb{C}$, $r_n \rightsquigarrow s$ means that $\frac{1}{j} \sum_{n < j} r_n$ converges to s as $j \rightarrow \infty$.

Lemma 2.2 Fix $r_n \in \mathbb{C}$ for $n \in \omega$ and $s \in \mathbb{C}$. Assume that there is an $M \geq 0$ such that $|r_n| \leq M$ for all n , and that $\lim_{n \in E} r_n = s$ for some $E \subseteq \omega$ with $d(E) = 1$. Then $r_n \rightsquigarrow s$.

The following is proved exactly like Lemma 4.9 of [12], although the basic idea for the proof goes back to Weyl [15]§7.

Lemma 2.3 Let μ be a probability measure on X . Let $\varphi_n : X \rightarrow \mathbb{C}$, for $n \in \omega$, be measurable. Assume that $M \geq 0$, $|\varphi_n(x)| \leq M$ for all n and x , and the φ_n are orthogonal in $L^2(\mu)$. Then $\mu(\{x \in X : \varphi_n(x) \rightsquigarrow 0\}) = 1$.

Proof of Proposition 1.8. Use Lemma 2.2 and 2.3. Here, the φ_n map into \mathbb{T} , so $\mathcal{D}_{\mathcal{F}_\varphi}(0) = \emptyset$, so that $\widetilde{\mathcal{D}}_{\mathcal{F}_\varphi}$ is disjoint from $\{x \in X : \varphi_n(x) \rightsquigarrow 0\}$. \square

The next lemma is immediate from the Pontrjagin Duality Theorem:

Lemma 2.4 For compact abelian X and Y , if \widehat{Y} is isomorphic to a subgroup of \widehat{X} , then there is a continuous homomorphism π mapping X onto Y .

Given compact abelian X , we can choose Y so that \widehat{Y} is a countable subgroup of \widehat{X} . Then Y is second countable. This sometimes lets us reduce a statement about arbitrary X to a statement about second countable groups, as is illustrated in the proof below of Theorem 1.4(2). It is also useful to recall:

Lemma 2.5 If π is a continuous homomorphism mapping the compact group X onto Y , then π is both a closed map and an open map. Also, $\lambda_X(\pi^{-1}(E)) = \lambda_Y(E)$ for all Haar-measurable $E \subseteq Y$, where λ_X, λ_Y are the Haar probability measures on X, Y , respectively.

To prove Theorem 1.4(3), we need:

Lemma 2.6 Every infinite discrete abelian group G is a Haar null subset of \mathbf{bG} .

This lemma is immediate from Varopoulos [14], who proves a more general result. To prove the result directly for discrete abelian groups, note that for countable ones, the result is trivial. So for an arbitrary infinite discrete abelian G , take a homomorphism π from G onto a countable H , and then note that π induces $\mathfrak{b}\pi : \mathfrak{b}G \rightarrow \mathfrak{b}H$, with $G \subseteq (\mathfrak{b}\pi)^{-1}(H)$.

The following lemma is also needed for Theorem 1.4(3):

Lemma 2.7 *Let X be a compact abelian group with $G = \widehat{X}$, and fix $u \in X$ and a subgroup S of G . Let $K = \{x \in X : \forall \varphi \in S [\varphi(x) = \varphi(u)]\}$. Then $\lambda(K) = 1/|S|$, where λ is Haar measure on X .*

Proof. Let $\pi : X \rightarrow \widehat{S}$ be the natural map. Viewing X as the characters of G , we have

$$K = \{x \in X : \forall \varphi \in S [x(\varphi) = u(\varphi)]\} = \{x \in X : x|_S = u|_S\} = \pi^{-1}\{u|_S\} .$$

Here, $u|_S$ is a point in \widehat{S} . Since π preserves Haar measure (see Lemma 2.5), if S is infinite then $\lambda(K) = 0$, while if S is finite then $\lambda(K) = 1/|\widehat{S}| = 1/|S|$. \square

Proof of Theorem 1.4. Part (1) is clear from Proposition 1.8, since it is sufficient to prove it when B is countable.

For (2), we shall produce a perfect subset of X via a tree of open sets indexed by finite 0-1 sequences. List Q as $\{q_j : j \in \omega\}$. We now get distinct $\varphi_n \in \widehat{X}$ for $n \in \omega$ and $U_s \subseteq X$ for $s \in 2^{<\omega} = \bigcup_{n \in \omega} \{0, 1\}^n$ so that:

- $\Leftrightarrow U_s$ is open and nonempty.
- $\Leftrightarrow \text{cl}(U_{s \smallfrown 0}) \cap \text{cl}(U_{s \smallfrown 1}) = \emptyset$ and $\text{cl}(U_{s \smallfrown 0}), \text{cl}(U_{s \smallfrown 1}) \subseteq U_s$.
- $\Leftrightarrow |1 - \varphi_n(x)| < 1/n$ whenever $x \in \{q_j : j \leq n\} \cup \bigcup \{U_s : s \in 2^n\}$.

We do this by induction on n . φ_0 can be arbitrary and U_0 can be X . If we are given U_s for $s \in 2^n$ and $\varphi_0, \dots, \varphi_n$: First, choose distinct $p_{s \smallfrown 0}, p_{s \smallfrown 1} \in U_s$. Then choose $\varphi_{n+1} \notin \{\varphi_0, \dots, \varphi_n\}$ such that $|1 - \varphi_{n+1}(x)| < 1/(n+1)$ whenever $x \in \{q_j : j \leq n+1\} \cup \{p_t : t \in 2^{n+1}\}$; this is possible because $\mathbf{1} \in \widehat{X} \subset C_p(X, \mathbb{T})$ and is not isolated in \widehat{X} (see Lemma 1.2). Then, we may choose U_t for $t \in 2^{n+1}$ using the continuity of φ_{n+1} .

Let $K = \bigcap_{n \in \omega} \bigcup_{s \in 2^n} \text{cl}(U_s)$, and let $B = \{\varphi_n : n \in \omega\}$. Then $K \cup Q \subseteq \mathcal{C}_B$. K is not scattered, since it maps continuously onto the Cantor set, so its perfect kernel is non-empty.

We still need to get \mathcal{C}_B dense in X . If X is separable, this is trivial, since we may assume that the countable Q contains a dense subset of X . So for any X ,

choose a second countable Y with $\widehat{Y} < \widehat{X}$, and let $\pi : X \rightarrow Y$ be as in Lemma 2.4. For this separable Y , choose an infinite $B_Y \subseteq \widehat{Y}$ such that $\pi(Q) \subseteq \mathcal{C}_{B_Y}$, \mathcal{C}_{B_Y} is dense in Y , and \mathcal{C}_{B_Y} contains a perfect subset. Then $B = \{\varphi \circ \pi : \varphi \in B_Y\}$ satisfies (2); since π is an open map (Lemma 2.5), $\mathcal{C}_B = \pi^{-1}(\mathcal{C}_{B_Y})$ is dense in X .

For (3), define $\Theta_0 : \widetilde{\mathcal{C}}_B \rightarrow \mathbb{T}$ so that $\Theta_0(x)$ is the limit of the sequence $\langle \varphi(x) : \varphi \in B \rangle$ (which exists by definition of $\widetilde{\mathcal{C}}_B$). Note that Θ_0 is a homomorphism from the group $\widetilde{\mathcal{C}}_B$ into \mathbb{T} , so, since \mathbb{T} is divisible, it extends to a homomorphism $\Theta : X \rightarrow \mathbb{T}$. Then $\widetilde{\mathcal{C}}_B = \{x \in X : \langle \varphi(x) : \varphi \in B \rangle \rightarrow \Theta(x)\}$. Let X_d denote the group X with the discrete topology; then we can identify $\mathfrak{b}G$ with the compact group \widehat{X}_d . So, $\Theta \in \mathfrak{b}G$. We can view $G^\#$ as a dense subgroup of $\mathfrak{b}G$, so that each $x \in X$ can be identified with a continuous homomorphism on $\mathfrak{b}G$. With this identification, each $x \in \widetilde{\mathcal{C}}_B$ satisfies $\langle x(\varphi) : \varphi \in B \rangle \rightarrow x(\Theta)$, so that $x(\Phi) = x(\Theta)$ for each $\Phi \in \overline{B} \setminus B$. Thus, $\overline{B} \setminus B \subseteq \{\Phi \in \mathfrak{b}G : \forall x \in \widetilde{\mathcal{C}}_B [x(\Phi) = x(\Theta)]\}$, so that $\lambda(\overline{B} \setminus B) \leq 1/|\widetilde{\mathcal{C}}_B|$ by applying Lemma 2.7; the X, G, u, S in 2.7 becomes $\mathfrak{b}G, X, \Theta, \widetilde{\mathcal{C}}_B$ here. Finally, $\lambda(\overline{B}) \leq 1/|\widetilde{\mathcal{C}}_B|$ because $B \subseteq G$, which, by Lemma 2.6, is a Haar null set in $\mathfrak{b}G$. \square

Just getting a \mathcal{C}_B that contains a perfect set is trivial in the case that $G = \widehat{X}$ has an infinite subgroup H with infinite index. Let $Z = \bigcap \{\ker(\varphi) : \varphi \in H\}$. Then $Z \cong \widehat{G/H}$ is an infinite compact subgroup of X , and $Z = \mathcal{C}_H$.

3 Reduction to Stock

In this section, we show that it is sufficient to prove Theorem 1.9 in the case that X is the dual of one of the groups listed in the following lemma:

Lemma 3.1 *Every infinite abelian group contains a subgroup isomorphic to one of the following:*

- $\Leftrightarrow \mathbb{Z}$.
- $\Leftrightarrow \sum_{n \in \omega} \mathbb{Z}_{p_n}$, where the p_n are primes and $p_0 < p_1 < \dots$.
- $\Leftrightarrow \sum_{n \in \omega} \mathbb{Z}_p$, where p is a fixed prime.
- $\Leftrightarrow \mathbb{Z}_{p^\infty}$, for some prime p .

This lemma is part of the structure theory for infinite abelian groups (see Kaplansky [10], or Hewitt and Ross [9], or [12] §3). The duals of these four groups are, respectively, \mathbb{T} , $\prod_{n \in \omega} \mathbb{Z}_{p_n}$, $(\mathbb{Z}_p)^\omega$, and the p -adic integers; for the last one, see [9] §25.

Next, we use the $\pi : X \rightarrow Y$ obtained from Lemma 2.4 to translate a φ satisfying Theorem 1.9 for Y to a $\varphi \circ \pi$ satisfying Theorem 1.9 for X .

Lemma 3.2 *Let X and Y be compact abelian groups, with π a continuous homomorphism mapping X onto Y . Assume that $\varphi = \langle \varphi_n : n \in \omega \rangle$ is a sequence of distinct characters of Y such that $\mathcal{D}_{\mathcal{F}_\varphi}$ is not a subset of any countable union of the form $\bigcup_\ell \widetilde{\mathcal{C}}_{A^\ell}$, whenever each A^ℓ is an infinite subset of \widehat{Y} . Let $\varphi \circ \pi = \langle \varphi_n \circ \pi : n \in \omega \rangle$. Then, in X , $\mathcal{D}_{\mathcal{F}_{\varphi \circ \pi}}$ is not a subset of any countable union of the form $\bigcup_\ell \widetilde{\mathcal{C}}_{B^\ell}$, whenever each B^ℓ is an infinite subset of \widehat{X} . Also, if $\mathcal{D}_{\mathcal{F}_\varphi}$ is dense in Y then $\mathcal{D}_{\mathcal{F}_{\varphi \circ \pi}}$ is dense in X .*

Proof. Let $K = \ker(\pi)$. Since π is an epimorphism, $X/K \cong Y$, so characters of Y correspond to characters of X/K . Note also that each character in \widehat{X} restricts to one in \widehat{K} . Since $\widetilde{\mathcal{C}}_B$ gets bigger as B gets smaller, we may shrink each B^ℓ to a countable set. Shrinking again to $B^\ell = \{\psi_n^\ell : n \in \omega\}$, we may assume that for each ℓ , the $\psi_n^\ell|_K$, for $n \in \omega$, are either all the same or are all different.

Case 1: The $\psi_n^\ell|_K$, for $n \in \omega$, are all the same. So each $\psi_n^\ell \cdot (\psi_0^\ell)^{-1}$ is identically 1 on K , and hence yields a character $\delta_n^\ell \in \widehat{X/K} \cong \widehat{Y}$ (with $\psi_n^\ell \cdot (\psi_0^\ell)^{-1} = \delta_n^\ell \circ \pi$). Let $A^\ell = \{\delta_n^\ell : n \in \omega\} \subseteq \widehat{Y}$. By our assumption on φ , we can fix a $y \in Y$ such that $y \in \mathcal{D}_{\mathcal{F}_\varphi}$ and $y \notin \widetilde{\mathcal{C}}_{A^\ell}$ for all Case 1 ℓ . Note that if x is any element of $\pi^{-1}\{y\}$, then $x \in \mathcal{D}_{\mathcal{F}_{\varphi \circ \pi}}$. Also, such an x is not in $\widetilde{\mathcal{C}}_{B^\ell}$ for all Case 1 ℓ , because the non-convergence of $\langle \delta_n^\ell(y) : n \in \omega \rangle$ implies the non-convergence of $\langle \psi_n^\ell(x) \cdot (\psi_0^\ell(x))^{-1} : n \in \omega \rangle$, and hence the non-convergence of $\langle \psi_n^\ell(x) : n \in \omega \rangle$. We are thus done if we produce $x \in \pi^{-1}\{y\}$ so that $x \notin \widetilde{\mathcal{C}}_{B^\ell}$ for all Case 2 ℓ . Fix $x^* \in \pi^{-1}\{y\}$. Then our desired x will be an element of the coset $Kx^* = \pi^{-1}\{y\}$.

Case 2: The $\psi_n^\ell|_K$, for $n \in \omega$, are all different. For all Case 2 ℓ , define $f_n^\ell : K \rightarrow \mathbb{T}$ by $f_n^\ell(t) = \psi_n^\ell(tx^*) = \psi_n^\ell(t) \cdot \psi_n^\ell(x^*)$. Note that each f_n^ℓ is the product of a character $\psi_n^\ell|_K$ of K with a number $\psi_n^\ell(x^*)$, so that $\{f_n^\ell : n \in \omega\}$ is an orthogonal family in $L^2(K)$. It follows, by using Lemma 2.3, that $\widetilde{\mathcal{C}}_{\{f_n^\ell : n \in \omega\}}$ is a Haar null set in K . Choose t such that for each Case 2 ℓ the sequence $\langle f_n^\ell(t) : n \in \omega \rangle$ does not converge; then $tx^* \notin \widetilde{\mathcal{C}}_{B^\ell}$.

Finally, to prove that $\mathcal{D}_{\mathcal{F}_{\varphi \circ \pi}}$ is dense in X , use the facts that π is an open map by Lemma 2.5, and that $\mathcal{D}_{\mathcal{F}_{\varphi \circ \pi}} = \pi^{-1}(\mathcal{D}_{\mathcal{F}_\varphi})$. \square

4 Nice Groups

Definition 4.3 below isolates the key property of the groups \widehat{G} , for the groups G listed in Lemma 3.1.

Definition 4.1 *If X is a compact abelian group, then*

$$\widehat{X}(X) = \{\varphi(x) : \varphi \in \widehat{X} \text{ \& } x \in X\} .$$

Proposition 4.2 *$\widehat{X}(X)$ is a subgroup of \mathbb{T} .*

Proof. Let $G = \widehat{X}$. If G contains an element of infinite order, then $\widehat{X}(X) = \mathbb{T}$. Otherwise, $\widehat{X}(X)$ is the group generated by all $e^{2\pi i/p^n}$ such that p is prime and G contains an element of order p^n . \square

If G is of finite exponent (= bounded order), then $\widehat{X}(X)$ is finite; otherwise, $\text{cl}(\widehat{X}(X)) = \mathbb{T}$.

Definition 4.3 *The compact abelian group X is nice iff $|\widehat{X}| = \aleph_0$ and for all non-empty open $U \subseteq X$ and all $\varepsilon > 0$: $\text{cl}(\widehat{X}(X)) \subseteq N_\varepsilon(\varphi(U))$ for all but finitely many $\varphi \in \widehat{X}$. Here, $N_\varepsilon(S) = \{z \in \mathbb{T} : \exists w \in S [|z - w| < \varepsilon]\}$.*

Lemma 4.4 *If $G = \widehat{X}$ is an infinite torsion abelian group and $\{\varphi \in G : \varphi^k = 1\}$ is finite for each k , then X is nice.*

Proof. Note that $\text{cl}(\widehat{X}(X)) = \mathbb{T}$, so we fix a non-empty open $U \subseteq X$ and an $\varepsilon > 0$, and we must verify that $N_\varepsilon(\varphi(U)) = \mathbb{T}$ for all but finitely many $\varphi \in \widehat{X}$. Observe that $\varphi(X)$ is finite for all $\varphi \in G$. Translating U and shrinking it, we may assume that $U = \{x \in X : \forall \psi \in F [\psi(x) = 1]\}$, where F is a finite subgroup of G . Let $R_n = \{z \in \mathbb{T} : z^n = 1\}$, and fix m such that $N_\varepsilon(R_m) = \mathbb{T}$. For all but finitely many $\varphi \in G$, the order of $[\varphi]$ in G/F is at least m . Fix any such φ ; then for some $n \geq m$, $\varphi^n \in F$ but $\varphi^k \notin F$ whenever $0 < k < n$. Fix $y \in \text{Hom}(G/F, \mathbb{T})$ such that $y([\varphi]) = e^{2\pi i/n}$; this lifts to an $x \in \widehat{G} = \text{Hom}(G, \mathbb{T})$ such that $x(\varphi) = e^{2\pi i/n}$ and $x(\psi) = 1$ for all $\psi \in F$. Identifying \widehat{G} with X , we have $x \in U$ and $\varphi(x) = e^{2\pi i/n}$; so, since U is a group, $\varphi(U) \supseteq R_n$. Then $n \geq m$ yields $N_\varepsilon(\varphi(U)) = \mathbb{T}$. \square

Lemma 4.5 *\widehat{G} is nice whenever G is one of the groups listed in Lemma 3.1.*

Proof. Lemma 4.4 handles the duals of $\sum_{n \in \omega} \mathbb{Z}_{p^n}$ and \mathbb{Z}_{p^∞} . For $\mathbb{T} = \widehat{\mathbb{Z}}$, note that for a given U , $\varphi(U) = \mathbb{T}$ for all but finitely many φ .

For $G = \sum_{n \in \omega} \mathbb{Z}_p$ and $\widehat{G} = (\mathbb{Z}_p)^\omega$, follow the proof of Lemma 4.4. U and F are exactly the same. Now, $\widehat{X}(X) = R_p$, and $\varphi(U) = R_p$ for all $\varphi \notin F$. \square

We now proceed to prove Theorem 1.9 for nice groups.

Definition 4.6 Let X be a compact 2^{nd} countable abelian group with metric ρ and let $G = \widehat{X}$. A nice partition for (X, ρ) is a sequence $\langle \Phi_j : j \in \omega \rangle$ such that the Φ_j are finite disjoint nonempty sets whose union is G and, if we set

$$\rho_j(x, y) = \rho(x, y) + \sum \left\{ |\varphi(x) - \varphi(y)| : \varphi \in \bigcup_{k \leq j} \Phi_k \right\} ,$$

then for each j and all $\varphi \in \bigcup_{k \geq j+2} \Phi_k$, all $x \in X$, and all $z \in \text{cl}(\widehat{X}(X))$, there is a $y \in X$ with $\rho_j(x, y) < 2^{-j}$ and $|\varphi(y) - z| < 2^{-j}$.

Lemma 4.7 If X is a nice compact abelian group with metric ρ , then there is a nice partition for (X, ρ) .

Proof. List G as $\{\varphi_j : j \in \omega\}$. Now, define the Φ_j by induction. Let $\Phi_0 = \{\varphi_0\}$. Given Φ_k for $k \leq j$, we have the metric ρ_j on X , so for some finite m , we may cover X by open sets U_0, \dots, U_m of ρ_j -diameter less than $\varepsilon := 2^{-j}$. Now choose Φ_{j+1} so that $\text{cl}(\widehat{X}(X)) \subseteq N_\varepsilon(\varphi(U_\ell))$ for all ℓ and for all $\varphi \in G \setminus \bigcup_{k \leq j+1} \Phi_k$. Also make sure that $\varphi_j \in \bigcup_{k \leq j+1} \Phi_k$ so that G will be the union of all the Φ_j . \square

Definition 4.8 Suppose that $\Phi = \langle \Phi_j : j \in \omega \rangle$ is a nice partition for (X, ρ) . A sequence $\langle \varphi_n : n \in \omega \rangle$ from \widehat{X} is thin (with respect to Φ) iff each $\varphi_n \in \Phi_{j_n}$, where each $j_{n+1} \geq j_n + 2$.

Lemma 4.9 Assume that $\langle \varphi_n : n \in \omega \rangle$ is a thin sequence, ω is partitioned into two infinite sets, A, B , and $a, b \in \text{cl}(\widehat{X}(X))$. Then for some $x \in X$,

$$\varphi_n(x) \xrightarrow[n \in A]{} a \quad \text{AND} \quad \varphi_n(x) \xrightarrow[n \in B]{} b .$$

Proof. Choose $x_n \in X$ for $n \in \omega$ as follows: x_0, x_1 are arbitrary. Given x_{n-1} with $n \geq 2$, use $\varphi_n \in \bigcup_{k \geq j_{n-1}+2} \Phi_k$, plus $j_{n-1} \geq 2(n-1) \geq n$, to get x_n to satisfy:

- $\Leftrightarrow \rho_{j_{n-1}}(x_{n-1}, x_n) < 2^{-n}$.
- $\Leftrightarrow n \in A \Rightarrow |\varphi_n(x_n) - a| < 2^{-n}$.
- $\Leftrightarrow n \in B \Rightarrow |\varphi_n(x_n) - b| < 2^{-n}$.

Then each $\rho(x_{n-1}, x_n) < 2^{-n}$, so $\langle x_n : n \in \omega \rangle$ converges to some x . Now, fix $n \geq 1$, and we estimate $|\varphi_n(x_n) - \varphi_n(x)|$: For all $m > n$, $|\varphi_n(x_{m-1}) - \varphi_n(x_m)| \leq \rho_{j_{m-1}}(x_{m-1}, x_m) < 2^{-m}$. Thus, $|\varphi_n(x_n) - \varphi_n(x)| \leq \sum_{m=n+1}^{\infty} 2^{-m} = 2^{-n}$.

Now, if $n \in A$, then

$$|\varphi_n(x) - a| \leq |\varphi_n(x_n) - \varphi_n(x)| + |\varphi_n(x_n) - a| \leq 2^{-n} + 2^{-n} \rightarrow 0 .$$

The argument is the same for $n \in B$. \square

Lemma 4.10 *Let X be a compact 2^{nd} countable abelian group with metric ρ and let $G = \widehat{X}$. Suppose that $\Phi = \langle \Phi_j : j \in \omega \rangle$ is a nice partition for (X, ρ) and $\varphi = \langle \varphi_n : n \in \omega \rangle$, where each $\varphi_n \in \Phi_{3n}$. Let B_ℓ , for $\ell \in \omega$, be any infinite subsets of G . Then $\mathcal{D}_{\mathcal{F}_\varphi} \not\subseteq \bigcup_\ell \widetilde{\mathcal{C}}_{B_\ell}$.*

Proof. By a standard diagonal argument, get φ'_n for $n \in \omega$ and $E, F \subseteq \omega$ such that:

1. $\langle \varphi'_n : n \in \omega \rangle$ is thin with respect to Φ .
2. $d(\{n : \varphi'_n = \varphi_n\}) = 1$.
3. $E \cup F = \omega$ and $E \cap F = \emptyset$.
4. $d(E) = 1$.
5. For each ℓ , both $\{n \in E : \varphi'_n \in B_\ell\}$ and $\{n \in F : \varphi'_n \in B_\ell\}$ are infinite.

Fix $z \in \widehat{X}(X) \setminus \{1\}$. By (1)(3), we may apply Lemma 4.9 and fix $x \in X$ such that

$$\varphi'_n(x) \xrightarrow{n \in E} 1 \quad \text{AND} \quad \varphi'_n(x) \xrightarrow{n \in F} z \quad .$$

By (2)(4), $x \in \mathcal{D}_{\mathcal{F}_\varphi}$. By (5), $x \notin \widetilde{\mathcal{C}}_{B_\ell}$ for each ℓ . \square

Lemma 4.11 *Suppose that $\Phi = \langle \Phi_j : j \in \omega \rangle$ is a nice partition for (X, ρ) and $\varphi = \langle \varphi_n : n \in \omega \rangle$ from \widehat{X} is thin with respect to Φ . Let $B = \{\varphi_n : n \in \omega\}$. Then \mathcal{C}_B is dense in X , so that $\mathcal{D}_{\mathcal{F}_\varphi}$ is dense in X .*

Proof. This is similar to the proof of Lemma 4.9. Fix a non-empty open $U \subseteq X$. We must produce an $x \in U$ such that $\varphi_n(x) \rightarrow 1$. We may assume that $q \in X$ and $r \in \omega$ and $U = \{x \in X : \rho(x, q) < 2^{-r+1}\}$. Choose $x_n \in X$ for $n \in \omega$ as follows: $x_0 = x_1 = \dots = x_r = q$. Given x_{n-1} with $n \geq r+1$, get x_n to satisfy:

$$\begin{aligned} \Leftrightarrow \rho_{j_{n-1}}(x_{n-1}, x_n) &< 2^{-n}. \\ \Leftrightarrow |\varphi_n(x_n) - 1| &< 2^{-n}. \end{aligned}$$

Then $\langle x_n : n \in \omega \rangle$ converges to some x with $\rho(x, q) \leq 2^{-r}$, so $x \in U$. As in the proof of Lemma 4.9, $|\varphi_n(x) - 1| \rightarrow 0$. \square

Proof of Theorem 1.9. By Lemmas 4.10 and 4.11, the theorem holds for all nice groups, which by Lemma 4.5, includes the duals of all the groups listed in Lemma 3.1. Then, by Lemma 3.2, the theorem holds for all X . \square

Note that not every X with a countable dual is nice; see Example 5.2.

5 Remarks and Examples

The proof in §2 that $\tilde{\mathcal{D}}_{\mathcal{F}_\varphi}$ is null makes essential use of asymptotic density, via Lemma 2.2; one cannot replace \mathcal{F}_φ by an arbitrary filter \mathcal{F} , since $\tilde{\mathcal{D}}_{\mathcal{F}}$, or even the smaller $\mathcal{D}_{\mathcal{F}}$, might be all of X . By Proposition 1.2 of [7] and Lemma 1.2:

Proposition 5.1 *If X is any infinite compact abelian group, then there is a free filter \mathcal{F} on \hat{X} such that \mathcal{F} contains a countable set and $\mathcal{D}_{\mathcal{F}} = X$.*

It is not clear whether the nice groups are of interest in their own right, or just an artifact in the proof of Theorem 1.9. Not every dual of a countable discrete abelian group is nice:

Example 5.2 $\mathbb{Z}_4 \times (\mathbb{Z}_2)^\omega$ is not nice.

Proof. Elements of $X = \mathbb{Z}_4 \times (\mathbb{Z}_2)^\omega$ are of the form $\langle x, \vec{y} \rangle$, where $x \in \mathbb{Z}_4 = \{1, i, -1, -i\}$, $\mathbb{Z}_2 = \{\pm 1\}$, and $\vec{y} \in (\mathbb{Z}_2)^\omega$. $\hat{X}(X) = \{1, i, -1, -i\}$. Let $\varphi_n(x, \vec{y}) = x \cdot y_n$. Let $U = \{\langle x, \vec{y} \rangle : x = i\}$. Then the φ_n are distinct characters, and $\varphi_n(U) = \{\pm i\}$, so Definition 4.3 fails whenever $\varepsilon < \sqrt{2}$. \square

All the “ \mathcal{C} ” and “ \mathcal{D} ” sets discussed in this paper are Borel:

Proposition 5.3 *Let X be any compact abelian group. If $B \subseteq \hat{X}$ is countably infinite, then \mathcal{C}_B and $\tilde{\mathcal{C}}_B$ are $F_{\sigma\delta}$ sets. If $\varphi = \langle \varphi_n : n \in \omega \rangle$ is a sequence of distinct elements of \hat{X} , then $\mathcal{D}_{\mathcal{F}_\varphi}$ and $\tilde{\mathcal{D}}_{\mathcal{F}_\varphi}$ are $F_{\sigma\delta}$ sets.*

Proof. Let $B = \{\varphi_n : n \in \omega\}$. Then $x \in \tilde{\mathcal{C}}_B$ iff

$$\forall r \in \omega \exists s < r \exists k \in \omega \forall m > k \left[|\varphi_m(x) - e^{2\pi i s/r}| \leq \frac{\pi}{r} \right] ,$$

since $\mathbb{T} \subseteq \bigcup_{s < r} N_{\pi/r}(e^{2\pi i s/r})$. This displays $\tilde{\mathcal{C}}_B$ as a countable intersection of F_σ sets. The argument for \mathcal{C}_B is similar; just replace s by 0. Likewise, $x \in \tilde{\mathcal{D}}_{\mathcal{F}_\varphi}$ iff

$$\forall r \in \omega \exists s < r \exists k \in \omega \forall n > k \left[\frac{1}{n} \left| \{m < n : |\varphi_m(x) - e^{2\pi i s/r}| \leq \frac{\pi}{r}\} \right| \geq 1 - \frac{1}{r} \right] .$$

Again, replace s by 0 to see that $\mathcal{D}_{\mathcal{F}_\varphi}$ is an $F_{\sigma\delta}$ set. \square

It is natural to ask whether the countable $\{B_\ell : \ell \in \omega\}$ from Theorem 1.9 could be replaced by a family of \aleph_1 sets. Under CH, this is clearly false, since then $|X|$ may be \aleph_1 , in which case Theorem 1.4 implies that a union of the

form $\bigcup_{\alpha < \omega_1} \mathcal{C}_{B_\alpha}$ can be all of X . Assuming Martin's Axiom (MA), the proof of Theorem 1.9 implies that our $\mathcal{D}_{\mathcal{F}_\varphi}$ is not a subset of any union of the form $\bigcup_{\alpha < \kappa} \tilde{\mathcal{C}}_{B_\alpha}$ where $\kappa < 2^{\aleph_0}$. To see this, note that the countability of the family $\{B_\ell : \ell \in \omega\}$ was only used in two places. First, in handling Case 2 of Lemma 3.2, we used the fact that a compact group K is not covered by \aleph_0 null sets, and MA lets us replace the " \aleph_0 " by " $< 2^{\aleph_0}$ ". Second, the diagonal argument in the proof Lemma 4.10 will work with families of size less than 2^{\aleph_0} under MA.

It is also consistent with ZFC to have 2^{\aleph_0} arbitrarily large but $\mathbb{T} = \bigcup_{\alpha < \omega_1} \mathcal{C}_{B_\alpha}$. This proof resembles the standard construction of an ultrafilter of character \aleph_1 (see [11], Exercise VIII.A10). Start with 2^{\aleph_0} large in the ground model V and iterate forcing \aleph_1 times with finite supports, forming V_α for $\alpha \leq \omega_1$. When $\alpha < \omega_1$, let $\mathcal{F}_\alpha \in V_\alpha$ be a filter on $\mathbb{Z} = \widehat{\mathbb{T}}$ obtained from Proposition 5.1, and get $B_\alpha \in V_{\alpha+1}$ so that $B_\alpha \subset^* A$ for all $A \in \mathcal{F}_\alpha$. One can even make the B_α generate a P-point ultrafilter, so that in the final model V_{ω_1} , the \mathcal{F} of Proposition 5.1 could be a P-point of character \aleph_1 . To do this, make sure that each B_α is chosen so that 0 is a limit point of B_α in the Bohr topology of \mathbb{Z} . Note that the \mathcal{F} of Proposition 5.1 can never be a selective ultrafilter, since it would then contain thin sets and run afoul of Lemma 4.9.

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