Homeomorphisms with Small Twist*

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Abstract

We extend Baumgartner's result on isomorphisms of $\aleph_1$–dense subsets of $\mathbb{R}$ in two ways: First, the function can be made to be absolutely continuous. Second, one can replace $\mathbb{R}$ by $\mathbb{R}^n$.

1 Introduction

Definition 1.1 *For any topological space X, $\mathcal{H}(X)$ denotes the set of all homeomorphisms from X onto X, and a subset $A \subseteq X$ is κ–*dense *(in X) iff $|A \cap U| = \kappa$ for all non-empty open $U \subseteq X$.*

Then, for $X = \mathbb{R}$, we have

Theorem 1.2

a. *If D, E are $\aleph_0$–dense in $\mathbb{R}$, then there is an $f \in \mathcal{H}(\mathbb{R})$ such that $f(D) = E$.*
b. *Assuming* PFA*, if D, E are $\aleph_1$–dense in $\mathbb{R}$, then there is an $f \in \mathcal{H}(\mathbb{R})$ such that $f(D) = E$.*

Here, (a) is a classical result of Cantor, while (b) is due to Baumgartner [3, 4]. In both cases, the proof obtains an order isomorphism h from D onto E, which must then extend to a unique $f \in \mathcal{H}(\mathbb{R})$. In (b), Baumgartner's original proof [3] predates PFA; he simply showed that the result of the theorem, together with $\mathrm{MA} + \mathfrak{c} = \aleph_2$, can be obtained by iterated ccc forcing over any model of $\mathrm{ZFC} + \mathrm{GCH}$. Using his forcing, the PFA result is immediate by the "collapse the

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continuum trick" (see [4]) ; similar remarks hold for our uses of PFA in this paper. By Avraham and Shelah [2], the result in (b) does not follow from MA + $\mathfrak{c} = \aleph_2$ alone.

In this paper, we assume PFA and prove two extensions of (b). First, we show that both f and f^{-1} can be made to be absolutely continuous (AC). Absolute continuity for real-valued functions is discussed below, and in many analysis texts, such as Rudin [8]. It is easily seen (Example 2.3 below) that Baumgartner's forcing yields an f such that neither f nor f^{-1} is AC. If f is Lipschitz ($\forall x, z\, [\, |f(x) - f(z)| \le C|x - z|\,]$), then f must be AC, but one cannot improve (b) to make f and f^{-1} Lipschitz; a ZFC counter-example is described in [7], although this example is implicit in the earlier [1]. Note that in (a), it is easy to make f and f^{-1} Lipschitz, and also real-analytic; this seems to have been done first by Franklin [5] in 1925.

Our second extension of (b) replaces $\mathbb{R}$ by $\mathbb{R}^n$. One such extension is already known, and is due to Steprāns and Watson [9]:

Theorem 1.3 *For any infinite κ and any finite $n \ge 2$,* MA(κ) *implies that if D, E are κ–dense in $\mathbb{R}^n$, then there is an $f \in \mathcal{H}(\mathbb{R}^n)$ such that $f(D) = E$.*

This makes it appear that the result for $\mathbb{R}^n$, for $n \ge 2$, is "easier" than for $\mathbb{R}$. When $\kappa = \aleph_1$, we only need MA + $\mathfrak{c} = \aleph_2$, not PFA. When $\kappa = \aleph_2$ and $n = 1$, it is a well-known open question whether the result of Theorem 1.3 is even *consistent* with $\mathfrak{c} \ge \aleph_2$.

The "easiness" of $\mathbb{R}^n$ for $n \ge 2$ is explained by the fact that $\mathbb{R}^n$ has "more" homeomorphisms than $\mathbb{R}$. For example, every permutation of a finite subset of $\mathbb{R}^n$ extends to some $f \in \mathcal{H}(\mathbb{R}^n)$, while this is clearly false for $n = 1$, since every $f \in \mathcal{H}(\mathbb{R})$ is monotonic (either order-preserving or order-reversing); in fact, the proofs of (a) and (b) in Theorem 1.2 produce order-preserving functions.

Now, if we set $\kappa = \aleph_1$ and demand that our f in Theorem 1.3 be "order-preserving" (*suitably defined*), then we do get a harder result that follows from PFA but not from MA($\aleph_1$). As with the $n = 1$ results, we do not know if there is any consistent version of our results with $\kappa > \aleph_1$.

But, what is the right definition of "order-preserving"? One possibility might be order-preserving on each coordinate; i.e., for each $\vec{x}, \vec{z} \in \mathbb{R}^n$, and each coordinate $i = 0, \ldots, n-1$: $x_i < y_i$ iff $f(x_i) < f(y_i)$ for all i. But this is "wrong", in that there is a ZFC counter-example in $\mathbb{R}^2$ (Example 6.2). A "correct" definition, which leads to a PFA theorem, involves the notion of *twist*:

Definition 1.4 *For $\vec{v}, \vec{w} \in \mathbb{R}^n \backslash \{\vec{0}\}$:*

$$\angle(\vec{v}, \vec{w}) = \arccos(\, (\vec{v} \cdot \vec{w})/(\|\vec{v}\|\|\vec{w}\|)\,) \in [0, \pi] \quad .$$

So, we are thinking of $\vec{v}, \vec{w}$ as arrows pointing from the origin $\vec{0}$, and we are measuring the angle between them in the usual way. Note that we sometimes (not always) place arrows over elements of $\mathbb{R}^n$ to distinguish them from elements of $\mathbb{R}$.

Definition 1.5 *If $F \subseteq \mathbb{R}^n \times \mathbb{R}^n$, let*

$$\mathrm{twist}(F) = \{\angle(\vec{d_1} - \vec{d_0}, \vec{e}_1 - \vec{e}_0) : (\vec{d_0}, \vec{e}_0), (\vec{d_1}, \vec{e}_1) \in F \wedge \vec{d_0} \neq \vec{d_1} \wedge \vec{e}_0 \neq \vec{e}_1\} \quad .$$

Then, let $\mathrm{tw}(F) = \sup(\mathrm{twist}(F))$.

In our applications, F will usually be the graph of a bijection, although $\mathrm{dom}(F)$ and $\mathrm{ran}(F)$ may be proper subsets of $\mathbb{R}^n$.

Lemma 1.6 *For any $F \subseteq \mathbb{R}^n \times \mathbb{R}^n$:* $\mathrm{twist}(F) \subseteq [0, \pi]$, *and* $\mathrm{tw}(F) \in [0, \pi]$, *and* $\mathrm{twist}(\overline{F}) \subseteq \mathrm{cl}(\mathrm{twist}(F))$, *and* $\mathrm{tw}(\overline{F}) = \mathrm{tw}(F)$.

When $n = 1$, $\mathrm{twist}(F) \subseteq \{0, \pi\}$, and a bijection F is strictly increasing (i.e., order-preserving) iff $\mathrm{tw}(F) = 0$.

Then we shall prove

Proposition 1.7 *Assume* PFA. *Fix $\theta > \pi/2$ and $\aleph_1$–dense $D, E \subset \mathbb{R}^n$. Then there is an $f \in \mathcal{H}(\mathbb{R}^n)$ such that $f(D) = E$ and* $\mathrm{tw}(f) \leq \theta$.

The "PFA" is needed here, since it is consistent with $\mathrm{MA} + \mathfrak{c} = \aleph_2$ that the proposition fails for all $n \geq 1$ and all $\theta < \pi$ (Example 6.3).

The "$\theta > \pi/2$" is needed here, since for $\theta \leq \pi/2$ and $n \geq 2$, there is a ZFC counter-example (Example 6.1). Of course, when $n = 1$, Proposition 1.7 is just Baumgartner's result, and $\mathrm{tw}(f)$ can be 0.

But now, we wish to add into Proposition 1.7 the claim that f is AC. Since for $n \geq 2$, AC is not quite a standard notion, we shall define what we mean here:

Definition 1.8 *Let X be a Polish space with a σ-finite Borel measure μ, and fix $f \in \mathcal{H}(X)$. Then f is* absolutely continuous *(with respect to μ) iff for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all open U, $\mu(U) < \delta \to \mu(f(U)) < \varepsilon$. f is* bi-absolutely continuous *(BAC) iff f and f^{-1} are both absolutely continuous. When discussing $\mathbb{R}^n$, μ always refers to Lebesgue measure.*

When $X = \mathbb{R}$, f is a monotonic function, and this definition coincides with the usual definition of absolute continuity for real-valued functions. For general X and f: If f is BAC, then the induced measures are absolutely continuous ($\mu \ll \mu f \ll \mu f^{-1} \ll \mu$; that is, $\mu(B) = 0 \leftrightarrow \mu(f(B)) = 0 \leftrightarrow \mu(f^{-1}(B)) = 0$ for all Borel $B \subseteq X$). This implication is an equivalence when $\mu(X) < \infty$, but not in general; the map $x \mapsto x^3$ on $\mathbb{R}$ is a counter-example.

We can now combine our two extensions of Baumgartner's result:

Theorem 1.9 *Assume* PFA. *Fix* $\theta > \pi/2$ *and* $\aleph_1$*–dense* $D, E \subset \mathbb{R}^n$. *Then there is an* $f \in \mathcal{H}(\mathbb{R}^n)$ *such that* $f(D) = E$ *and* $\mathrm{tw}(f) \le \theta$ *and* f *is BAC.*

Proposition 1.7 is obvious from this. Theorem 1.9 is proved at the end of Section 3. We shall prove the $n = 1$ case first (Lemma 3.6); here, the "$\mathrm{tw}(f) \le \theta$" is trivial, making the proof quite a bit simpler; we shall then use the notation in that proof to motivate the terminology in the general proof. Actually, our proof for the $n > 1$ case uses some properties of our forcing poset that are not proved until Sections 4 and 5.

2 The Basic Poset

We describe here a natural modification of Baumgartner's poset, obtained by replacing $\mathbb{R}$ by $\mathbb{R}^n$ and replacing "order preserving" by a restriction on twists, and we shall prove that our poset is ccc. Since we plan to use PFA with the "collapse the continuum trick" (or else just do an iterated forcing argument over a model of GCH), it is sufficient to assume CH, fix θ, D, E, and produce a ccc poset $\mathbb{P}$ that forces an appropriate f. For constructing ccc posets in our forcing arguments, we use the standard setup with elementary submodels, following approximately the terminology in [6]:

Definition 2.1 *Let* $D, E \subseteq \mathbb{R}^n$ *be* $\aleph_1$*–dense. Fix* κ*, a suitably large regular cardinal. Let* $\langle M_\xi : 0 < \xi < \omega_1 \rangle$ *be a continuous chain of countable elementary submodels of* $H(\kappa)$*, with* $D, E \in M_1$ *and each* $M_\xi \in M_{\xi+1}$*. Let* $M_0 = \emptyset$*. For* $x \in \bigcup_\xi M_\xi$*, let* $\mathrm{ht}(x)$*, the* height *of* x*, be the* ξ *such that* $x \in M_{\xi+1} \backslash M_\xi$*.*

By setting $M_0 = \emptyset$, we ensure that under CH, $\mathrm{ht}(x)$ is defined whenever $x \in \mathbb{R}^n$ or x is a Borel subset of $\mathbb{R}^n$. Observe that $\{d \in D : \mathrm{ht}(d) = \xi\}$ and $\{e \in E : \mathrm{ht}(e) = \xi\}$ are both countable and dense for each $\xi < \omega_1$. Note that $\mathrm{ht}(\,(x, y)\,) = \max(\mathrm{ht}(x), \mathrm{ht}(y))$.

Definition 2.2 *Fix* $\theta \in (0, \pi)$ *and* $\aleph_1$*–dense* $D, E \subset \mathbb{R}^n$*. Assume* CH *and use the notation from Definition 2.1 for the elementary submodels. Then, let* $\mathbb{P}_0^\theta$ *be the set of all* p *satisfying:*

P1. $p \in [D \times E]^{<\omega}$ *is a bijection from* $\mathrm{dom}(p)$ *onto* $\mathrm{ran}(p)$*.*

P2. $\mathrm{tw}(p) < \theta$*.*

P3. For each $(d, e) \in p$*,* $\mathrm{ht}(d), \mathrm{ht}(e)$ *differ by a finite non-zero ordinal.*

P4. $(d_0, e_0) \in p \wedge (d_1, e_1) \in p \wedge (d_0, e_0) \neq (d_1, e_1) \Rightarrow \mathrm{ht}(\,(d_0, e_0)\,) \neq \mathrm{ht}(\,(d_1, e_1)\,)$*.*

Define $q \le p$ *iff* $q \supseteq p$*; so* $\mathbb{1} = \emptyset$*. When* $n = 1$*,* $\mathbb{P}_0 = \mathbb{P}_0^\theta$ *for some (any)* $\theta \in (0, \pi)$*.*

Consider the one-dimensional version of this, so in the ground model V: D, E are $\aleph_1$–dense subsets of $\mathbb{R}$. It is easy to see that the sets $\{p : d \in \mathrm{dom}(p)\}$ and $\{p : e \in \mathrm{ran}(p)\}$ are dense for all $d \in D$ and $e \in E$, so in $V[G]$, $\bigcup G$ is an order-preserving bijection from D onto E. Viewing $\bigcup G$ as a subset of $\mathbb{R} \times \mathbb{R}$, let $f = \mathrm{cl}(\bigcup G)$. Then, in $V[G]$ we have $f \in \mathcal{H}(\mathbb{R})$ and $f(D) = E$.

Since the definition of $\mathbb{P}_0$ contains nothing relevant to absolute continuity, this cannot suffice to prove Theorem 1.9:

Example 2.3 *With f as above, neither f nor f^{-1} is absolutely continuous.*

Proof. For $p \in \mathbb{P}_0 \backslash \{\mathbb{1}\}$, let $d_p^0 = \min(\mathrm{dom}(p))$ and $d_p^1 = \max(\mathrm{dom}(p))$ and $e_p^0 = p(d_p^0) = \min(\mathrm{ran}(p))$ and $e_p^1 = p(d_p^1) = \max(\mathrm{ran}(p))$; then, let $h_p \in \mathcal{H}(\mathbb{R})$ be the natural piecewise linear extension of p obtained by linear interpolation, giving it a slope of 1 outside of $[d_p^0, d_p^1]$. Let $h_{\mathbb{1}}(x) = x$.

Working in $V[G]$ and using standard density arguments, we can chose $p_n \in G$ for $n \in \omega$ such that, setting $h_n = h_{p_n}$: The h_n converge to f uniformly on compact sets and the h_n^{-1} converge to f^{-1} uniformly on compact sets. Let $a_n^i = d_{p_n}^i$ and $b_n^i = e_{p_n}^i$ $(i = 0, 1)$, so h_n maps (a_n^0, a_n^1) onto (b_n^0, b_n^1).

We may furthermore choose the p_n such that for each n: $a_n^0, b_n^0 < -n$ and $a_n^1, b_n^1 > +n$, and for all $x \in (a_n^0, a_n^1) \setminus \mathrm{dom}(p_n)$: either $h_n'(x) < 2^{-n}/(a_n^1 - a_n^0)$ or $h_n'(x) > 2^n \cdot (b_n^1 - b_n^0)$. Then we have $(a_n^0, a_n^1) \setminus \mathrm{dom}(p_n)$ partitioned into two open subsets, U_n, W_n (each a finite union of intervals), where $x \in U_n \to h_n'(x) < 2^{-n}/(a_n^1 - a_n^0)$ and $x \in W_n \to h_n'(x) > 2^n \cdot (b_n^1 - b_n^0)$. Note that $\mu(W_n) \le 2^{-n}$, so $N := \bigcap_m \bigcup_{n>m} W_n$ is a null G_δ; but $\mu(h_n(U_n)) \le 2^{-n}$, so $h_n(W_n)$ is "most of" (b_n^0, b_n^1). Using this, we see that $\mu(\mathbb{R} \backslash f(N)) = 0$, so f is not AC. Likewise, f^{-1} maps a null set onto the complement of a null set, so f^{-1} is not AC.

We shall eventually modify $\mathbb{P}_0^\theta$ by adding some side conditions, obtaining a proof of Theorem 1.9, but we shall conclude this section by proving that $\mathbb{P}_0^\theta$ is ccc. This is a straightforward variant of Baumgartner's argument:

Lemma 2.4 *Fix $\theta > \pi/2$ and $t \in \omega$. Assume* CH*, and assume that:*

1. $p_\alpha = \{(d_\alpha^0, e_\alpha^0), \ldots, (d_\alpha^{t-1}, e_\alpha^{t-1})\}$ *satisfies* $(P1)(P3)(P4)$ *above for each* $\alpha < \omega_1$.
2. $d_\alpha^i \ne d_\beta^j$ *and* $e_\alpha^i \ne e_\beta^j$ *unless* $\alpha = \beta$ *and* $i = j$.

Then there are $\alpha \ne \beta$ such that $\angle(d_\beta^i - d_\alpha^i, e_\beta^i - e_\alpha^i) < \theta$ for all $i < t$. Hence, $\mathbb{P}_0^\theta$ is ccc.

Proof. The ccc follows from the rest of the lemma by a standard delta system argument.

Now, induct on t. The case $t = 0$ is trivial, so assume the result for t, and we shall prove it for $t+1$; so now $p_\alpha = \{(d^0_\alpha, e^0_\alpha), \ldots, (d^t_\alpha, e^t_\alpha)\}$. Permuting and thinning the sequence if necessary, we may assume that each $\mathrm{ht}(p_\alpha) = \mathrm{ht}(e^t_\alpha) > \mathrm{ht}(d^t_\alpha)$, and that $\alpha < \beta \rightarrow \mathrm{ht}(p_\alpha) < \mathrm{ht}(p_\beta)$. Note that $\mathrm{ht}(p_\alpha) > \mathrm{ht}(d^i_\alpha)$ and $\mathrm{ht}(p_\alpha) > \mathrm{ht}(e^i_\alpha)$ for all $i < t$.

Identify each p_α with a point in $(\mathbb{R}^n)^{2t+2}$, and let $K = \mathrm{cl}\{p_\alpha : \alpha < \omega_1\} \subseteq (\mathbb{R}^n)^{2t+2}$. For each α and each $y \in \mathbb{R}^n$, obtain $p_\alpha / y \in (\mathbb{R}^n)^{2t+2}$ by replacing the e^t_α by y in p_α. Let $K_\alpha = \{y \in \mathbb{R}^n : p_\alpha/y \in K\}$. By CH, fix ζ such that $K \in M_\zeta$.

For $\alpha \ge \zeta$: K_α is uncountable because $K_\alpha \in M_{\mathrm{ht}(p_\alpha)}$, $e^t_\alpha \in K_\alpha$, and $e^t_\alpha \notin M_{\mathrm{ht}(p_\alpha)}$. Fix $\widehat{e}_\alpha \neq \widetilde{e}_\alpha$ in $K_\alpha \setminus \{e^0_\alpha, e^1_\alpha, \ldots, e^t_\alpha\}$. Since $\theta > \pi/2$, $\varepsilon := \theta - \pi/2 > 0$. Now, fix disjoint basic open neighborhoods U, V of $\widehat{e}_\alpha, \widetilde{e}_\alpha$ respectively so that $\angle(x_1 - y_1,\, x_2 - y_2) < \varepsilon/2$ for all $x_1, x_2 \in U$ and all $y_1, y_2 \in V$.

Of course, U, V depend on α, but we may fix an uncountable $S \subseteq \omega_1 \backslash \zeta$ such that they have the same values for all $\alpha \in S$. Then, applying induction, fix $\alpha \neq \beta$ in S such that $\angle(d^i_\beta - d^i_\alpha, e^i_\beta - e^i_\alpha) < \theta$ for all $i < t$. Then, fix any $x \in U$ and any $y \in V$. Then either $\angle(d^t_\beta - d^t_\alpha, y - x) \le \pi/2$ or $\angle(d^t_\beta - d^t_\alpha, x - y) \le \pi/2$, since the sum of the two angles is π. In any case, $x, \widehat{e}_\alpha, \widehat{e}_\beta \in U$ and $y, \widetilde{e}_\alpha, \widetilde{e}_\beta \in V$.

If $\angle(d^t_\beta - d^t_\alpha, y - x) \le \pi/2$, use $\widehat{e}_\alpha \in K_\alpha$ and $\widetilde{e}_\beta \in K_\beta$;

approximate $\{(d^0_\alpha, e^0_\alpha), \ldots, (d^t_\alpha, \widehat{e}_\alpha)\}$ and $\{(d^0_\beta, e^0_\beta), \ldots, (d^t_\beta, \widetilde{e}_\beta)\}$
by $\{(d^0_\mu, e^0_\mu), \ldots, (d^t_\mu, e^t_\mu)\}$ and $\{(d^0_\nu, e^0_\nu), \ldots, (d^t_\nu, e^t_\nu)\}$

Fix μ, ν such that $e^t_\mu \in U$ and $e^t_\nu \in V$ and $\angle(d^i_\nu - d^i_\mu, e^i_\nu - e^i_\mu) < \theta$ for all $i < t$ and $\angle(d^t_\nu - d^t_\mu, d^t_\beta - d^t_\alpha) < \varepsilon/2$. Then $\angle(d^t_\nu - d^t_\mu, e^t_\nu - e^t_\mu) \le \angle(d^t_\beta - d^t_\alpha, y - x) + \angle(d^t_\nu - d^t_\mu, d^t_\beta - d^t_\alpha) + \angle(e^t_\nu - e^t_\mu, y - x) < \theta$.

If $\angle(d^t_\beta - d^t_\alpha, x - y) \le \pi/2$, the argument is essentially the same, using $\widetilde{e}_\alpha \in K_\alpha$ and $\widehat{e}_\beta \in K_\beta$.

Proposition 1.7 is false when $\theta \le \pi/2$ and $n \ge 2$; see Example 6.1. For an easy counter-example to the lemma in $\mathbb{R}^2$, for suitable D, E: For $\alpha < \omega_1$, let $p_\alpha = \{(d_\alpha, e_\alpha)\}$, where the d_α are distinct points on the x-axis and the e_α are distinct points on the y-axis with $\mathrm{ht}(e_\alpha) = \mathrm{ht}(d_\alpha) + 1$. Then $\{p_\alpha : \alpha < \omega_1\}$ is an antichain in $\mathbb{P}^\theta_0$.

3 On Absolute Continuity

Here, we make some further remarks on absolute continuity and give a proof of the $n = 1$ case of Theorem 1.9.

Our forcing arguments will obtain "generic" functions as limits of absolutely continuous functions. But such limits are not in general absolutely continuous; for example, in $\mathbb{R}$, *every* continuous function on $[0, 1]$ is a uniform limit of polynomials (which are clearly absolutely continuous). We shall prove absolute continuity by applying Lemma 3.2.

Lemma 3.1 *If $f_j \to f$ pointwise, all f_j are measurable functions, $U \subseteq X$ is open, and $\mu(f_j^{-1}(U)) \leq \varepsilon$ for all j, then $\mu(f^{-1}(U)) \leq \varepsilon$.*

Proof. By pointwise convergence, $f^{-1}(U) \subseteq \bigcup_{m\in\omega} \bigcap_{j\geq m} f_j^{-1}(U)$.

Applying this to f^{-1}:

Lemma 3.2 *Assume that $f_j \in \mathcal{H}(X)$ for all $j \in \omega$ and $f_j^{-1} \to f^{-1}$ pointwise, where $f \in \mathcal{H}(X)$. Assume also that for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all open U and* all *j, $\mu(U) < \delta \to \mu(f_j(U)) < \varepsilon$. Then f is absolutely continuous.*

When $X = \mathbb{R}$, one way to obtain the hypotheses of this lemma is to bound uniformly the derivatives of the f_j. For general $\mathbb{R}^n$, we use the Jacobian. We review here some standard notation:

If $f : \mathbb{R}^n \to \mathbb{R}^n$, then $\partial_i f$ (where $i < n$) denotes the partial derivative of f with respect to the i^{th} variable. Then $\partial_i f : \mathbb{R}^n \to \mathbb{R}^n$, assuming that this derivative exists everywhere. As usual, $C^1(\mathbb{R}^n, \mathbb{R}^n)$ denotes the set of all $f : \mathbb{R}^n \to \mathbb{R}^n$ such that each $\partial_i f$ exists everywhere and is continuous.

We shall use J_f to denote the Jacobian *matrix*; so $J_f : \mathbb{R}^n \to \mathbb{R}^{n^2}$, and $J_f(x)$ is an $n \times n$ matrix whose j^{th} column is $\partial_j f(x)$ (viewed as a column vector). Recall that if f and f^{-1} are C^1 bijections, then $J_{f^{-1}}(f(x)) = (J_f(x))^{-1}$.

Also, if f is 1-1 and C^1 on U, then $\mu(f(U)) = \int_U |\det J_f(x)|$. Thus we could obtain the hypotheses of Lemma 3.2 if we had a uniform bound to all the $|\det J_{f_j}(x)|$. However, in our forcing argument, this turns out to be impossible for the same reason that we cannot get f and f^{-1} to be Lipschitz in Theorem 1.9. We shall get a somewhat weaker condition on the f_j; $|\det J_{f_j}(x)| < 2$ will hold "most of the time", that is, $\mu(f_j(\{x : |\det J_{f_j}(x)| \geq 2\}))$ will be finite. We plan to apply Lemma 3.4 below to each f_j. We state it so that it applies both to C^1 functions on $\mathbb{R}^n$ and to piecewise linear functions on $\mathbb{R}$.

Definition 3.3 *For $f : \mathbb{R}^n \to \mathbb{R}^n$ and $\ell \in (0, \infty)$, let $W_\ell^f = \{x \in \mathbb{R}^n : \ell \leq |\det J_f(x)|\}$ and $Z_\ell^f = \{x \in \mathbb{R}^n : \ell - 1 \leq |\det J_f(x)| \leq \ell\}$.*

Lemma 3.4 *Fix $f \in \mathcal{H}(\mathbb{R}^n)$, and assume that f is C^1 except on some finite set. Assume also that $\int_{W_2^f} |\det J_f(x)|\, dx < \infty$. Fix $\varepsilon > 0$. Then choose $k \geq 2$ so that $\int_{W_k^f} |\det J_f(x)|\, dx < \varepsilon/2$. Let $\delta = \varepsilon/(2k)$. Then for all Borel sets U, $\mu(U) < \delta \to \mu(f(U)) < \varepsilon$.*

Proof. Let $U = A \cup B$, where $A = U \backslash W_k^f$ and $B = U \cap W_k^f$. Then $\mu(f(A)) \leq k\mu(A) < k\delta = \varepsilon/2$ and $\mu(f(B)) \leq \mu(f(W_k^f)) = \int_{W_k^f} |\det J_f(x)|\, dx < \varepsilon/2$, so $\mu(f(A \cup B)) \leq \varepsilon$. ☕

Our generic f will not be differentiable, but it will be a limit of functions f_j to which Lemma 3.4 will apply. To make the lemma apply *uniformly*, so that we can use Lemma 3.2, we shall have a uniform bound $\Upsilon(\ell)$ to each $\mu(Z_\ell^{f_j})$, and apply:

Lemma 3.5 *Fix $f \in \mathcal{H}(\mathbb{R}^n)$, and assume that f is C^1 except on some finite set. Then for all $k \geq 2$:*

$$\tfrac{1}{3}\textstyle\sum_{\ell>k} \ell\mu(Z_\ell^f) \leq \mu(f(W_k^f)) = \int_{W_k^f} |\det J_f(x)|\, dx \leq \sum_{\ell>k} \int_{Z_\ell^f} |\det J_f(x)|\, dx \leq \sum_{\ell>k} \ell\mu(Z_\ell^f) \quad .$$

Proof. The "=" holds by the change-of-variables formula, the second "$\leq$" holds because $W_k^f = \bigcup_{\ell>k} Z_\ell^f$, and the third "$\leq$" holds because $|\det J_f(x)| \leq \ell$ for all $x \in Z_\ell^f$. For the first "$\leq$": note that each point x is in no more than two different Z_ℓ^f, and $|\det J_f(x)| \geq \ell - 1$ for all $x \in Z_\ell^f$, so that

$$\int_{W_k^f} |\det J_f(x)|\, dx \geq \tfrac{1}{2}\textstyle\sum_{\ell>k} \int_{Z_\ell^f} |\det J_f(x)|\, dx \geq \tfrac{1}{2}\sum_{\ell>k}(\ell-1)\mu(Z_\ell^f) \quad ,$$

and now use $\frac{1}{2}(\ell - 1) \geq \frac{1}{3}\ell$, which holds because $\ell \geq k + 1 \geq 3$. ☕

It might seem more elegant to let $Z_\ell^f = \{x \in \mathbb{R}^n : \ell - 1 \leq |\det J_f(x)| < \ell\}$. Then, the Z_ℓ^f would partition W_k^f, and the $\frac{1}{3}$ in the lemma could be replaced by $\frac{2}{3}$. But, our forcing arguments (such as the proof of Lemma 3.6) will use the fact that since $[\ell-1, \ell]$ is closed, if $x \notin Z_\ell^f$, then also $x \notin Z_\ell^g$ whenever the derivatives of f, g are sufficiently close to each other.

In the proof of Theorem 1.9, we shall modify the poset $\mathbb{P}_0^\theta$ to force an f that is BAC. To do this, each forcing condition p will have a side condition Υ_p that will enable us to apply Lemma 3.4 to f. First, we describe the one-dimensional case, where $\det J_f(x)$ is just $f'(x)$:

Lemma 3.6 *Theorem 1.9 holds when $n = 1$.*

Proof. As remarked in Section 2, it is enough to assume CH and construct a ccc poset and prove that $V[G]$ contains the required f. Let $\mathbb{P}$ be the set of all pairs $p = (\sigma_p, \Upsilon_p)$ such that

1. $\sigma_p \in \mathbb{P}_0$ and $\Upsilon_p \in (\mathbb{Q} \cap (0,\infty))^{<\omega}$; let $m_p = \mathrm{dom}(\Upsilon_p)$.
2. $\sum\{\ell \Upsilon_p(\ell) : \ell \geq 3 \ \& \ \ell < m_p\} < 1$.
3. Whenever $3 \leq \ell < m_p$: $\mu(Z_\ell^{h_{\sigma_p}}) < \Upsilon_p(\ell)$ and $\mu(Z_\ell^{h_{\sigma_p}^{-1}}) < \Upsilon_p(\ell)$.
4. $1/\max(2, m_p - 1) < h'_{\sigma_p}(x) < \max(2, m_p - 1)$ for all $x \notin \mathrm{dom}(\sigma_p)$.

In (3), h_σ is as defined in the proof of Example 2.3. Define $q \leq p$ iff $\sigma_q \leq \sigma_p$ and $\Upsilon_q \leq \Upsilon_p$, so $\mathbb{1} = (\emptyset, \emptyset)$.

Working in $V[G]$, let $f = \mathrm{cl}(\bigcup\{\sigma_p : p \in G\})$; $f(D) = E$ because $\{p : d \in \mathrm{dom}(\sigma_p)\}$ and $\{p : e \in \mathrm{ran}(\sigma_p)\}$ are dense whenever $d \in D$ and $e \in E$. Let $\Upsilon = \bigcup\{\Upsilon_p : p \in G\}$; $\mathrm{dom}(\Upsilon) = \omega$ because, by (4), the sets $\{p : m_p > \ell\}$ are dense. Note that $\sum_{\ell \geq 3} \ell \Upsilon(\ell) \leq 1$. We next prove that f is AC (the proof for f^{-1} is similar):

First note that for all $p \in G$ and all $\ell \geq 3$, $\mu(Z_\ell^{h_{\sigma_p}}) < \Upsilon(\ell)$: For $\ell < m_p$, this is clear by (3), while for $\ell \geq m_p$, $Z_\ell^{h_{\sigma_p}} = \emptyset$ by (4).

For $\varepsilon > 0$, choose $\delta = \delta_\varepsilon$ as follows: choose $k \geq 2$ so that $\sum_{\ell > k} \Upsilon(\ell) < \varepsilon/2$; then let $\delta = \varepsilon/(2k)$. Now, for $p = (\sigma_p, \Upsilon_p) \in G$, if $h = h_{\sigma_p}$ and $k \geq 2$: $\int_{W_k^h} h'(x)\, dx \leq \sum_{\ell > k} \ell \mu(Z_\ell^h) \leq \sum_{\ell > k} \ell \Upsilon(\ell)$ by Lemma 3.5. By Lemma 3.4, $\mu(U) < \delta_\varepsilon \to \mu(h(U)) < \varepsilon$ for all Borel U.

Next, choose $p_j \in G$ for $j \in \omega$ such that $h_{\sigma_{p_j}} \to f$ and $h_{\sigma_{p_j}}^{-1} \to f^{-1}$ pointwise. To do this, choose p_j so that $\mathrm{dom}(\sigma_{p_j})$ and $\mathrm{ran}(\sigma_{p_j})$ both meet the interval $[a2^{-j}, (a+1)2^{-j}]$ for all $a \in \mathbb{Z} \cap [-2^{2j}, 2^{2j}]$. Then, f is AC by Lemma 3.2.

Back in V, we need to prove that $\mathbb{P}$ is ccc, so fix $p_\alpha \in \mathbb{P}$ for $\alpha < \omega_1$; we shall find $\alpha \neq \beta$ with $p_\alpha \not\perp p_\beta$. WLOG, each $p_\alpha = (\sigma_\alpha, \Upsilon)$, with $m = \mathrm{dom}(\Upsilon) \geq 3$. We may also assume that each $|\sigma_\alpha| = t \geq 1$, and $\sigma_\alpha = \{(d_\alpha^i, e_\alpha^i) : i < t\}$. Further, we may assume that $d_\alpha^i < d_\beta^j$ and $e_\alpha^i < e_\beta^j$ holds whenever $i < j$ and $\alpha, \beta < \omega_1$.

Now, since $\mathbb{P}_0$ is ccc, fix $\alpha \neq \beta$ with $\sigma_\alpha \not\perp \sigma_\beta$; we shall get a $q = (\sigma_\alpha \cup \sigma_\beta, \hat{\Upsilon})$ such that $q \leq p_\alpha$ and $q \leq p_\beta$. So $\hat{m} = \mathrm{dom}(\hat{\Upsilon}) \geq m$ and $\hat{\Upsilon} \supseteq \Upsilon$. Taking $\hat{\Upsilon} = \Upsilon$ need not work because then q may fail to be in $\mathbb{P}$ because (3) or (4) could fail. To partly handle (3), we assume that there is some fixed rational $\varepsilon > 0$ such that $\mu(Z_\ell^{h_{\sigma_\alpha}}) < \Upsilon(\ell) - \varepsilon$ and $\mu(Z_\ell^{h_{\sigma_\alpha^{-1}}}) < \Upsilon(\ell) - \varepsilon$ holds for each α whenever $3 \leq \ell < m_p$, and that $\sum\{\ell \Upsilon(\ell) : \ell \geq 3 \ \& \ \ell < m\} < 1 - \varepsilon$, and that the σ_α are close enough together that for each α, β, $|d_\alpha^i - d_\beta^i| < \varepsilon/(4t)$ and $|e_\alpha^i - e_\beta^i| < \varepsilon/(4t)$. Furthermore, assume that for each i with $i + 1 < t$, and each integer ℓ, if the slope $(e_\alpha^{i+1} - e_\alpha^i)/(d_\alpha^{i+1} - d_\alpha^i) \notin [\ell - 1, \ell]$ holds for some α, then $(e_\beta^{i+1} - e_\alpha^i)/(d_\beta^{i+1} - d_\alpha^i) \notin [\ell - 1, \ell]$ holds for all α, β; and, likewise, for the slope of the inverse, $(d_\alpha^{i+1} - d_\alpha^i)/(e_\alpha^{i+1} - e_\alpha^i)$. This cures the problem with (3) for $\ell < m$.

However, (4) might fail for q because there is no way to bound, below or above, the slope between pairs of points (d_α^i, e_α^i) and (d_β^i, e_β^i). Let $\hat{m}$ be the smallest number $\geq m$ that makes (4) hold. If $\hat{m} = m$, we are done. Otherwise:

Let $\sigma = \sigma_\alpha \cup \sigma_\beta$. When $m \le \ell < \hat{m}$, let $c_\ell = |C_\ell|$, where $C_\ell = C_\ell^A \cup C_\ell^B$ and

$$\begin{array}{l} C_\ell^A = \{i < t : \ (d_\alpha^i, e_\alpha^i) \ne (d_\beta^i, e_\beta^i) \ \wedge \ (e_\beta^i - e_\alpha^i)/(d_\beta^i - d_\alpha^i) \in [\ell - 1, \ell]\} \\ C_\ell^B = \{i < t : \ (d_\alpha^i, e_\alpha^i) \ne (d_\beta^i, e_\beta^i) \ \wedge \ (d_\beta^i - d_\alpha^i)/(e_\beta^i - e_\alpha^i) \in [\ell - 1, \ell]\} \ . \end{array}$$

Let $\hat{\Upsilon}(\ell) = (c_\ell \varepsilon)/(2t\ell)$. Note that $C_\ell^A \cap C_k^B = \emptyset$, so no i lies in more than two of the C_ℓ, so $\sum_{m \le \ell < \hat{m}} c_\ell \le 2t$, and hence $\sum_{m \le \ell < \hat{m}} \ell \hat{\Upsilon}(\ell) \le \varepsilon$, which gives us (2); that is, $\sum\{\ell \hat{\Upsilon}(\ell) : \ell \ge 3 \ \& \ \ell < \hat{m}\} < 1$. To verify (3) when $m \le \ell < \hat{m}$, note that, using $|e_\alpha^i - e_\beta^i| < \varepsilon/(4t)$: $\mu(Z_\ell^{h_\sigma}) \le \sum_{i \in C_\ell^A} |d_\beta^i - d_\alpha^i| \le c_\ell \cdot \varepsilon/(4t(\ell - 1)) < c_\ell \cdot \varepsilon/(2t\ell)$; to bound $\mu(Z_\ell^{h_{\sigma^{-1}}})$, use C_ℓ^B.

In the higher dimensional case, we have no natural analog of h_σ; instead, our side conditions will include a function chosen from $\mathcal{F}_\theta$, defined below. First, a remark on norms; we use the Pythagorean norm on vectors in $\mathbb{R}^n$ and the operator norm on matrices:

Definition 3.7 *For $\vec{v} \in \mathbb{R}^n$, let $\|\vec{v}\| = (\sum_{i<n} (v_i)^2)^{1/2}$, and when Y is an $n \times n$ matrix, let $\|Y\| = \sup\{\|Y\vec{v}\| : \vec{v} \in S^{n-1}\}$.*

Definition 3.8 *When $\theta > 0$, let $\mathcal{F}_\theta = \mathcal{F}_\theta^n$ denote the set of all f such that:*

1. *f is a bijection from $\mathbb{R}^n$ onto $\mathbb{R}^n$.*
2. *f and f^{-1} are C^1.*
3. *$\exists r \ \exists \vec{c} \ \forall \vec{x} \ [\|\vec{x}\| \ge r \to f(\vec{x}) = \vec{c} + \vec{x}]$.*
4. *$\mathrm{tw}(f) < \theta$.*

Applying (2)(3),

Lemma 3.9 *If $f \in \mathcal{F}_\theta$, then $f^{-1} \in \mathcal{F}_\theta$, and f and f^{-1} are BAC.*

We remark that replacing "bijection" by "injection" in (1) results in an equivalent definition:

Lemma 3.10 *Assume that $f : \mathbb{R}^n \to \mathbb{R}^n$ is 1-1 and continuous and satisfies* (3) *above. Then f is a bijection.*

Proof. If $n = 1$, this is obvious by the Intermediate Value Theorem, so assume that $n > 1$. Now, assume that $\vec{d} \notin \mathrm{ran}(f)$. Replacing f by $\vec{x} \mapsto f(\vec{x}) - \vec{d}$, we may assume that $\vec{0} \notin \mathrm{ran}(f)$.

Define $\rho(\vec{y}) = \vec{y}/\|\vec{y}\|$, so ρ is the natural retraction of $\mathbb{R}^n \backslash \{0\}$ onto S^{n-1}. For $t \in [0, \infty)$, define $h_t : S^{n-1} \to S^{n-1}$ by $h_t(\vec{v}) = \rho(f(t\vec{v}))$. Then h_0 is the constant map $\vec{v} \mapsto \rho(f(\vec{0}))$. Fix $r, \vec{c}$ as in (3). For $t \gg \max(r, \|\vec{c}\|)$, $h_t(\vec{v}) =$

$(\vec{c} + t\vec{v})/\|\vec{c} + t\vec{v}\| \approx t\vec{v}/t = \vec{v}$, so h_t converges uniformly to the identity map as $t \to \infty$. But then, the identity map on S^n is homotopic to a constant map, which is impossible. ☕

Another simple remark:

Lemma 3.11 *If $f \in \mathcal{F}_\theta$, then $\det J_f(\vec{x}) > 0$ for all $\vec{x}$.*

Proof. $\det J_f(\vec{x}) \neq 0$ for all $\vec{x}$ by (2), and $\det J_f(\vec{x}) = 1$ for large enough $\vec{x}$ by (3), so use the fact that $\mathbb{R}^n$ is connected. ☕

Some more notation on norms:

Definition 3.12 *For $f : \mathbb{R}^n \to \mathbb{R}^n$, $\|f\| = \sup\{\|f(x)\| : x \in \mathbb{R}^n\}$, and $\|J_f\| = \sup\{\|J_f(x)\| : x \in \mathbb{R}^n\}$.*

For $f, g \in \mathcal{F}_\theta$, let $d(f, g) = \max(\|f - g\|, \|f^{-1} - g^{-1}\|)$. Then, the ball $B(f, \varepsilon) = \{g \in \mathcal{F}_\theta : d(f, g) < \varepsilon\}$.

Of course $\|f\|$ and/or $\|J_f\|$ may be ∞, and $\|J_f\|$ is only defined when f is differentiable. When $f, g \in \mathcal{F}_\theta$, $\|f\| = \infty$, but $d(f, g) < \infty$ and $1 \leq \|J_f\| < \infty$.

For forcing the f of Theorem 1.9, it will be convenient to use the distance function d, since it preserves the symmetry between f and f^{-1}:

Definition 3.13 *Following the terminology of Definition 2.2, and assuming* CH, *let $\mathbb{P}^\theta$ be $\mathbb{1}$ together with the set of all quadruples $p = (\sigma_p, h_p, \varkappa_p, \Upsilon_p)$ such that:*

1. *$\sigma_p \in \mathbb{P}^\theta_0$ and $\Upsilon_p \in (\mathbb{Q} \cap (0, \infty))^{<\omega}$; let $m_p = \mathrm{dom}(\Upsilon_p)$.*
2. *$\sum\{\ell \Upsilon_p(\ell) : \ell \geq 3 \ \&\ \ell < m_p\} < 1$.*
3. *$h_p \in \mathcal{F}_\theta$ and $h_p \supseteq \sigma_p$.*
4. *$\varkappa_p$ is a positive rational number.*
5. *Whenever $3 \leq \ell < m_p$: $\mu(Z_\ell^{h_p}) < \Upsilon_p(\ell)$ and $\mu(Z_\ell^{h_p^{-1}}) < \Upsilon_p(\ell)$.*
6. *$1/\max(2, m_p - 1) < \det J_{h_p}(x) < \max(2, m_p - 1)$ for all x.*

Define $q \leq p$ iff $p = \mathbb{1}$ or p, q are quadruples with $\sigma_q \supseteq \sigma_p$ and $\Upsilon_q \supseteq \Upsilon_p$ and $\varkappa_q \leq \varkappa_p$ and $B(h_q, \varkappa_q) \subseteq B(h_p, \varkappa_p)$.

So, h_p is an approximation to the f that we are constructing, and $\varkappa_p$ is a "promise" that this f will satisfy $d(f, h_p) \leq \varkappa_p$. There is no natural $\mathbb{1}$ in this poset, so we added one artificially, on top of all the "natural" forcing conditions. Note that $(\sigma, h, \varkappa', \Upsilon) \leq (\sigma, h, \varkappa, \Upsilon)$ always holds whenever $\varkappa' \leq \varkappa$. Also, by (6):

Lemma 3.14 *$\{p : m_p > \ell\}$ is dense for each ℓ.*

Also, we note that we can make a "small change" to h_p and obtain an extension of p:

Lemma 3.15 *For each* $p = (\sigma, h, \varkappa_p, \Upsilon_p) \in \mathbb{P}^\theta$*, there is a rational* $\zeta = \zeta_p > 0$ *such that for all* $g \in \mathcal{F}_\theta$*:*

If $d(g,h) < \varkappa_p$*, and* $g \supseteq \sigma$*, and* $\mu(\overline{S}), \mu(\overline{T}) \le \zeta$*, where* $S = \{x : g(x) \neq h(x)\}$ *and* $T = \{y : g^{-1}(y) \neq h^{-1}(y)\}$, then *there is a* $q \le p$ *of the form* $q = (\sigma, g, \varkappa_q, \Upsilon_q)$*.*

Proof. Choose ζ so that: (A) $\zeta < \Upsilon_p(\ell) - \mu(Z^h_\ell)$ and $\zeta < \Upsilon_p(\ell) - \mu(Z^{h^{-1}}_\ell)$ for all $\ell < m_p$, and (B) $7\zeta < 1 - \sum\{\ell\Upsilon_p(\ell) : \ell \ge 3 \ \& \ \ell < m_p\}$.

For $q \le p$: We need $\varkappa_q \le \varkappa_p$ and $B(g, \varkappa_q) \subseteq B(h, \varkappa_p)$, and these are satisfied if we just choose $\varkappa_q < \varkappa_p - d(g,h)$.

But we also need $\Upsilon_q \supseteq \Upsilon_p$ (so $m_q \ge m_p$), and we must be careful to define q to satisfy $(1-6)$. For (6), choose any $m_q \ge \max(3, m_p)$ such that $1/(m_q - 1) < \det J_g(x) < (m_q - 1)$ for all x.

For (5): (A) implies that (5) (for $\ell < m_p$) continues to hold with g replacing h. If $m_q = m_p$, we are now done, so assume that $m_q > m_p$. Also, assume that $m_q \ge 4$, since otherwise (5) and (2) are vacuous.

To ensure (5) when $\max(3, m_p) \le \ell < m_q$: choose rational $\Upsilon_q(\ell)$ such that $\mu(Z^g_\ell) + \mu(Z^{g^{-1}}_\ell) < \Upsilon_q(\ell) < \mu(Z^g_\ell) + \mu(Z^{g^{-1}}_\ell) + \zeta/m_q$. But now for (2): We've added $\sum\{\ell\Upsilon_q(\ell) : \max(3, m_p) \le \ell < m_q\}$ to the $\sum$ in (2). This amount is bounded above by ζ (from the ζ/m_q terms) plus

$$\sum_{\ell > k} [\ell\mu(Z^g_\ell) + \ell\mu(Z^{g^{-1}}_\ell)] \le 3[\mu(g(W^g_k)) + \mu(g^{-1}(W^{g^{-1}}_k))] \le 3[\mu(\overline{T}) + \mu(\overline{S})] \le 6\zeta\,,$$

where $k = \max(3, m_p) - 1$ (see Lemma 3.5), so we are done by (B).

To verify the second "$\le$" above, use $g(W^g_k) \subseteq \overline{T}$ and $g^{-1}(W^{g^{-1}}_k) \subseteq \overline{S}$. To verify $g(W^g_k) \subseteq \overline{T}$, fix $x \in W^g_k$. Then $\det J_g(x) \ge \max(3, m_p) - 1$. But also $\det J_h(x) < \max(2, m_p - 1)$, so $J_g(x) \neq J_h(x)$, and hence $x \in \mathrm{cl}(S)$ so $g(x) \in \mathrm{cl}(g(S))$; but $g(S) = h(S) = T$ because g and h are bijections. ☕

We now need the following two lemmas, whose proofs are a bit more complex than the corresponding results used in the proof of Lemma 3.6:

Lemma 3.16 *For* $\vec{d} \in D$ *and* $\vec{e} \in E$*, both sets* $\{p : \vec{d} \in \mathrm{dom}(\sigma_p)\}$ *and* $\{p : \vec{e} \in \mathrm{ran}(\sigma_p)\}$ *are dense in* $\mathbb{P}^\theta$*.*

Lemma 3.17 $\mathbb{P}^\theta$ *is ccc whenever* $\theta > \pi/2$*.*

These lemmas will be proved in Sections 4 and 5, after we prove some more facts about twists and Jacobians.

Proof of Theorem 1.9. As in the proof of Lemma 3.6, it is enough to assume CH, construct $\mathbb{P}^\theta$ (which is ccc by Lemma 3.17), and show that $V[G]$ contains the required f. We again have $f = \mathrm{cl}(\bigcup\{\sigma_p : p \in G\})$ and $\Upsilon = \bigcup\{\Upsilon_p : p \in G\}$. Since f and f^{-1} are uniform limits of continuous bijections, f is a continuous bijection of $\mathbb{R}^n$ onto $\mathbb{R}^n$. $\mathrm{tw}(f) \le \theta$ by Lemma 1.6. Also, $f(D) = E$ by Lemma 3.16, and absolute continuity for f and f^{-1} is proved as in Lemma 3.6. ☕

4 Twists and Jacobians

Definition 4.1 *$p = (\sigma, h, \varkappa, \Upsilon) \in \mathbb{P}^\theta$ is* nice *iff for all $(d, e) \in \sigma$, $h(x) = x + e - d$ holds in some neighborhood of d.*

Lemma 4.2 *The set of all nice p is dense in $\mathbb{P}^\theta$.*

This will be used in the proof of ccc (Lemma 3.17). That proof will use the same basic idea as the ccc proof from Lemma 3.6, which relied on establishing "$\sigma_\alpha \not\perp \sigma_\beta \to p_\alpha \not\perp p_\beta$". In the proof of Lemma 3.17, we can now say WLOG that all the p_α are nice. The fact that h_α and h_β are just translations near the various $(d, e) \in \sigma_\alpha \cup \sigma_\beta$ will aid in the proof of $p_\alpha \not\perp p_\beta$.

We shall prove Lemma 4.2 later in this section, after some preliminaries.

Because we are using the operator norm on the Jacobian, there is a Lipschitz condition in terms of $\|J_f\|$ when $\|J_f\| < \infty$:

Lemma 4.3 *If $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ then $\|f(c) - f(a)\| \le \|J_f\| \, \|c - a\|$ for all $c, a \in \mathbb{R}^n$.*

Proof. Let $b = c - a$. $\|f(c) - f(a)\|$ is no more than the length of the path from $f(a)$ to $f(c)$ defined by $t \mapsto f(a + tb)$ for $t \in [0, 1]$. This length equals $\int_0^1 \|\frac{d}{dt} f(a + tb)\| \, dt = \int_0^1 \|J_f(a + tb)\, b\| \, dt \le \int_0^1 \|J_f\| \|b\| \, dt = \|J_f\| \|b\|$. ☕

Using J_f, we can compute a "local twist":

Definition 4.4 *If Y is a non-singular matrix, let*

$$\mathrm{twist}(Y) = \{\angle(\vec{v}, Y\vec{v}) : \vec{v} \in S^{n-1}\} = \{\angle(\vec{v}, Y\vec{v}) : \vec{v} \in \mathbb{R}^n \backslash \{\vec{0}\}\} \quad .$$

Then, let $\mathrm{tw}(Y) = \sup(\mathrm{twist}(Y)) \in [0, \pi]$.

Observe that for $f \in \mathcal{F}_\theta$, $\mathrm{tw}(J_f(\vec{x})) < \theta$ for all $\vec{x}$. Also, note that $\mathrm{twist}(Y) = \mathrm{twist}(Y^{-1})$. Also, if f is the function $\vec{v} \mapsto Y\vec{v}$, then $\mathrm{twist}(Y) = \mathrm{twist}(f)$ and $\mathrm{tw}(Y) = \mathrm{tw}(f)$.

Next, a remark on elementary geometry. Let $\vec{v}$ be the center of the Earth and $\vec{x}$ a point on its surface, and let $\vec{w}$ be the center of the Moon and $\vec{y}$ a point on its surface. Then the lines $\overrightarrow{\vec{v}\vec{w}}$ and $\overrightarrow{\vec{x}\vec{y}}$ point in "almost" the same direction, and the following lemma gives a crude upper bound to the angle between them:

Lemma 4.5 *In $\mathbb{R}^n$: say $\|\vec{w} - \vec{v}\| = T$ (the* distance*), and $\|\vec{x} - \vec{v}\| = r$ and $\|\vec{y} - \vec{w}\| = s$ (the* two radii*), and assume that $T \geq r + s$. Let $\beta = \angle(\vec{w} - \vec{v}, \vec{y} - \vec{x})$. Then $\beta \leq \pi(r+s)/(2T)$.*

Proof. $\beta = \angle(\vec{w} - \vec{v},\ (\vec{y} + \vec{v} - \vec{x}) - \vec{v})$. Consider $\triangle ABC$, where A, B, C are the points $\vec{y} + \vec{v} - \vec{x}$, $\vec{v}$, $\vec{w}$, respectively. Let a, b be the lengths of the sides opposite A, B respectively, and let α be the angle at A; β is the angle at B. Note that $b = \|\vec{y} + \vec{v} - \vec{x} - \vec{w}\| \leq r + s \leq T = a$.

By the "law of sines", $b/\sin(\beta) = a/\sin(\alpha)$, so $\sin(\beta) = (b/a)\sin(\alpha) \leq b/a$. Also, $\beta < \pi/2$ because $b \leq a$, and $0 \leq t \leq \pi/2 \to \sin(t) \geq (2/\pi)\,t$, so $\beta \leq (\pi/2)\,(b/a) \leq (\pi/2)\,((r+s)/a)$. ☕

In many (but not all) of our applications, one of r, s will be 0. We remark that a precise upper bound is $\beta \leq \arcsin((r+s)/T)$, but the one in the lemma is simpler and will suffice in all our arguments.

We shall eventually prove the following, which is the "pure $\mathcal{F}_\theta$" analog of Lemma 4.2.

Lemma 4.6 *Assume that $f \in \mathcal{F}_\theta$ and $f(d) = e$ and $\varepsilon > 0$. Then there exists a $g \in \mathcal{F}_\theta$ such that $d(f, g) < \varepsilon$, and $g(d) = e$, and $g(x) = f(x)$ whenever $\|x - d\| \geq \varepsilon$ or $\|f(x) - e\| \geq \varepsilon$, and $g(x) = x - d + e$ holds in some neighborhood of d.*

So, g is close to f, but equals a simple translation near d. A rough idea of the proof: By translating the domain and range, we may assume that $d = e = 0$; then we need to get $g(x) = x$ for x near 0. We first modify f slightly to get a function h such that $h(x) = Ax$ near 0, where $A = J_f(0)$. We then get g by "morphing" A to I near 0. This "morphing" requires some further discussion of matrices:

Definition 4.7 *For $n \geq 1$, $\mathcal{M}^n$ denotes the space of all $n \times n$ real matrices; this has the topology of $\mathbb{R}^{n^2}$. Then, for $\theta > 0$, define $\mathcal{N}^n_\theta = \{A \in \mathcal{M}^n : \det A > 0\ \&\ \mathrm{tw}(A) < \theta\}$.*

Some easy closure properties:

Lemma 4.8 *$A \in \mathcal{N}^n_\theta \leftrightarrow A^{-1} \in \mathcal{N}^n_\theta \leftrightarrow cA \in \mathcal{N}^n_\theta \leftrightarrow O^{-1}AO \in \mathcal{N}^n_\theta$ whenever $c > 0$ and O is an orthogonal matrix.*

$\mathcal{N}^n_\theta$ is clearly open in $\mathcal{M}^n$, and $I \in \mathcal{N}^n_\theta$. But:

Question 4.9 *Is $\mathcal{N}^n_\theta$ connected when $0 < \theta < \pi$?*

The answer is trivially "yes" for $n = 1$. It is also "yes" for $n = 2$, as can be proved by direct computation, using Lemma 4.8 to simplify the form of the matrix. The following observation makes this question irrelevant for our work here:

Lemma 4.10 *If $f \in \mathcal{F}_\theta$ and $\vec{a} \in \mathbb{R}^n$ and $A = J_f(\vec{a})$, then $A \in \mathcal{N}^n_\theta$ and there is a C^∞ path $\Gamma : [0, 1] \to \mathcal{N}^n_\theta$ such that $\Gamma(0) = I$ and $\Gamma(1) = A$.*

Proof. To get a continuous Γ, fix $r, \vec{c}$ as in (3) of Definition 3.8, and then fix $\vec{d}$ with $\|\vec{d}\| > r$. Then let $\Gamma(t) = J_f(t\vec{a} + (1-t)\vec{d})$. Then, observe that (just because $\mathcal{N}^n_\theta$ is open in $\mathcal{M}^n$), whenever A, B lie in the same connected component of $\mathcal{N}^n_\theta$, they are connected by a C^∞ path lying in $\mathcal{N}^n_\theta$. ☕

The following lemma expresses the basic matrix morphing:

Lemma 4.11 *Assume that $h(\vec{v}) = A(\|\vec{v}\|)\vec{v}$, where $A : [0, \infty) \to \mathcal{M}^n$ and for each $r \in [0, \infty)$, $A(r)$ is non-singular and $\mathrm{tw}(A(r)) < \theta$. Assume that $M := \sup\{\|A(r)^{-1}\| : r \in [0, \infty)\} < \infty$. Fix $\varepsilon \in (0, \pi/2)$ and assume that:*

$$\|A((1+\sigma)r) - A(r)\| < (\varepsilon\sigma)/(\pi M) \qquad (\forall \sigma, r > 0) \quad . \tag{$*$}$$

Then h is 1-1 and $\mathrm{tw}(h) < \theta + \varepsilon/2$. Furthermore,

$$\|h(\vec{v}_1) - h(\vec{v}_0)\| \geq \|\vec{v}_1 - \vec{v}_0\|/(2M) \tag{$\dagger$}$$

for all $\vec{v}_0, \vec{v}_1$.

Proof. First, we establish ($\dagger$), which implies that h is 1-1. Let $A_i = A(\vec{v}_i)$ for $i = 0, 1$. Observe:

$$\|A_1(\vec{v}_1 - \vec{v}_0)\| \geq \|\vec{v}_1 - \vec{v}_0\|/\|A_1^{-1}\| \geq \|\vec{v}_1 - \vec{v}_0\|/M \tag{1}$$

Since $\dagger$ is clear from (1) when $\|\vec{v}_1\| = \|\vec{v}_0\|$ or $\vec{v}_0 = \vec{0}$, we may assume that $\|\vec{v}_0\| = r$ and $\|\vec{v}_1\| = (1+\sigma)r$, where $\sigma, r > 0$. Then ($\dagger$) follows using (2)(1)(3):

$$\|h(\vec{v}_1) - h(\vec{v}_0)\| = \|A_1\vec{v}_1 - A_0\vec{v}_0\| = \|A_1(\vec{v}_1 - \vec{v}_0) + (A_1 - A_0)\vec{v}_0\| \tag{2}$$
$$\|(A_1 - A_0)\vec{v}_0\| \leq (\varepsilon\sigma r)/(\pi M) \leq \|\vec{v}_1 - \vec{v}_0\|\varepsilon/(\pi M) \leq \|\vec{v}_1 - \vec{v}_0\|/(2M) \tag{3}$$

For $\mathrm{tw}(h) < \theta + \varepsilon/2$, we must show that $\angle(\vec{v}_1 - \vec{v}_0, h(\vec{v}_1) - h(\vec{v}_0)) < \theta + \varepsilon/2$ whenever $\vec{v}_1 \neq \vec{v}_0$. This is clear if $\|\vec{v}_1\| = \|\vec{v}_0\|$ or if one of $\vec{v}_1, \vec{v}_0$ is $\vec{0}$, so we may assume that $\vec{v}_0, \vec{v}_1, A_0, A_1, r, \sigma$ are as above, and we must show that

$$\angle(\vec{v}_1 - \vec{v}_0, A_1\vec{v}_1 - A_0\vec{v}_0) < \theta + \varepsilon/2$$

Now, using $\mathrm{tw}(A(r)) < \theta$, we know that $\angle(\vec{v}_1 - \vec{v}_0, A_1\vec{v}_1 - A_1\vec{v}_0) < \theta$, so we now use Lemma 4.5 to show that

$$\beta := \angle(A_1\vec{v}_1 - A_1\vec{v}_0, A_1\vec{v}_1 - A_0\vec{v}_0) \leq \varepsilon/2 \quad .$$

The "distance" is $T = \|A_1\vec{v}_1 - A_1\vec{v}_0\| \geq \|\vec{v}_1 - \vec{v}_0\|/M \geq \sigma r/M$, using (1), and the two "radii" are 0 and $\|A_1\vec{v}_0 - A_0\vec{v}_0\| \leq r \cdot (\varepsilon\sigma)/(\pi M)$ by $(*)$, so that $\beta \leq \pi \cdot r \cdot (\varepsilon\sigma)/(\pi M) \div 2\sigma r/M = \varepsilon/2$.

We shall obtain the $A(r)$ using a path in $\mathcal{N}^n_\theta$, with the aid of the following:

Lemma 4.12 *Given $P, Q, \zeta > 0$, with $P < Qe^{-1/\zeta}$, there is a non-decreasing C^∞ function $\varphi : \mathbb{R} \to [0, 1]$ such that $\varphi(x) = 0$ whenever $x \leq P$, and $\varphi(x) = 1$ whenever $x \geq Q$, and $\varphi((1 + \sigma)x) - \varphi(x) \leq \zeta\sigma$ whenever $\sigma, x > 0$.*

Proof. Fix P', Q', ζ' such that $P < P' < Q' = e^{1/\zeta'}P' < Q$ and $0 < \zeta' < \zeta$. Now, let $\psi(x)$ be 0 when $x \leq P'$, 1 when $x \geq Q'$, and $\zeta' \log(x/P')$ when $P' \leq x \leq Q'$. Then $\psi((1 + \sigma)x) - \psi(x) \leq \zeta'\sigma$ whenever $\sigma, x > 0$, but ψ does not satisfy the lemma because, although it is continuous, it is not C^1.

To obtain a C^∞ function, fix $a > 0$ such that $a < Q - Q'$ and $a < P' - P$ and $a \leq (\zeta - \zeta')P/(1 + \zeta)$, and convolve ψ with a smooth function supported on $[-a, a]$. Let $\delta : \mathbb{R} \to [0, 1]$ be a C^∞ function such that $\delta(t) = 0$ whenever $|t| \geq a$ and $\delta(t) = \delta(-t)$ for all t and $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$. Then let

$$\varphi(x) = \int_{-\infty}^{\infty} \delta(t)\psi(x - t)\, dt = \int_{-\infty}^{\infty} \delta(x - u)\psi(u)\, du \quad .$$

Then φ satisfies everything required except possibly for $\varphi((1 + \sigma)x) - \varphi(x) \leq \zeta\sigma$ whenever $\sigma, x > 0$. Rewrite this as the equivalent

$$0 < x < y \quad \to \quad \varphi(y) - \varphi(x) \leq \zeta(y - x)/x \quad . \tag{$*$}$$

This is clear when $y \leq P$ (since then $\varphi(y) - \varphi(x) = 0$), so assume always that $y > P$. Also, $(*)$ is clear when $\zeta(y-x)/x \geq 1$, which is equivalent to $\zeta y \geq (1+\zeta)x$. Using $y > P$, we may assume now also that $\zeta P < (1 + \zeta)x$. This implies that

$x - a > 0$ (using our third assumption on a), which justifies the following, using $0 < u < v \to \psi(v) - \psi(u) \le \zeta'(v-u)/u$:

$$\varphi(y) - \varphi(x) = \int_{-\infty}^{\infty} \delta(t)[\psi(y-t) - \psi(x-t)]\,dt \le \zeta' \int_{-\infty}^{\infty} \delta(t)[(y-x)/(x-t)]\,dt \quad .$$

This will give us $(*)$ *if* we know that

$$\forall t \in [-a, a] \; \big(\zeta'[(y-x)/(x-t)] \le \zeta(y-x)/x\big) \quad . \qquad (\dagger)$$

But $(\dagger)$ is equivalent to $\zeta'/\zeta \le \min\{(x-t)/x : t \in [-a,a]\}$, and this min is just $1 - a/x$, so we shall have $(\dagger)$ if $a/x \le 1 - \zeta'/\zeta = (\zeta - \zeta')/\zeta$. Since we are assuming that $x > \zeta P/(1+\zeta)$, we just need $a \le (\zeta - \zeta')P/(1+\zeta)$, which was our third assumption on a.

Lemma 4.13 *Lemma 4.6 holds in the special case that* $\vec{d} = \vec{e} = \vec{0}$ *and* $f(\vec{x}) = A\vec{x}$ *in some neighborhood of* $\vec{0}$.

Proof. Fix $\hat{\theta} \in (0, \theta)$ such that $f \in \mathcal{F}_{\hat{\theta}}$; make sure that $\theta - \hat{\theta} < \pi/2$. Then, applying Lemma 4.10, let $\Gamma : [0,1] \to \mathcal{N}^n_{\hat{\theta}}$ be a C^∞ path in $\mathcal{N}^n_{\hat{\theta}}$ with $\Gamma(0) = I$ and $\Gamma(1) = A$. Note that a smooth path is also Lipschitz, so fix $K > 0$ such that $\|\Gamma(t_0) - \Gamma(t_1)\| \le K|t_0 - t_1|$ for all $t_0, t_1 \in [0,1]$. Also fix $R > 0$ such that $f(\vec{v}) = A\vec{v}$ whenever $\|\vec{v}\| \le R$. Let $M = \sup\{\|(\Gamma(t))^{-1}\| : t \in [0,1]\}$. Let $C = \inf\{\|f(\vec{v})\| : \|\vec{v}\| \ge R\}$. Let $J = \sup\{\|\Gamma(t)\| : t \in [0,1]\}$; note that $J \ge 1$. Then, choose Q, ζ satisfying:

a. $0 < \zeta < (\theta - \hat{\theta})/(\pi M K) \le 1/(2KM)$.
b. $0 < Q < R$
c. $0 < \pi J Q/(C - \|A\|Q) < (\theta - \hat{\theta})$.
d. $JQ < \varepsilon/2$ and $\forall \vec{x}\,[\|\vec{x}\| \le Q \to \|f(\vec{x})\| < \varepsilon/2]$.

Fix $P \in (0, Qe^{-1/\zeta})$, and then fix φ as in Lemma 4.12. Let $A(r) = \Gamma(\varphi(r))$. Then $A(r) = I$ for $r \le P$ and $A(r) = A$ for $r \ge Q$. Define $h(\vec{v}) = A(\|\vec{v}\|)\vec{v}$. By Lemma 4.11, $\mathrm{tw}(h) < \theta$ and h is 1-1 if we can show:

$$\|\Gamma(\varphi((1+\sigma)r)) - \Gamma(\varphi(r))\| < \frac{(\theta - \hat{\theta})}{\pi M}\sigma \qquad (\forall \sigma, r > 0) \quad .$$

But this follows from (a) above, using $\varphi((1+\sigma)r) - \varphi(r) \le \zeta\sigma$ and our Lipschitz constant K, which implies that $\|\Gamma(\varphi((1+\sigma)r)) - \Gamma(\varphi(r))\| \le \zeta K \sigma$.

Note that $h(\vec{v}) = f(\vec{v})$ whenever $Q \le \vec{v} \le R$. Let $g(\vec{v})$ be $h(\vec{v})$ when $\|\vec{v}\| \le R$ and $f(\vec{v})$ when $\|\vec{v}\| \ge Q$.

To show that g is 1-1: fix v_0, v_1 with $v_0 \neq v_1$; we must show that $g(v_0) \neq g(v_1)$. Let $r_i = \|\vec{v}_i\|$. We may assume that $r_0 \leq r_1$. But also, $g(v_0) \neq g(v_1)$ is clear whenever $g\restriction\{\vec{v}_0, \vec{v}_1\}$ equals either $f\restriction\{\vec{v}_0, \vec{v}_1\}$ or $h\restriction\{\vec{v}_0, \vec{v}_1\}$, so we may assume that $r_0 < Q$ and $r_1 > R$. Then $\|g(v_0)\| = \|A(r_0)v_0\| \leq JQ$ and $\|g(v_1)\| = \|f(v_1)\| \geq C$, so $g(v_0) \neq g(v_1)$ because $JQ < C$ by (c).

To prove that $\mathrm{tw}(g) < \theta$, fix v_0, v_1, r_0, r_1 as above with $v_0 \neq v_1$; we must show that that $\angle(\vec{v}_1 - \vec{v}_0, g(\vec{v}_1) - g(\vec{v}_0)) < \theta$. By the same reasoning, we may assume that $r_0 < Q$ and $r_1 > R$.

Now, we have $\angle(\vec{v}_1 - \vec{v}_0, f(\vec{v}_1) - f(\vec{v}_0)) < \hat{\theta}$, and shall use Lemma 4.5 to conclude that $\angle(\vec{v}_1 - \vec{v}_0, g(\vec{v}_1) - g(\vec{v}_0))$ by verifying that

$$\beta := \angle(f(\vec{v}_1) - f(\vec{v}_0), g(\vec{v}_1) - g(\vec{v}_0)) \leq \theta - \hat{\theta} \quad .$$

Note that $g(\vec{v}_1) = f(\vec{v}_1)$, while $g(\vec{v}_0) = h(\vec{v}_0) = A(r_0)\vec{v}_0$ and $f(\vec{v}_0) = A\vec{v}_0$. Then the "distance" is $T = \|f(\vec{v}_1) - f(\vec{v}_0)\| \geq C - \|A\|Q$, and the two "radii" are $\|f(\vec{v}_1) - g(\vec{v}_1)\| = 0$ and $\|f(\vec{v}_0) - g(\vec{v}_0)\| = \|(A - A(r_0))\vec{v}_0\| \leq 2JQ$, so $\beta \leq \pi \cdot 2JQ \div 2(C - \|A\|Q) \leq \theta - \hat{\theta}$ by (c).

To prove that $g(\vec{x}) = f(\vec{x})$ whenever $\|\vec{x}\| \geq \varepsilon$ or $\|f(\vec{x})\| \geq \varepsilon$: For $\|\vec{x}\| \geq \varepsilon$, just use $Q < \varepsilon$, by (d). For $\|f(\vec{x})\| \geq \varepsilon$, use (d), which implies that $\|f(\vec{x})\| \geq \varepsilon \to \|\vec{x}\| \geq Q \to g(\vec{x}) = f(\vec{x})$.

To prove that $\|g - f\| < \varepsilon$, use (d) to show that $\|\vec{x}\| \leq Q$ implies that $\|g(x) - f(x)\| \leq \|g(\vec{x})\| + \|f(\vec{x})\| \leq JQ + \|f(\vec{x})\| < \varepsilon/2 + \varepsilon/2$.

To prove that $\|g^{-1} - f^{-1}\| < \varepsilon$: We want $f(\vec{x}) = g(\vec{z}) \to \|\vec{x} - \vec{z}\| < \varepsilon$. Since f, g are both 1-1, this is trivial unless $f(\vec{x}) \neq g(\vec{x})$ and $f(\vec{z}) \neq g(\vec{z})$. Then $\|\vec{x}\|, \|\vec{z}\| < Q$, so apply the fact that $Q < \varepsilon/2$.

Finally, we must prove that g^{-1} is C^1. Since f^{-1} is C^1, it is sufficient to prove that h^{-1} is C^1. Since h is a C^1 bijection, it is sufficient to prove that J_h is everywhere non-singular, which follows if we show that h^{-1} is Lipschitz; but this is clear from Lemma 4.11.

Next, we need to show that every function in $\mathcal{F}_\theta$ is close to some $f \in \mathcal{F}_\theta$ such that $f(\vec{x}) = A\vec{x}$ in some neighborhood of $\vec{0}$. We first show that every "small modification" of a function in $\mathcal{F}_\theta$ also lies in $\mathcal{F}_\theta$.

Lemma 4.14 *Fix $f \in \mathcal{F}_\theta$, and fix $\hat{\theta} \in (\mathrm{tw}(f), \theta)$ with $\theta - \hat{\theta} < \pi/2$. Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function such that $\exists r\, \forall \vec{x}\, [\|\vec{x}\| \geq r \to g(\vec{x}) = \vec{0}]$. Assume also*

$$\|g(\vec{v}_1) - g(\vec{v}_0)\| \leq \frac{2}{\pi}(\theta - \hat{\theta})\|f(\vec{v}_1) - f(\vec{v}_0)\| \qquad (\forall \vec{v}_0, \vec{v}_1 \in \mathbb{R}^n) \quad . \qquad (✰)$$

Then $f + g \in \mathcal{F}_\theta$. Furthermore, $d(f, f + g) \leq \|g\| \cdot \|J_{f^{-1}}\|$.

Proof. Let $h = f + g$. It is clear that h is C^1 and 1-1 and satisfies (3) of Definition 3.8. It follows from Lemma 3.10 that h is a bijection. It is easy to see from (✰) that $J_h(\vec{x})$ must be non-singular, so that h^{-1} is also C^1.

To prove that $\mathrm{tw}(h) < \theta$, we must show that $\angle(\vec{v}_1 - \vec{v}_0, h(\vec{v}_1) - h(\vec{v}_0)) < \theta$. whenever $\vec{v}_0 \neq \vec{v}_1$. Now $\angle(\vec{v}_1 - \vec{v}_0, f(\vec{v}_1) - f(\vec{v}_0)) < \hat{\theta}$, so we apply Lemma 4.5 to show that

$$\beta := \angle(f(\vec{v}_1) - f(\vec{v}_0), h(\vec{v}_1) - h(\vec{v}_0)) \leq \theta - \hat{\theta} \quad .$$

Here, $h = f + g$, so $\beta = \angle(f(\vec{v}_1) - f(\vec{v}_0), f(\vec{v}_1) - [f(\vec{v}_0) + g(\vec{v}_0) - g(\vec{v}_1)])$. Then the "distance" is $T = \|f(\vec{v}_1) - f(\vec{v}_0)\|$ and the two radii are 0 and $\|g(\vec{v}_1) - g(\vec{v}_0)\|$, so $\beta \leq \pi \cdot \|g(\vec{v}_1) - g(\vec{v}_0)\| \div 2\|f(\vec{v}_1) - f(\vec{v}_0)\| \leq \theta - \hat{\theta}$, using (✰).

Regarding $d(f, f+g)$ and referring to Definition 3.12: It is obvious that $\|f - (f+g)\| = \|g\| \leq \|g\| \cdot \|J_{f^{-1}}\|$, but to bound $\|f^{-1} - (f+g)^{-1}\|$: say $f^{-1}(\vec{y}) = \vec{x}$ and $(f+g)^{-1}(\vec{y}) = \vec{z}$. Then $f(\vec{x}) = \vec{y} = f(\vec{z}) + g(\vec{z})$. Now

$$\|g\| \geq \|f(\vec{z}) - (f(\vec{z}) + g(\vec{z}))\| = \|f(\vec{z}) - f(\vec{x})\| \geq \|\vec{x} - \vec{z}\| / \|J_{f^{-1}}\| \quad ,$$

so $\|\vec{x} - \vec{z}\| \leq \|g\| \cdot \|J_{f^{-1}}\|$. ☕

Lemma 4.15 *Fix $f \in \mathcal{F}_\theta$, and assume that $f(\vec{0}) = \vec{0}$. Let $A = J_f(\vec{0})$. Fix any $\varepsilon > 0$. Then there exists an $h \in \mathcal{F}_\theta$ such that $d(f, h) < \varepsilon$, and $h(\vec{0}) = \vec{0}$, and $h(\vec{x}) = f(\vec{x})$ whenever $\|\vec{x}\| \geq \varepsilon$ or $\|f(\vec{x})\| \geq \varepsilon$, and $h(\vec{x}) = A\vec{x}$ holds in some neighborhood of $\vec{0}$.*

Proof. Fix $\hat{\theta} \in (\mathrm{tw}(f), \theta)$, with $\theta - \hat{\theta} < \pi/2$. h will be $f + g$, where (✰) of Lemma 4.14 holds. Let $L = \|J_{f^{-1}}\|$; so $L \geq 1$. Then, applying Lemma 4.3, $\|f^{-1}(\vec{y}_1) - f^{-1}(\vec{y}_0)\| \leq L\|\vec{y}_0 - \vec{y}_1\|$ holds for all $\vec{y}_0, \vec{y}_1, \vec{x}_0, \vec{x}_1$. Shrinking ε if necessary, we may assume that $\varepsilon \leq 2(\theta - \hat{\theta})/\pi$; then (✰) will follow from:

$$\|g(\vec{v}_1) - g(\vec{v}_0)\| \leq (\varepsilon/L)\|\vec{v}_1 - \vec{v}_0\| \qquad (\forall \vec{v}_1, \vec{v}_0 \in \mathbb{R}^n) \quad . \tag{✫}$$

Also, $d(f, h) \leq \|g\| L$ by Lemma 4.14, and we shall in fact get $\|g\| < \varepsilon/L$.

Choose P, Q, R, ζ with $0 < P < Q < R$ and $\zeta > 0$, and choose $\psi : \mathbb{R} \to [0, 1]$ to satisfy:

a. $\zeta \leq \varepsilon/(2L)$.
b. $R \leq \varepsilon/2$; and $\|A\vec{x} - f(\vec{x})\| < \varepsilon/L$ and $\|J_f(\vec{x}) - A\| \leq \zeta$ whenever $\|\vec{x}\| \leq R$.
c. $Q \leq R/2$, and $\|A\vec{x}\| + \|f(\vec{x})\| \leq (\varepsilon/L)(R/2)$ whenever $\|\vec{x}\| \leq Q$.
d. ψ is C^∞ and non-increasing, and $\psi(t) = 1$ for all $t \leq P$, and $\psi(t) = 0$ for all $t \geq Q$, and $\psi(t) - \psi((1+\sigma)t) \leq \zeta\sigma$ whenever $\sigma, t > 0$.

There are such P, ψ as in (d) by Lemma 4.12. Note that $\varepsilon < 1$ and $\zeta, R < 1/2$ and $Q < 1/4$. Let $g(\vec{x}) = \psi(\|\vec{x}\|)(A\vec{x} - f(\vec{x}))$. Then $\|g\| < \varepsilon/L$ by (b). We need $g(\vec{x}) = \vec{0}$ (and hence $h(\vec{x}) = f(\vec{x})$) whenever $\|\vec{x}\| \geq \varepsilon$ or $\|f(\vec{x})\| \geq \varepsilon$. When $\|\vec{x}\| \geq \varepsilon$, use $Q \leq R \leq \varepsilon$. When $\|f(\vec{x})\| \geq \varepsilon$, note that $\|\vec{x}\| \geq Q$ because $\|\vec{x}\| < Q \to \|f(\vec{x})\| < \varepsilon$ by (c). Now, we are done if we verify (✫):

Let $r_i = \|\vec{v}_i\|$. We may assume that $r_0 \leq r_1$. We may also assume that $r_0 \leq Q$, since otherwise (✫) is trivial.

If $r_1 \geq R$, then $g(\vec{v}_1) = \vec{0}$ and $\|\vec{v}_0 - \vec{v}_1\| \geq (R - Q)$, so it is sufficient to verify

$$\|g(\vec{v}_0)\| \leq (\varepsilon/L)(R - Q) \quad ,$$

which follows from (c) above.

From now on, assume that $r_1 \leq R$. Define $\vec{w}(\vec{v}_0, \vec{v}_1)$ by:

$$\vec{w}(\vec{v}_0, \vec{v}_1) = f(\vec{v}_1) - f(\vec{v}_0) - A(\vec{v}_1 - \vec{v}_0) = k(\vec{v}_1) - k(\vec{v}_0) \ ; \ k(\vec{v}) = f(\vec{v}) - A\vec{v} \ .$$

Note that $J_k(\vec{v}) = J_f(\vec{v}) - A$. Then, $\|\vec{w}(\vec{v}_0, \vec{v}_1)\| \leq \zeta\|\vec{v}_1 - \vec{v}_0\|$ when $\|\vec{v}_1\|, \|\vec{v}_0\| \leq R$; to see this, use (b) above and Lemma 4.3. Let $r = r_0$ and $r_1 = ((1+\sigma)r)$. Now, using $r \leq Q$ along with (d) and $\sigma r = r_1 - r_0 \leq \|\vec{v}_1 - \vec{v}_0\|$:

$$\begin{aligned}
&\|g(\vec{v}_1) - g(\vec{v}_0)\| = \\
&\left\|\psi(r_1)\left[A(\vec{v}_1 - \vec{v}_0) - f(\vec{v}_1) + f(\vec{v}_0)\right] \ + \ (\psi(r_1) - \psi(r_0))(A\vec{v}_0 - f(\vec{v}_0))\right\| \leq \\
&\|w(\vec{v}_0, \vec{v}_1)\| \ + \ |\psi((1+\sigma)r) - \psi(r)| \cdot \|w(\vec{v}_0, \vec{0})\| \leq \\
&\zeta\|\vec{v}_1 - \vec{v}_0\| \ + \ \zeta\sigma \cdot \zeta Q \leq (\zeta + \zeta^2 Q/r)\|\vec{v}_1 - \vec{v}_0\| \ .
\end{aligned}$$

So, we are done because $(\zeta + \zeta^2 Q/r) \leq (1+\zeta) \cdot \varepsilon/(2L) \leq \varepsilon/L$.

Proof of Lemma 4.6. First, replacing f by $\vec{x} \mapsto f(\vec{x} + \vec{d}) - \vec{e}$, it is sufficient to prove the lemma in the case that $\vec{d} = \vec{e} = \vec{0}$. Then, using Lemma 4.15 (to get h) followed by Lemma 4.13 (to get g from h) yields everything required except that we'll have $d(f, g) < 2\varepsilon$; so, apply these two lemmas with $\varepsilon/2$ replacing ε.

Proof of Lemma 3.16. We show that $W := \{p : \vec{d} \in \mathrm{dom}(\sigma_p)\}$ is dense. Fix $p = (\sigma, h, \varkappa, \Upsilon) \in \mathbb{P}^\theta$ with $\vec{d} \notin \mathrm{dom}(\sigma)$; we shall find a $q = (\sigma_q, h_q, \varkappa_q, \Upsilon_q) \leq p$ with $q \in W$. Fix $\ell \in \omega$ such that $\xi := \mathrm{ht}(\vec{d}) + \ell \neq \mathrm{ht}(\vec{z})$ for all $\vec{z} \in \mathrm{dom}(\sigma) \cup \mathrm{ran}(\sigma)$. Let $E_\xi = \{\vec{e} \in E : \mathrm{ht}(\vec{e}) = \xi\}$.

Let $\vec{c} = h(\vec{d})$. Fix $\hat{\theta} \in (\mathrm{tw}(h), \theta)$, with $\theta - \hat{\theta} < \pi/2$. Let $M = \|J_h\|$ and $L = \|J_{h^{-1}}\|$; so $M, L \geq 1$. Fix $Q, P, \psi, \varepsilon, \vec{e}, \vec{a}$ so that:

a. $Q < \min\{\|\vec{d}-\vec{d'}\| : \vec{d'} \in \mathrm{dom}(\sigma)\}$ and $\mu(B(\vec{0}, MQ)) < \zeta_p$ (see Lemma 3.15).
b. $0 < P < Q$ and $\psi : \mathbb{R} \to [0,1]$ is a C^∞ non-increasing function, and $\psi(t) = 1$ for all $t \leq P$, and $\psi(t) = 0$ for all $t \geq Q$.
c. $0 < \varepsilon < 2(\theta - \hat{\theta})/(\pi L \|\psi'\|)$, and $\varepsilon < \varkappa_p / L$.
d. $\vec{e} = \vec{c} + \vec{a} \in E_\xi$ and $\|\vec{a}\| < \varepsilon$.

Let $h_q(\vec{d}+\vec{v}) = h(\vec{d}+\vec{v}) + \psi(\|\vec{v}\|)\vec{a}$; $h_q \supset \sigma$ by (a). Let $\sigma_q = \sigma_p \cup \{(\vec{d}, \vec{e})\}$. Then $h_q \supset \sigma_q \supset \sigma_p$. Now apply Lemma 4.14, with $f = h$ and $g(\vec{d}+\vec{v}) = \psi(\|\vec{v}\|)\vec{a}$. This yields $h_q \in \mathcal{F}_\theta$ and $d(h, h_q) \leq L\|g\| \leq L\|\vec{a}\| < L\varepsilon$ (using (d)).

But to see that Lemma 4.14 applies here, we need to verify (✰); that is, $\|g(\vec{d}+\vec{v}_1) - g(\vec{d}+\vec{v}_0)\| \leq (2(\theta - \hat{\theta})/\pi) \|h(\vec{d}+\vec{v}_1) - h(\vec{d}+\vec{v}_0)\|$. Let $r_i = \|\vec{v}_i\|$; we may assume that $r_0 \leq r_1$. Then $\|g(\vec{d}+\vec{v}_1) - g(\vec{d}+\vec{v}_0)\| \leq \varepsilon \|\psi'\|(r_1 - r_0)$ and $\|h(\vec{d}+\vec{v}_1) - h(\vec{d}+\vec{v}_0)\| \geq \|\vec{v}_1 - \vec{v}_0\|/L \geq (r_1 - r_0)/L$, so (✰) holds by (c).

We obtain $\varkappa_q$ and Υ_q by using Lemma 3.15. This lemma requires both $d(h, h_q) < \varkappa_p$ (which holds by (c)) and $\mu(\overline{S}), \mu(\overline{T}) \leq \zeta_p$. For this second inequality, apply (a) and note that $S \subseteq B(\vec{d}, Q)$ and $T \subseteq h(S) \subseteq B(\vec{c}, MQ)$.

Observe that $\sigma_q \in P_0^\theta$: (P2) holds because $\sigma_q \subset h_q$, and (P3)(P4) hold by (d) and our choice of ξ.

Proof of Lemma 4.2. If $p \in \mathbb{P}^\theta$ and $m = |\sigma_p|$, then we use Lemma 4.6 m times to construct $p = q_0 \geq q_1 \geq q_2 \cdots \geq q_m$, where q_m is nice. All q_i have the same $\sigma_{q_i} = \sigma_p$, but h_{q_i} will be a translation in some neighborhood of i many of the $(\vec{d}, \vec{e}) \in \sigma_p$. Given q_i, we use Lemma 4.6 to construct $h_{q_{i+1}}$ from h_{q_i}. Note that Lemma 4.6 lets us ensure that $h_{q_{i+1}}$ and h_{q_i} are close enough to be able to use Lemma 3.15 to build an appropriate $\varkappa_{q_{i+1}}$ and $\Upsilon_{q_{i+1}}$.

The following consequence of Lemma 4.14 will be useful:

Lemma 4.16 *Fix $\theta \in (0, \pi)$. To each $f \in \mathcal{F}_\theta$, one can assign positive rationals ε_f and δ_f and M_f such that:*

Whenever $f, g \in \mathcal{F}_\theta$ with $\delta_f = \delta_g = \delta$ and $\varepsilon_f = \varepsilon_g = \varepsilon$ and $M_f = M_g = M$: If $\|f - g\| < \varepsilon$ and $\|J_f - J_g\| < \delta$ then $(f+g)/2 \in \mathcal{F}_\theta$. Furthermore, $d(f, (f+g)/2) \leq \|g - f\| \cdot \|J_{f^{-1}}\|/2$.

Proof. Choose $M_f \geq \|J_{f^{-1}}\|$; then $M_f \geq 1$. Choose $\varepsilon_f < (\theta - \mathrm{tw}(f))/\pi$. Then $\varepsilon_f < 1$, and $f \in \mathcal{F}_{\hat{\theta}}$, where $\theta - \hat{\theta} > \pi \varepsilon_f$. Choose $\delta_f < 4\varepsilon_f / M_f$.

Now use Lemma 4.14. Let $h = (g - f)/2$, so $(f + g)/2 = f + h$. Then (✰) requires $\|h(\vec{v}_1) - h(\vec{v}_0)\| \leq \frac{2}{\pi}(\theta - \hat{\theta})\|f(\vec{v}_1) - f(\vec{v}_0)\|$, which will be satisfied if we have $\|h(\vec{v}_1) - h(\vec{v}_0)\| \leq 2\varepsilon \|f(\vec{v}_1) - f(\vec{v}_0)\|$. Now $\|J_h\| = \|J_f - J_g\|/2 < \delta/2$ so $\|h(\vec{v}_1) - h(\vec{v}_0)\| \leq \|v_1 - v_0\| \cdot \delta/2$, while $\|f(\vec{v}_1) - f(\vec{v}_0)\| \geq \|v_1 - v_0\|/M$, so we are done because $\delta/2 \leq 2\varepsilon/M$.

5 Proof of ccc

Our proof imitates the ccc proof in Lemma 3.6. We start with p_α for $\alpha < \omega_1$ and prove that two of them are compatible. By Lemma 4.2, we may assume that all the p_α are nice. We now apply some preliminary thinning. Since there are only $\aleph_0$ possibilities for $\varkappa_p$ and Υ_p, we may assume that each $p_\alpha = (\sigma_\alpha, h_\alpha, \varkappa, \Upsilon)$. We may assume that $|\sigma_\alpha| = t$ for all α, so $\sigma_\alpha = \{(d^0_\alpha, e^0_\alpha), \ldots, (d^{t-1}_\alpha, e^{t-1}_\alpha)\}$. We may also assume that there is a fixed $\hat\theta \in (\pi/2, \theta)$ such that all $p_\alpha \in \mathbb{P}^{\hat\theta}$.

By niceness, WLOG there is a fixed $r > 0$ such that each h_α is a translation on each $B(d^i_\alpha, r)$; so $h_\alpha(x) = x + e^i_\alpha - d^i_\alpha$ whenever $\|x - d^i_\alpha\| \le r$; hence also $h^{-1}_\alpha(y) = y + d^i_\alpha - e^i_\alpha$ whenever $\|y - e^i_\alpha\| \le r$. We choose our r small enough so that also $\|d^i_\alpha - d^j_\alpha\| \gg r$ and $\|e^i_\alpha - e^j_\alpha\| \gg r$ whenever $i \ne j$. WLOG, the σ_α are close to some common condensation point $\{(d^0, e^0), \ldots, (d^{t-1}, e^{t-1})\}$, so that $\|d^i_\alpha - d^i\| \ll r$ and $\|e^i_\alpha - e^i\| \ll r$, and hence also $\|d^i - d^j\| \gg r$ and $\|e^i - e^j\| \gg r$ whenever $i \ne j$.

Also, WLOG (by Lemma 4.16), all $(h_\alpha + h_\beta)/2 \in \mathcal{F}_{\hat\theta}$. After a bit more thinning, we apply Lemma 2.4 to fix $\alpha \ne \beta$ such that σ_α and σ_β are compatible in $\mathcal{P}^{\hat\theta}_0$. Then $\sigma := \sigma_\alpha \cup \sigma_\beta \in \mathcal{P}^{\hat\theta}_0$. We now construct a $q \in P^\theta$ with $q \le p_\alpha$ and $q \le p_\beta$. Let $\sigma_q = \sigma$. Let $\hat h = (h_\alpha + h_\beta)/2$. Although $\hat h \in \mathcal{F}_{\hat\theta}$, we cannot let $\hat h_q = \hat h$ because $\hat h$ need not extend σ; but it is "close enough" to σ that we may vary it slightly to obtain our $h_q \supset \sigma$ with $h_q \in \mathcal{F}_\theta$. Finally, we make sure that our r was chosen to be small enough that the argument of Lemma 3.15 can be applied to choose $\varkappa_q$ and Υ_q.

The hardest part of the argument is modifying $\hat h$ to obtain h_q. We shall have $h_q(x) = \hat h(x)$ unless x is near some d^i_α, d^i_β. More specifically, let $\hat d^i = (d^i_\alpha + d^i_\beta)/2$ and $\hat e^i = (e^i_\alpha + e^i_\beta)/2$. Using $\|d^i_\alpha - d^i_\beta\| \ll r$, we have

$$\hat h(x) = (x + e^i_\alpha - d^i_\alpha + x + e^i_\beta - d^i_\beta)/2 = x + \hat e^i - \hat d^i$$

when $\|x - d^i_\beta\| \le r/2$, and likewise $\hat h^{-1}(y) = y + \hat d^i - \hat e^i$ when $\|y - e^i_\beta\| \le r/2$. In particular, $\hat h(\hat d^i) = \hat e^i$. Then we shall have $h_q(x) = \hat h(x)$ unless $\|x - \hat d^i\| \le r/2$ for some i, so the changes are only within the various $B(\hat d^i, r/2)$. We need to make sure that we can make these changes without bringing $\mathrm{tw}(h_q)$ above θ. Using $\|d^i - d^j\| \gg r$, the changes to $\hat h$ inside the various $B(\hat d^i, r/2)$ will not interfere with each other.

Focusing on one i: if $d^i_\alpha = d^i_\beta$, then let $h_q \restriction B(\hat d^i, r/2) = \hat h \restriction B(\hat d^i, r/2)$. Now, assume that $d^i_\alpha \ne d^i_\beta$ and hence $e^i_\alpha \ne e^i_\beta$; we need to get $h_q(d^i_\alpha) = e^i_\alpha$ and $h_q(d^i_\beta) = e^i_\beta$. Since $\hat h(\hat d^i) = \hat e^i$, we can temporarily change coordinates in the domain and range and assume that $\hat d^i = \hat e^i = \vec 0$, so that now $\hat h(x) = x$ for

$x \in B(\vec{0}, r/2)$. Then, let $d = d^i_\alpha$ and $e = e^i_\alpha$, so $d^i_\beta = -d$ and $e^i_\beta = -e$, and we need to get $h_q(d) = e$ and $h_q(-d) = -e$. WLOG $K := \|e\|/\|d\| \geq 1$; otherwise, we can interchange d with e and $\hat{h}$ with $\hat{h}^{-1}$ in the argument. We remark that there is no a priori upper bound to K in this argument.

In changing $\hat{h}$ to h_q within $B(\vec{0}, r/2)$ we have two tasks: *expand* and *rotate*: That is, we must rotate d by angle $\angle(d, e)$ so that it points in direction e; note that $\angle(d, e) = \angle(d^i_\alpha - d^i_\beta, e^i_\alpha - e^i_\beta) \leq \mathrm{tw}(\sigma) < \hat{\theta}$. At the same time, we must expand d by a factor of K so that it has length $\|e\|$.

The following lemma involves a pure rotation, without expansion:

Lemma 5.1 *Given $\pi/2 \leq \hat{\theta} < \theta < \pi$ and $0 < r_0 < r_1$ with $r_1/r_0 > e^{5/(\theta - \hat{\theta})}$, and given $\vec{d}, \vec{e}$ with $0 < \|\vec{d}\| = \|\vec{e}\| < r_0$ and $\angle(\vec{d}, \vec{e}) < \hat{\theta}$:*

There is an $f \in \mathcal{F}_\theta$ such that $f(\vec{d}) = \vec{e}$ and $f(\vec{x}) = A(\|\vec{x}\|)\,\vec{x}$ for all $\vec{x}$, where $A : \mathbb{R} \to \mathrm{SO}(n)$ is a C^∞ function, with $A(r) = I$ whenever $r \geq r_1$ and $A(r) = A(0)$ whenever $r \leq r_0$.

Proof. Let $\varrho = \angle(\vec{d}, \vec{e}) < \hat{\theta}$. We may assume that our coordinates are chosen so that $\vec{d}, \vec{e}$ are in the x_1, x_2 plane, with $\vec{e}$ obtained by rotating by ϱ in the positive direction. Then $A(r) = R_{\psi(r)}$, where R_α is just rotation by angle α in the x_1, x_2 plane, and $\psi \in C^\infty(\mathbb{R}, [0, \varrho])$ is a non-increasing function with $\psi(r) = \varrho$ when $r \leq r_0$ and $\psi(r) = 0$ when $r \geq r_1$. Then we are done if we show that we can choose ψ so that $f \in \mathcal{F}_\theta$.

Let $\zeta = \frac{2}{\pi}(\theta - \hat{\theta})/\varrho$. Let $\psi = \varrho\psi_0$, where $\psi_0 \in C^\infty(\mathbb{R}, [0, 1])$ is chosen so that $\psi_0(r) - \psi_0((1 + \sigma)r) \leq \zeta\sigma$ whenever $\sigma, r > 0$. This is possible by Lemma 4.12 because $r_1/r_0 > e^{1/\zeta}$. Now $\psi(r) - \psi((1 + \sigma)r) \leq \frac{2}{\pi}(\theta - \hat{\theta})\sigma$ whenever $\sigma, r > 0$.

To prove that $\mathrm{tw}(f) < \theta$, we fix $\vec{x}_0 \neq \vec{x}_1$, with $\vec{y}_i = f(\vec{x}_i)$, and show that $\gamma := \angle(\vec{x}_1 - \vec{x}_0, \vec{y}_1 - \vec{y}_0) \leq \varrho + (\theta - \hat{\theta})$. If $\|\vec{x}_0\| = \|\vec{x}_1\|$ or $\|\vec{x}_0\| = 0$ then $\gamma \leq \varrho$, so we may assume that $0 < r = \|\vec{x}_0\| < (1 + \sigma)r = \|\vec{x}_1\|$. Then $\vec{y}_0 = R_{\psi(r)}\vec{x}_0$ and $\vec{y}_1 = R_{\psi((1+\sigma)r)}\vec{x}_1$. Let $\vec{y}_0^{\,*} = R_{\psi((1+\sigma)r)}\vec{x}_0$. Then $\angle(\vec{x}_1 - \vec{x}_0, \vec{y}_1 - \vec{y}_0^{\,*}) = \psi((1 + \sigma)r) \leq \varrho$. Also, $\|\vec{y}_1 - \vec{y}_0^{\,*}\| = \sigma r$ and $\|\vec{y}_0 - \vec{y}_0^{\,*}\| \leq r[\psi((1 + \sigma)r) - \psi(r)]$, so $\|\vec{y}_0 - \vec{y}_0^{\,*}\|/\|\vec{y}_1 - \vec{y}_0^{\,*}\| \leq \frac{2}{\pi}(\theta - \hat{\theta})$. Then we use Lemma 4.5 to conclude that

$$\beta := \angle(\vec{y}_1 - \vec{y}_0^{\,*}, \vec{y}_1 - \vec{y}_0) \leq \theta - \hat{\theta} \quad ,$$

and hence that $\gamma \leq \varrho + (\theta - \hat{\theta})$. Here, the "distance" is $T = \|\vec{y}_1 - \vec{y}_0^{\,*}\| \geq \sigma r$ and the two radii are 0 and $\|\vec{y}_0 - \vec{y}_0^{\,*}\| \leq r[\psi(r) - \psi((1 + \sigma)r)] \leq r[\frac{2}{\pi}(\theta - \hat{\theta})\sigma]$, so $\beta \leq \pi \cdot r[\frac{2}{\pi}(\theta - \hat{\theta})\sigma] \div 2\sigma r = \theta - \hat{\theta}$.

We next consider the twist of a pure expansion, without rotation:

Lemma 5.2 *Assume that $f(\vec{x}) = \nu(\|\vec{x}|)\vec{x}$, where $\nu : [0, \infty) \to [0, \infty)$ and the map $r \mapsto \nu(r)r$ is strictly increasing. Then $\mathrm{tw}(f) \leq \pi/2$.*

Proof. We prove that $\gamma := \angle(\vec{x}_1 - \vec{x}_0, \nu(\|\vec{x}_1\|)\vec{x}_1 - \nu(\|\vec{x}_0\|)\vec{x}_0) \le \pi/2$ whenever $\vec{x}_0 \ne \vec{x}_1$. Since $\gamma = 0$ when $\|\vec{x}_0\| = \|\vec{x}_1\|$ or $\|\vec{x}_0\| = 0$, so we may assume that $0 < r = \|\vec{x}_0\| < s = \|\vec{x}_1\|$. Now, we may work entirely in the plane of $\vec{x}_0, \vec{x}_1$, which we identify with $\mathbb{C}$, and we may assume that $\vec{x}_1$ is on the positive x-axis. We can now write $\vec{x}_0 = re^{i\delta}$ and $\vec{x}_1 = s$. Then $\nu(\|\vec{x}_0\|)\vec{x}_0 = r'e^{i\delta}$ and $\nu(\|\vec{x}_1\|)\vec{x}_1 = s'$, where $r < s$ and hence $r' < s'$. Then $\gamma = \angle(s - re^{i\delta}, s' - r'e^{i\delta}) = \angle(1 - ue^{i\delta}, 1 - ve^{i\delta})$, where $u = r/s < 1$ and $v = r'/s' < 1$. Then $\gamma < \pi/2$ because both $1 - ue^{i\delta}$ and $1 - ve^{i\delta}$ lie in the same quadrant: namely quadrant I if $\delta \in (\pi, 2\pi)$ and IV if $\delta \in (0, \pi)$. If $\delta = 0$ or $\delta = \pi$, then $\gamma = 0$. ☕

Putting these two lemmas together:

Lemma 5.3 *Given $\pi/2 \le \hat{\theta} < \theta < \pi$ and $f \in \mathcal{F}_{\hat{\theta}}$ and $0 < r_0 < r_4$ with $r_4/r_0 > e^{5/(\theta-\hat{\theta})}[2\pi/(\theta - \hat{\theta})]$ and $f(\vec{x}) = \vec{x}$ whenever $\|\vec{x}\| \le r_4$, and given $\vec{d}, \vec{e}$ with either $\vec{d} = \vec{e} = \vec{0}$ or $0 < \|\vec{d}\|, \|\vec{e}\| < r_0$ and $\angle(\vec{d}, \vec{e}) < \hat{\theta}$:*

There is a $g \in \mathcal{F}_\theta$ such that $g(\vec{d}) = \vec{e}$, and $g(\vec{x}) = f(\vec{x})$ whenever $\|\vec{x}\| \ge r_4$, and $g(\vec{x}) = \nu(\|\vec{x}\|)A(\|\vec{x}\|)\,\vec{x}$ whenever $\|\vec{x}\| \le r_4$, where $A : \mathbb{R} \to \mathrm{SO}(n)$ and $\nu : [0, \infty) \to [0, \infty)$ are C^∞ functions, and the map $r \mapsto \nu(r)r$ is strictly increasing.

Proof. If $\vec{d} = \vec{e} = \vec{0}$ then we can let $g = f$. Then WLOG, $0 < \|\vec{d}\| \le \|\vec{e}\| < r_0$, since if $\|\vec{e}\| < \|\vec{d}\|$, we can interchange $\|\vec{d}\|, \|\vec{e}\|$ and use the proof below to construct f^{-1}. Let $K = \|\vec{e}\|/\|\vec{d}\| \in [1, \infty)$.

Choose r_1, r_2, r_3 with $r_0 < r_1 < r_2 < r_3 < r_4$ and $r_1/r_0 > e^{5/(\theta-\hat{\theta})}$ and $r_2/r_1 > 2$ and $r_4/r_3 > \pi/(\theta - \hat{\theta})$. Define $s_i = r_i/K$ for $i = 0, 1, 2$.

Choose ν so that $\nu(r) = K$ for $r \le s_2$ and $\nu(r) = 1$ for $r \ge r_3$. We can make $r \mapsto \nu(r)r$ strictly increasing and with positive slope because $K \cdot s_2 = r_2 < 1 \cdot r_3$.

As in the proof of Lemma 5.1, let $A(r) = R_{\psi(r)}$, where R_α is rotation by angle α, and $\psi \in C^\infty(\mathbb{R}, [0, \varrho])$, where $\varrho = \angle(\vec{d}, \vec{e}) < \hat{\theta}$. Again, ψ is a non-increasing function; but now $\psi(r) = \varrho$ when $r \le s_0$ and $\psi(r) = 0$ when $r \ge s_1$, and $\psi(r) - \psi((1+\sigma)r) \le \frac{2}{\pi}(\theta - \hat{\theta})\sigma$ whenever $\sigma, r > 0$. There is such a ψ by Lemma 4.12 because $s_1/s_0 = r_1/r_0 > e^{\pi\varrho/(2(\theta-\hat{\theta}))}$.

This defines g. To prove that $g \in \mathcal{F}_\theta$, we fix $\vec{x}_0 \ne \vec{x}_1$, with $\vec{y}_i = g(\vec{x}_i)$, and show that $\gamma := \angle(\vec{x}_1 - \vec{x}_0, \vec{y}_1 - \vec{y}_0) < \theta$. We may assume that $\|\vec{x}_0\| \le \|\vec{x}_1\|$, and we consider the various cases for the values of $\|\vec{x}_0\|, \|\vec{x}_1\|$:

If $\|\vec{x}_1\| \ge r_4$ and $\|\vec{x}_0\| \ge r_3$, then $\gamma < \hat{\theta}$ because $\mathrm{tw}(f) < \hat{\theta}$ and each $\vec{y}_i = g(\vec{x}_i) = f(\vec{x}_i)$.

If $\|\vec{x}_1\| \ge r_4$ and $\|\vec{x}_0\| \le r_3$: Then $\|\vec{y}_1\| \ge r_4$ and $\|\vec{y}_0\| \le r_3$. Also, $\angle(\vec{x}_1 - \vec{0}, \vec{y}_1 - \vec{0}) = \angle(\vec{x}_1 - \vec{0}, f(\vec{x}_1) - f(\vec{0})) < \hat{\theta}$ because $\mathrm{tw}(f) < \hat{\theta}$. We shall show that

$$\angle(\vec{x}_1 - \vec{0}, \vec{x}_1 - \vec{x}_0) < (\theta - \hat{\theta})/2 \quad \text{and} \quad \angle(\vec{y}_1 - \vec{0}, \vec{y}_1 - \vec{y}_0) < (\theta - \hat{\theta})/2 \ \ .$$

Applying Lemma 4.5, the "distance" T is either $\|\vec{x}_1 - \vec{0}\|$ or $\|\vec{y}_1 - \vec{0}\|$, so $T \geq r_4$, and the two radii are 0 and one of $\|\vec{x}_0\|$,$\|\vec{y}_0\|$, so each of the two angles is bounded by $\pi \cdot r_3 \div 2r_4 < (\theta - \hat{\theta})/2$ because $r_3/r_4 < (\theta - \hat{\theta})/\pi$.

In the remaining cases, $\|\vec{x}_0\| \leq \|\vec{x}_1\| \leq r_4$.

If $0 \leq \|\vec{x}_0\| \leq \|\vec{x}_1\| \leq s_2$, then $\gamma < \theta$, as in the proof of Lemma 5.1.

If $s_1 \leq \|\vec{x}_0\| \leq \|\vec{x}_1\| \leq r_4$, then $\gamma \leq \pi/2 < \theta$ by Lemma 5.2.

All that remains is the case that $0 \leq \|\vec{x}_0\| \leq s_1$ and $s_2 \leq \|\vec{x}_1\| \leq r_4$: Then $\angle(\vec{x}_1 - \vec{0}, \vec{y}_1 - \vec{0}) = 0$. Also, $\angle(\vec{x}_1 - \vec{0}, \vec{x}_1 - \vec{x}_0) \leq (\pi/2)(s_1/s_2)$ and $\angle(\vec{y}_1 - \vec{0}, \vec{y}_1 - \vec{0}) \leq (\pi/2)(r_1/r_2)$, so $\gamma < \pi/2$ because $s_2/s_1 = r_2/r_1 > 2$.

Note that our argument requires no lower bound to r_3/r_2; we just need $r_2 < r_3$. If $r_2 \approx r_3$ then $\|J_{f^{-1}}(\vec{y})\| \gg 1$ for some $\vec{y}$ with $r_2 < \|\vec{y}\| < r_3$, but our proof does not maintain any upper bound on $\|J_f\|$ and $\|J_{f^{-1}}\|$ anyway. ☕

Before we choose $\varkappa_q$ and Υ_q, we need some more preliminaries:

Definition 5.4 *For $f, g \in \mathcal{F}_\theta$, let*
$\Delta(f,g) = \max(\|f - g\|, \|J_f - J_g\|, \|f^{-1} - g^{-1}\|, \|J_{f^{-1}} - J_{g^{-1}}\|)$. *Let* $B_\Delta(f, \varepsilon) = \{g \in \mathcal{F}_\theta : d_\Delta(f,g) < \varepsilon\}$, *and* $B(f,\varepsilon) = B_d(f,\varepsilon) = \{g \in \mathcal{F}_\theta : d(f,g) < \varepsilon\}$.

Note that $\Delta(f,g) \geq d(f,g)$ (see Definition 3.12), and $\Delta(f,g) = \Delta(f^{-1}, g^{-1})$, and $d(f,g) = d(f^{-1}, g^{-1})$ Also, both $(\mathcal{F}_\theta, d)$ and $(\mathcal{F}_\theta, \Delta)$ are separable metric spaces, and neither is complete. Although Δ might seem more "natural" then d as a metric on our space $\mathcal{F}_\theta$ of C^1 functions, our generic function f will not be C^1, and is a limit of $\langle h_p : p \in G\rangle$ only with respect to d, not Δ.

Definition 5.5 *Fix $f \in \mathcal{F}_\theta$, and let $K \subset (1, \infty)$ be closed in $\mathbb{R}$. Then* $Z^f_K = \{x \in \mathbb{R}^n : \det J_f(x) \in K\}$.

Note that Z^f_K is compact because $J_f(x) = I$ outside a bounded set. Also, $Z^f_\ell = Z^f_{[\ell-1,\ell]}$ ($\ell \geq 3$) and $W^f_\ell = Z^f_{[\ell,\infty)}$ ($\ell \geq 2$) (see Definition 3.3).

Lemma 5.6 *Fix $K \subset (1,\infty)$ such that K is closed in $\mathbb{R}$, and fix $f \in \mathcal{F}_\theta$:*

1. For all $\zeta > 0$, there is an open $U \supset K$ such that $\mu(Z^f_{\overline{U}}) < \mu(Z^f_K) + \zeta$.

2. $\forall \zeta > 0\, \exists \varepsilon > 0\, \forall g \in \mathcal{F}_\theta\; [\Delta(f,g) < \varepsilon \rightarrow$
$\mu(Z^g_K) < \mu(Z^f_K) + \zeta \;\wedge\; \mu(Z^{g^{-1}}_K) < \mu(Z^{f^{-1}}_K) + \zeta]$.

Proof. For (1): Get open $U_m \supset K$ with all $\overline{U}_m \subset (1,\infty)$ and $\overline{U}_m \searrow K$. Then $\mu(Z^f_{\overline{U}_m}) \searrow \mu(Z^f_K)$ because the $\mu(Z^f_{\overline{U}_m})$ are finite.

For (2): By symmetry between f, f^{-1}, we need only consider the "$\mu(Z^g_K) < \mu(Z^f_K) + \zeta$" part of the conjunction. Note that this "$<$" might be *much* less; for example, K may be a singleton with $\mu(Z_K^f) > 0$; but there may be g arbitrarily close to f with $Z_K^g = \emptyset$.

First fix an open U with $K \subseteq U \subseteq \overline{U} \subseteq (1, \infty)$ and $\mu(Z_{\overline{U}}^f) < \mu(Z_K^f) + \zeta$. Then it is sufficient to choose $\varepsilon > 0$ so that $\forall g \in \mathcal{F}_\theta\, [\Delta(f,g) < \varepsilon \to Z_K^g \subseteq Z_{\overline{U}}^f]$. First fix $r > 0$ such that $J_f(x) = I$ whenever $\|x\| \geq r$. Then, since $K \subset (1, \infty)$ we can fix $\varepsilon_0 > 0$ such that $\det J_g(x) \notin K$ whenever $\|x\| \geq r$ and $\Delta(f,g) < \varepsilon_0$. Then we shall choose our desired ε so that $\varepsilon \leq \varepsilon_0$. If there is no such ε, then get a sequence $g_m \to f$ (wrt Δ) and $x_m \in Z_K^{g_m} \setminus Z_{\overline{U}}^f$ with all $\Delta(f, g_m) < \varepsilon_0$. Then all $\|x_m\| < r$, so, passing to a sub-sequence, we may assume that $x_m \to x$. Then $\det J_f(x_m) \to \det J_f(x) \notin U$ (since U is open) and $\det J_{g_m}(x_m) \to \det J_f(x) \in K$ (since K is closed), which contradicts $K \subseteq U$. ☕

Note that K need not be bounded here; in particular, it could be some $[c, \infty)$ with $c > 1$, so this lemma applies to the W_ℓ^f.

Note the following related to Lemmas 4.14 and 4.16. In both of them, we are starting with an $f \in \mathcal{F}_\theta$ and we are constructing a new function $k \in \mathcal{F}_\theta$, and we easily verify that $\|f - k\|$, $\|f^{-1} - k^{-1}\|$, and $\|J_f - J_k\|$ are "small", and we want to show that $\|J_{f^{-1}} - J_{k^{-1}}\|$ is "small", so that $\Delta(f,k)$ is small. Applied to Lemma 4.16, $k = (f+g)/2$, where g is "near to" f.

Lemma 5.7 *For each $f \in \mathcal{F}_\theta$, and each $\varepsilon > 0$, there is a $\delta \in (0, \varepsilon)$ such that: For all $k \in \mathcal{F}_\theta$, if $\|f - k\| < \delta$, $\|f^{-1} - k^{-1}\| < \delta$, and $\|J_f - J_k\| < \delta$, then $\|J_{f^{-1}} - J_{k^{-1}}\| < \varepsilon$, and hence $\Delta(f,k) < \varepsilon$.*

Proof. To bound $\|J_{f^{-1}} - J_{k^{-1}}\|$, fix y and we bound $\|J_{f^{-1}}(y) - J_{k^{-1}}(y)\|$. Let $f^{-1}(y) = x$ and $k^{-1}(y) = z$, so $f(x) = k(z) = y$, and $\|J_{f^{-1}}(y) - J_{k^{-1}}(y)\| = \|(J_f(x))^{-1} - (J_k(z))^{-1}\| \leq \|(J_f(x))^{-1} - (J_f(z))^{-1}\| + \|(J_f(z))^{-1} - (J_k(z))^{-1}\|$.

For the first summand: Given f, the map $x \mapsto (J_f(x))^{-1}$ is continuous on $\mathbb{R}^n$, and hence uniformly continuous (since $J_f(x) = I$ outside a bounded set), so choose $\delta > 0$ small enough that $\forall x, z \left[\|x - z\| < \delta \to \|(J_f(x))^{-1} - (J_f(z))^{-1}\| < \varepsilon/2\right]$. Now note that $\|x - z\| = \|f^{-1}(y) - k^{-1}(y)\| \leq \|f^{-1} - k^{-1}\| < \delta$.

For the second summand, let $2a = \inf\{\det(J_f(z)) : z \in \mathbb{R}^n\}$. For δ small enough, $\forall k, z\, [\|J_f - J_k\| < \delta \to \det(J_k(z)) \geq a]$. A still smaller δ ensures that for all such k and all z, $\|(J_f(z))^{-1} - (J_k(z))^{-1}\| < \varepsilon/2$. Here, we use the standard formula for the entries of the matrix Y^{-1} as a polynomial in the entries of Y divided by $\det(Y)$. ☕

We remark that the proof for the first summand is not uniform in f, in that δ really does depend on f. Define $\Delta_0(f,g) = \max(\|f - g\|, \|J_f - J_g\|, \|f^{-1} - g^{-1}\|)$. This lemma shows that Δ and Δ_0 yield the same topology on $\mathcal{F}_\theta$. But, they are are not equivalent metrics. A sequence that is Cauchy with respect to Δ_0 may fail to be Cauchy with respect to Δ.

Proof of Lemma 3.17. We begin with the details of the thinning argument. We start with $p_\alpha = (\sigma_\alpha, h_\alpha, \varkappa_\alpha, \Upsilon_\alpha)$, for $\alpha < \omega_1$, with $m_\alpha = \mathrm{dom}(\Upsilon_\alpha)$. Then,

1. WLOG, all $m_\alpha \geq 4$, and all $|\sigma_\alpha| \geq 1$, and all p_α are nice.
2. WLOG: all Υ_α are the same Υ; and all $\varkappa_\alpha$ are the same $\varkappa$; so $p_\alpha = (\sigma_\alpha, h_\alpha, \varkappa, \Upsilon)$; and all $|\sigma_\alpha| = t \geq 1$. Let $m = \mathrm{dom}(\Upsilon) \geq 4$, and let $\sigma_\alpha = \{(d^i_\alpha, e^i_\alpha) : i < t\}$.
3. $\hat\theta \in (\pi/2, \theta)$, and WLOG all $p_\alpha \in \mathbb{P}^{\hat\theta}$ and all $(h_\alpha + h_\beta)/2 \in \mathcal{F}_{\hat\theta}$.
4. WLOG: there is a fixed $r > 0$ such that each h_α is a translation on each $B(d^i_\alpha, r)$; so $h_\alpha(x) = x + e^i_\alpha - d^i_\alpha$ whenever $\|x - d^i_\alpha\| \leq r$; hence also $h^{-1}_\alpha(y) = y + d^i_\alpha - e^i_\alpha$ whenever $\|y - e^i_\alpha\| \leq r$.
5. WLOG: there is some fixed rational $\varepsilon > 0$ such that $\mu(Z^{h_\alpha}_\ell) < \Upsilon(\ell) - \varepsilon$ and $\mu(Z^{h^{-1}_\alpha}_\ell) < \Upsilon(\ell) - \varepsilon$ holds for each α whenever $3 \leq \ell < m$, and $\sum\{\ell\Upsilon(\ell) : 3 \leq \ell < m\} < 1 - 3\varepsilon$, and $Z^{h_\alpha}_{[m-1-\varepsilon,\infty)} = Z^{h^{-1}_\alpha}_{[m-1-\varepsilon,\infty)} = \emptyset$.
6. $\sigma = \{(d^0, e^0), \ldots, (d^{t-1}, e^{t-1})\}$ is a condensation point of $\{\sigma_\alpha : \alpha < \omega_1\}$ (considering these σ_α as points in $(\mathbb{R}^n)^{2t}$), and h is a condensation point of $\{h_\alpha : \alpha < \omega_1\}$ (with respect to the metric Δ). Also, $\sigma \in \mathbb{P}^{\hat\theta}_0$ and $h \in \mathcal{F}_{\hat\theta}$ and $\mu(Z^h_\ell) < \Upsilon(\ell) - \varepsilon$ and $\mu(Z^{h^{-1}}_\ell) < \Upsilon(\ell) - \varepsilon$ whenever $3 \leq \ell < m$, and $Z^h_{[m-1-\varepsilon,\infty)} = Z^{h^{-1}}_{[m-1-\varepsilon,\infty)} = \emptyset$.
7. WLOG: $\|d^i - d^j\| > 8\pi r/(\theta - \hat\theta)$ and $\|e^i - e^j\| > 8\pi r/(\theta - \hat\theta)$ whenever $i \neq j$, and $\mu(B(\vec{0}, r)) < \varepsilon/(2t)$. Also, $r < \varkappa/8$ and $\mu(B(\vec{0}, r)) < \varkappa/8$.
8. ν is small enough so that for all $g \in \mathcal{F}_\theta$, if $\Delta(g, h) < \nu$ then $\mu(Z^g_\ell) < \mu(Z^h_\ell) + \varepsilon/2$ and $\mu(Z^{g^{-1}}_\ell) < \mu(Z^{h^{-1}}_\ell) + \varepsilon/2$ whenever $3 \leq \ell < m$. Also, for all such g, $Z^g_{[m-1-\varepsilon/2,\infty)} = \emptyset$ and $Z^{g^{-1}}_{[m-1-\varepsilon/2,\infty)} = \emptyset$. Also, $\nu < \varkappa/8$.
9. WLOG: $\Delta(h, (h_\alpha + h_\beta)/2) < \nu$ for all α, β.
10. Let $r_4 = r/2$, and choose $r_0 \in (0, r_4)$ so that $r_4/r_0 > e^{5/(\theta-\hat\theta)}[2\pi/(\theta - \hat\theta)]$. WLOG, $\|d^i_\alpha - d^i\| < r_0/8$ and $\|e^i_\alpha - e^i\| < r_0/8$ for all α, i.

To justify some of these steps:

For (1): use the facts that $\{p : m_p \geq 4\}$ is dense (Lemma 3.14), and $\{p : |\sigma_p| \geq 1\}$ is dense (e.g., by Lemma 3.16), and the nice p are dense (Lemma 4.2).

For (3): Use Lemma 4.16.

For (5): Note that $\sup_x \det J_{h_\alpha}(x) = \max_x \det J_{h_\alpha}(x) < m-1$, using Definition 3.13 and the fact that $\det J_{h_\alpha}(x) = 1$ outside a bounded set.

For (6): use separability of the spaces involved. To ensure that $\sigma \in \mathbb{P}^{\hat\theta}_0$ and $h \in \mathcal{F}_{\hat\theta}$, etc., we may take σ to be one of the σ_α and take h to be one of the h_α.

For (7): shrink r if necessary.

For (8), see Lemma 5.6. Regarding getting $Z^g_{[m-1-\varepsilon/2,\infty)} = \emptyset$: we have $\forall x\,[\det J_h(x) < m-1-\varepsilon]$, so if $\|J_g - J_h\|$ is small enough, we'll have $\forall x\,[\det J_g(x) < m-1-\varepsilon/2]$.

For (9): $(h_\alpha + h_\beta)/2 = h + (\,(h_\alpha - h) + (h_\beta - h)\,)/2$. By making all the $\Delta(h, h_\alpha)$ small enough, we can make each $d(h, (h_\alpha + h_\beta)/2)$ and $\|J_h - J_{(h_\alpha+h_\beta)/2}\|$ arbitrarily small (see Lemma 4.14). Now apply Lemma 5.7 to guarantee that $\Delta(h, (h_\alpha + h_\beta)/2)$ is small.

We remark that the r_0, r_4 in (10) correspond to the r_0, r_4 in Lemma 5.3.

Now, to verify the ccc, fix $\alpha \neq \beta$ such that σ_α and σ_β are compatible in $\mathbb{P}^{\hat\theta}_0$. Then $\sigma := \sigma_\alpha \cup \sigma_\beta \in \mathbb{P}^{\hat\theta}_0$. We show that $p \not\perp q$ (in $\mathbb{P}^\theta$) by constructing a $q \in \mathbb{P}^\theta$ such that $q \le p_\alpha$ and $q \le p_\beta$. Let $\sigma_q = \sigma$. Let $\hat h = (h_\alpha + h_\beta)/2$. Then $\hat h \in \mathcal{F}_{\hat\theta}$ by (3), but we must modify $\hat h$ to obtain h_q. To do this, we apply Lemma 5.3 t times in parallel.

Let $\hat d^i = (d^i_\alpha + d^i_\beta)/2$ and $\hat e^i = (e^i_\alpha + e^i_\beta)/2$. Then $\hat h(\hat d^i) = \hat e^i$, and $\hat h$ is translation, $\hat h(x) = x + \hat e^i - \hat d^i$, mapping $B(\hat d^i, r_4)$ onto $B(\hat e^i, r_4)$. Also, by (8)(9), $\Delta(h, \hat h) < \nu$ and $\mu(Z^{\hat h}_\ell) < \mu(Z^h_\ell) + \varepsilon/2$ and $\mu(Z^{\hat h^{-1}}_\ell) < \mu(Z^{h^{-1}}_\ell) + \varepsilon/2$ whenever $3 \le \ell < m$. Hence, $\mu(Z^{\hat h}_\ell) < \Upsilon(\ell) - \varepsilon/2$ and $\mu(Z^{\hat h^{-1}}_\ell) < \Upsilon(\ell) - \varepsilon/2$

We let $h_q(x) = \hat h(x)$ for $x \notin \bigcup_i B(\hat d^i, r_4)$. For each i, $h_q \restriction B(\hat d^i, r_4)$ is obtained from $\hat h \restriction B(\hat d^i, r_4)$ by one application of Lemma 5.3 (temporarily changing coordinates and assuming that $\hat d^i = \hat e^i = \vec 0$). Now that we have h_q, we must verify that $\mathrm{tw}(h_q) < \theta$; that is $\angle(x_1 - x_0, y_1 - y_0) < \theta$, where $y_0 = h_q(x_0)$ and $y_1 = h_q(x_1)$. This is only a problem if $x_0 \in B(\hat d^i, r_4)$ and $x_1 \in B(\hat d^j, r_4)$, where $i \neq j$. Now $\|d^i_\alpha - d^i\| < r_0/8$ and $\|d^i_\beta - d^i\| < r_0/8$, so $\|\hat d^i - d^i\| < r_0/8$. Thus, $\|x_0 - d^i\| < r$. Likewise $\|x_1 - d^j\|, \|y_0 - e^i\|, \|y_1 - e^j\| < r$. Since $\angle(d^j - d^i, e^j - e^i) < \hat\theta$, (using $\sigma \in \mathbb{P}^{\hat\theta}_0$), it is sufficient to note by Lemma 4.5 that $\angle(d^j - d^i, x_1 - x_0) \le (\theta - \hat\theta)/8$ and $\angle(e^j - e^i, y_1 - y_0) \le (\theta - \hat\theta)/8$. For the first angle, the "distance" $T = \|d^i - d^j\| > 8\pi r/(\theta - \hat\theta)$, and the two radii are $< r$, so the angle is bounded by $\pi \cdot 2r \div 16\pi r/(\theta - \hat\theta)$.

Finally, we choose $\varkappa_q$ and Υ_q using the method of proof of Lemma 3.15; see also the corresponding argument in the proof of Lemma 3.6.

For $q \le p_\alpha, p_\beta$, we need $\varkappa_q \le \varkappa$ and $B(h_q, \varkappa_q) \subseteq B(h_\alpha, \varkappa) \cap B(h_\beta, \varkappa)$, and these are satisfied if we choose $\varkappa_q < \varkappa - \max(d(h_q, h_\alpha), d(h_q, h_\beta))$; this number is positive by (7)(8)(9): Note that $d(h_q, h_\alpha) \le d(h_q, \hat h) + d(\hat h, h) + d(h, h_\alpha)$. By (9), $d(h, \hat h) \le \nu$ and $d(h, h_\alpha) \le \nu$, and $\nu < \varkappa/8$ by (8), and $d(h_q, \hat h) \le 2r_4 = r < \varkappa/8$ by (7).

Also, for q to be in $\mathbb{P}^\theta$, we are required to choose $m_q \ge m$ so that $1/(m_q - 1) < \det J_{h_q}(x) < (m_q - 1)$ for all x; then, for $m \le \ell < m_q$, we need to choose $\Upsilon_q(\ell)$ to satisfy: $\sum\{\ell \Upsilon_q(\ell) : \ell \ge 3 \;\&\; \ell < m_q\} < 1$, as well as $\mu(Z^{h_q}_\ell) < \Upsilon_q(\ell)$ and

$\mu(Z_\ell^{h_q^{-1}}) < \Upsilon_q(\ell)$ whenever $3 \le \ell < m_q$.

When $\ell < m$: We already have $\Upsilon_q(\ell) = \Upsilon(\ell)$. By (6), $\mu(Z_\ell^h) < \Upsilon(\ell) - \varepsilon$. By (9), $\Delta(h, \hat{h}) < \nu$, so by (8), $\mu(Z_\ell^{\hat{h}}) < \Upsilon(\ell) - \varepsilon/2$. But $h_q, \hat{h}$ differ on a set of measure no more than $t \cdot \mu(B(\vec{0}, r)) < \varepsilon/2$ (by (7)), so $\mu(Z_\ell^{h_q}) < \Upsilon(\ell)$. The same argument works for h_q^{-1}.

When $m \le \ell < m_q$: We choose $\Upsilon_q(\ell)$ just large enough to make $\mu(Z_\ell^{h_q}) < \Upsilon_q(\ell)$ and $\mu(Z_\ell^{h_q^{-1}}) < \Upsilon_q(\ell)$ hold, making sure that $\sum_{m \le \ell < m_q} \ell \Upsilon_q(\ell) < 3\varepsilon$. Then q will satisfy (2) of Definition 3.13 by (5). This choice is possible if we have $\sum_{m \le \ell < \infty} \ell \mu(Z_\ell^{h_q}) < 3\varepsilon/2$ and $\sum_{m \le \ell < \infty} \ell \mu(Z_\ell^{h_q^{-1}}) < 3\varepsilon/2$; then we can choose $\Upsilon_q(\ell) > \mu(Z_\ell^{h_q}) + \mu(Z_\ell^{h_q^{-1}})$. For the first $\sum$: by Lemma 3.5, $\sum_{m \le \ell < \infty} \ell \mu(Z_\ell^{h_q}) < 3\mu(h_q(W_{m-1}^{h_q}))$. Now $W_{m-1}^{\hat{h}} = \emptyset$ (see (8)), which implies that $3\mu(h_q(W_{m-1}^{h_q})) \le 3t \cdot \mu(B(\vec{0}, r)) < 3\varepsilon/2$. The argument for the second $\sum$ is the same. ☕

Observe that in building h_q from $\hat{h}$, we lose any bound that we had on the Jacobians; in particular, $d(h_q, \hat{h})$ is small but $\Delta(h_q, \hat{h})$ isn't.

6 Examples and Remarks

We provide here the examples mentioned in the previous sections.

The following shows that the "$\theta > \pi/2$" in Proposition 1.7 cannot be replaced by "$\theta \ge \pi/2$":

Example 6.1 *There are $\aleph_1$–dense $D, E \subseteq \mathbb{R}^2$ such that no bijection $f : D \to E$ satisfies* $\mathrm{tw}(f) \le \pi/2$*.*

Proof. Let $E = \widehat{E} \times \widehat{E}$, where $\widehat{E}$ is an $\aleph_1$–dense subset of $\mathbb{R}$. Let $D \subseteq \mathbb{R}^2$ be any $\aleph_1$–dense set of the form $\bigcup_{n \in \omega} \widehat{D}_n \times \{y_n\}$, where each $\widehat{D}_n \subseteq \mathbb{R}$.

Now, fix a 1-1 function $f : D \to E$ with $\mathrm{tw}(f) \le \pi/2$, and we shall show that f is not onto. For this, it is sufficient to show that for each $n \in \omega$, there is a countable $A_n \subseteq \widehat{E}$ such that $|(f(\widehat{D}_n \times \{y_n\}))_t| \le 1$ for all $t \in \widehat{E} \backslash A_n$; here, $(X)_t = \{u : (t, u) \in X\}$.

Fix n. For $x \in \widehat{D}_n$, let $f(x, y_n) = (g_n(x), h_n(x))$, where $g_n, h_n : \widehat{D}_n \to \widehat{E}$. Then $g_n : \widehat{D}_n \to \mathbb{R}$ is non-decreasing (using $\mathrm{tw}(f) \le \pi/2$), so each $g_n^{-1}\{t\}$ is a convex subset of $\widehat{D}_n$, so $A_n := \{t : |g_n^{-1}\{t\}| \ge 2\}$ is countable. If $t \in \widehat{E} \backslash A_n$, then there is at most one x such that $g_n(x) = t$, which implies that $|(f(\widehat{D}_n \times \{y_n\}))_t| \le 1$. ☕

Example 6.2 *In Example 6.1, D and E can be taken so that the two coordinate projections π_0 and π_1 are both 1-1 on D and on E. Then note that no bijection $f : D \to E$ is order-preserving on each coordinate (i.e., $\pi_i(d') < \pi_i(d)$ iff $\pi_i(f(d')) < \pi_i(f(d))$ for $i = 0, 1$).*

Proof. To get D, E, start with D_0, E_0 satisfying Example 6.1, and obtain D, E by rotating D, E by some angle α chosen to make π_0, π_1 1-1. Such an α obviously exists under $\neg$CH, but in any case, it is easy to choose $\widehat{E}$ and the $\widehat{D}_n$ and the y_n in the proof so that $\alpha = 40°$ works.

For the "note that", observe that if $\angle(d' - d, e' - e) \geq \pi/2$, then $d' - d$ and $e' - e$ lie in different quadrants.

We next point out that Proposition 1.7, and hence also Theorem 1.9, cannot be proved from $\mathrm{MA} + \mathfrak{c} = \aleph_2$ alone:

Example 6.3 *It is consistent with $\mathrm{MA} + \mathfrak{c} = \aleph_2$ that there are $\aleph_1$–dense $D, E \subset \mathbb{R}^2$ such that $\pi \in \mathrm{twist}(f)$ whenever f is a bijection from D onto E.*

Proof. Work in a model of $\mathrm{MA} + \mathfrak{c} = \aleph_2$ in which there is a 2-entangled subset of $\mathbb{R}$ of size $\aleph_1$ (see [1, 2]), and partition this set into disjoint pieces A_q and B_q for $q \in \mathbb{Q}$. We may assume that all A_q and B_q are $\aleph_1$–dense in $\mathbb{R}$.

Then, let $D = \bigcup_q A_q \times \{q\}$ and $E = \bigcup_q B_q \times \{q\}$. Say $f : D \to E$ is a bijection. Then fix $q, r \in \mathbb{Q}$ and $\widehat{A} \in [A_q]^{\aleph_1}$ and $\widehat{B} \in [B_q]^{\aleph_1}$ and a bijection $g : A \to B$ such that the map $(x, q) \mapsto (g(x), r)$ is a sub-function of f. By entangledness, g is not order-preserving, so choose $a < a'$ in A such that $g(a) > g(a')$.

If $d = (a, q)$ and $d' = (a', q)$ then $\angle(d' - d, f(d') - f(d)) = \pi$.

It is easy to modify Examples 6.1, 6.2, and 6.3 to replace $\mathbb{R}^2$ by $\mathbb{R}^n$ for any $n \geq 2$.

Question 6.4 *Forcing with $\mathbb{P}_0^\theta$, with $\theta \in (\pi/2, \pi)$, are $\{p : d \in \mathrm{dom}(p)\}$ and $\{p : e \in \mathrm{ran}(p)\}$ dense for all $d \in D$ and $e \in E$?*

If the answer is "yes", then we could dispense with the side conditions in the proof of Proposition 1.7, resulting in a much simpler proof, but we needed the side conditions anyway in the proof of Theorem 1.9 to ensure that the generic function is BAC.

The interest of this question for forcing is only when $\theta > 90°$, but a simple example in the plane shows that the answer is "no" with $\theta = 18°$: Let $p = \{(d_i, e_i) : i < 3\}$, where $d_0 = (0, 10)$, $e_0 = (0, -9)$, $d_1 = e_1 = (0, -10)$, and $d_2 = e_2 = (0, 11)$. Then $\mathrm{tw}(p) = 0$, so $p \in \mathbb{P}^\theta$. Let $d = (10, 0)$ and suppose that

$p \cup \{(d,e)\} \in \mathbb{P}_0^\theta$. Let $e = (x,y)$. The requirements $\angle(d - d_0, e - e_0) \le 18^\circ$ and $\angle(d - d_1, e - e_1) \le 18^\circ$ imply that $0 \le x \le 1$ and $-10 \le y \le -9$. But then we have $\angle(d - d_2, e - e_2) \ge \angle((10,0) - (0,11),\ (1,-9) - (0,11)) \approx 39^\circ$.

We remark that in Theorem 1.9, if we change "$\aleph_1$–dense" to "$\aleph_0$–dense" (i.e., countable and dense), then we actually get a ZFC theorem, since we only need to meet $\aleph_0$ dense sets in our (slightly modified) forcing poset. But then, it is simpler not to use forcing at all; the desired function f can be obtained as a uniform limit of a sequence of functions, similarly to the argument in Franklin [5]. At the same time, the function can be made C^∞, by making all the derivatives converge uniformly. Also, we can now get $\text{tw}(f) \le \theta$ for *any* given $\theta > 0$; the requirement $\theta > \pi/2$, which was needed for the ccc proof, can be dropped.

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