A Compact Homogeneous S-space*

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June 26, 2003

Abstract

Under the continuum hypothesis, there is a compact homogeneous strong S-space.

1 Introduction

A space X is *hereditarily separable* (HS) iff every subspace is separable. An *S-space* is a regular Hausdorff HS space with a non-Lindelöf subspace. A space X is *homogeneous* iff for every $x, y \in X$ there is a homeomorphism f of X onto X with $f(x) = y$. Under CH, several examples of S-spaces have been constructed, including topological groups (see [5]) and compact S-spaces (see [8]). It is asked in [1] (Problem I.5) and in [6] whether there are compact homogeneous S-spaces. As we shall show in Theorem 4.2, there are under CH. This cannot be done in ZFC, since there are no compact S-spaces under $MA + \neg CH$ (see [13]); there are no S-spaces at all under PFA (see [14]).

In Section 2, we use a slightly modified version of the construction in [8, 11] to refine the topology of any given second countable space, and turn it into a first countable *strong S-space* (i.e., each of its finite powers is an S-space). In Section 3, we show that if the original space is compact, then there is a natural compactification of the new space which is also a first countable strong S-space. If in addition the original space is zero-dimensional, then the ω^{th} power of this compactification will be homogeneous by Motorov [10], proving Theorem 4.2.

*2000 Mathematics Subject Classification: Primary 54G20, 54D30. Key Words and Phrases: S-space, homogeneous space.

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2 A Strong S-Space

If τ is a topology on X, we write τ^I for the corresponding product topology on X^I; likewise if $\tau' \subseteq \tau$ is a base we write $(\tau')^I$ for the natural corresponding base for τ^I. If $E \subseteq X$, then $\mathrm{cl}(E, \tau)$ denotes the closure of E with respect to the topology τ. This notation will be used when we are discussing two different topologies on the same set X.

The following two lemmas are well-known; the second is Lemma 7.2 in [11]:

Lemma 2.1 *If X is HS and Y is second countable, then $X \times Y$ is HS*

Lemma 2.2 *X^ω is HS iff X^n is HS for all $n < \omega$.*

The next lemma, an easy exercise, is used in the proof of Theorem 2.4:

Lemma 2.3 *If $(x, y) \in X \times Y$ and $S \subseteq X \times Y$, then $(x, y) \in \mathrm{cl}(S)$ iff $y \in \mathrm{cl}(\pi(S \cap (U \times Y)))$ for all neighborhoods U of x, where $\pi : X \times Y \to Y$ is projection.*

The following is proved (essentially) in [11], but our proof below may be a bit simpler:

Theorem 2.4 *Assume CH. Let ρ be a second countable T_3 topology on X, where $|X| = \aleph_1$. Then there is a finer topology τ on X such that (ω_1, τ) is a first countable locally compact strong S-space.*

Proof. WLOG, $X = \omega_1$. For $\eta < \omega_1$ we write ρ_η for the topology of η as a subspace of (ω_1, ρ). Applying *CH*, list $\bigcup_{0<n<\omega} [(\omega_1)^n]^{\leq\omega}$ as $\{S_\mu : \mu \in \omega_1\}$, so that each $S_\mu \subseteq \mu^{n(\mu)}$ for some $n(\mu)$ with $0 < n(\mu) < \omega$.

For $\eta \leq \omega_1$ we construct a topology τ_η on η by induction on η so as to make the following hold for all $\xi < \eta \leq \omega_1$:

1. $\tau_\xi = \tau_\eta \cap \mathcal{P}(\xi)$.
2. τ_η is first countable, locally compact, and T_3.
3. $\tau_\eta \supseteq \rho_\eta$.

Note that (1) implies in particular that $\xi \in \tau_\eta$; that is, ξ is open. Thus, if $\tau = \tau_{\omega_1}$, then (ω_1, τ) is not Lindelöf. Also by (1), τ_η for limit η is determined from the τ_ξ for $\xi < \eta$. So, we need only specify what happens at successor ordinals.

For $n \geq 1$ and $\xi < \omega_1$, let $Iseq(n, \xi)$ be the set of all $f \in (\omega_1)^n$ which satisfy $f(0) < f(1) < \cdots < f(n-1) = \xi$. The following condition states our requirement on $\tau_{\xi+1}$:

4. For each $\mu < \xi$ and each $f \in Iseq(n, \xi)$, where $n = n_\mu$:
$$f \in \mathrm{cl}(S_\mu,\, (\tau_{\xi+1})^{n-1} \times \rho) \implies f \in \mathrm{cl}(S_\mu,\, (\tau_{\xi+1})^n) \quad .$$

If $n = n_\mu = 1$, then $(\tau_{\xi+1})^{n-1} \times \rho$ just denotes ρ. That is, (4) requires

$$\xi \in \mathrm{cl}(E, \rho) \implies \xi \in \mathrm{cl}(E, \tau_{\xi+1}) \tag{$*$}$$

for all E in the countable family $\{S_\mu : \mu < \xi \;\&\; n(\mu) = 1\}$. It is standard (see [8]) that one may define $\tau_{\xi+1}$ so that this holds. Now, consider (4) in the case $n = n_\mu \geq 2$. By (2), τ_ξ is second countable, so let τ'_ξ be a countable base for τ_ξ. Applying Lemma 2.3, (4) will hold if whenever $U = U_0 \times \cdots \times U_{n-2} \in (\tau'_\xi)^{n-1}$ is a neighborhood of $f \restriction (n-1)$,

$$\xi \in \mathrm{cl}(\pi(S_\mu \cap (U \times (\xi+1))), \rho) \implies \xi \in \mathrm{cl}(\pi(S_\mu \cap (U \times (\xi+1))), \tau_{\xi+1}) \quad ,$$

where $\pi : \xi^{n-1} \times (\xi+1) \to (\xi+1)$ is projection. But this is just a requirement of the form $(*)$ for countably many more sets E, so again there is no problem meeting it.

Now, we need to show that τ^n is *HS* for each $0 < n < \omega$. We proceed by induction, so assume that τ^m is *HS* for all $m < n$. Fix $A \subseteq (\omega_1)^n$; we need to show that A is τ^n-separable. Applying the induction hypothesis, we may assume that each $f \in A$ has all coordinates distinct. Also, since permutation of coordinates induces a homeomorphism of $(\omega_1)^n$, we may assume that each $f \in A$ is strictly increasing; that is, $f \in Iseq(n, \xi)$, where $\xi = f(n-1)$. By the induction hypothesis and Lemma 2.1, A is separable in $\tau^{n-1} \times \rho$. We can then fix μ such that $n(\mu) = n$, $S_\mu \subseteq A$, and S_μ is $\tau^{n-1} \times \rho$-dense in A. Now, say $f \in A$ with $\xi = f(n-1) > \mu$. Applying (4), we have $f \in \mathrm{cl}(S_\mu,\, \tau^n)$. Thus, $A \setminus \mathrm{cl}(S_\mu,\, \tau^n)$ is countable, so A is τ^n-separable. ♠

3 Compactification

We need the following generalization of the Aleksandrov duplicate construction. Similar generalizations have been described elsewhere; see in particular [2], which also gives references to the earlier literature.

Definition 3.1 *If φ is a continuous map from the T_2 space Y into X, then $Y\dot{\cup}_\varphi X$ denotes the disjoint union of X and Y, given the topology which has as a base:*

a. *All open subsets of Y, together with*
b. *All $[U,K] := U \cup (\varphi^{-1}U \setminus K)$, where U is open in X and K is compact in Y.*

Our main interest here is in the case where X is compact and Y is locally compact. Then, if $|X| = 1$, we have the 1-point compactification of Y, and if Y is discrete and φ is a bijection we have the Aleksandrov duplicate of X.

Lemma 3.2 *Let $Z = Y\dot{\cup}_\varphi X$, with X and Y Hausdorff:*

1. *X is closed in Z, Y is open in Z, and both X and Y inherit their original topology as subspaces of Z.*
2. *If Y is locally compact, then Z is Hausdorff.*
3. *If X is compact, then Z is compact.*
4. *If X and Y are first countable, X is compact, Y is locally compact, and each $\varphi^{-1}(x)$ is compact, then Z is first countable.*
5. *If X and Y are zero dimensional, X is compact, and Y is locally compact, then Z is zero dimensional.*
6. *If X is second countable and Y^ω is HS, then Z^ω is HS.*

Proof. For (3): If $\mathcal{U}$ is a basic open cover of Z, then there are $n \in \omega$ and $[U_i, K_i] \in \mathcal{U}$ for $i < n$ such that $\bigcup_{i<n} U_i = X$. Thus, $\bigcup_{i<n}[U_i, K_i]$ contains all points of Z except for (possibly) the points in the compact set $\bigcup_{i<n} K_i \subseteq Y$.

For (4): Z is compact Hausdorff and of countable pseudocharacter.

For (5): Z is compact Hausdorff and totally disconnected.

For (6): By Lemma 2.2, it is sufficient to prove that each Z^n is HS. But Z^n is a finite union of subspaces of the form $X^j \times Y^k$, which are HS by Lemma 2.1. ♠

4 Homogeneity

The following was proved by Dow and Pearl [4]:

Theorem 4.1 *If Z is first countable and zero dimensional, then Z^ω is homogeneous.*

Actually, we only need here the special case of this result where Z is compact and has a dense set of isolated points; this was announced (without proof) earlier by Motorov [10].

Note that by Šapirovskiĭ [12], any compact *HS* space must have countable π-weight (see also [7], Theorem 7.14), so if it is also homogeneous, it must have size at most $2^{\aleph_0}$ by van Douwen [3]. Under *CH* this implies, by the Čech – Pospíšil Theorem, that the space must be first countable.

Theorem 4.2 (*CH*) *There is a (necessarily first countable) zero-dimensional compact homogeneous strong S-space.*

Proof. Let X be the Cantor set 2^ω with its usual topology, let Y be 2^ω with the topology constructed in Theorem 2.4, let φ be the identity, and let $Z = Y\dot{\cup}_\varphi X$. By Lemma 3.2, Z, and hence also Z^ω, are zero-dimensional first countable compact strong S-spaces; Z^ω is homogeneous by Theorem 4.1. ♠

No compact topological group can be an S-space or an L-space. However under *CH* there are, by [9], compact L-spaces which are right topological groups (i.e. they admit a group operation such that multiplication on the right by a fixed element defines a continuous map). We do not know whether there can be compact S-spaces which are right topological groups.

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