Induced ideals in Cohen and random extensions*

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Abstract

We show that the following is consistent (relative to the consistency of a measurable cardinal): There is no real valued measurable cardinal below continuum and there is a finitely additive extension $m : \mathcal{P}([0,1]) \to [0,1]$ of Lebesgue measure whose null ideal is a sigma ideal. We also show that there is a countable partition of $[0,1]$ into interior free sets under the m-density topology of any such extension.

1 Introduction

We investigate some questions around a problem of Juhász. Our main results are the following: Let κ be a measurable cardinal in V. Let G be a generic filter for a finite support iteration of random forcing of length κ. Then in $V[G]$, there is no real valued measurable cardinal below continuum and there is a finitely additive extension $m : \mathcal{P}([0,1]) \to [0,1]$ of Lebesgue measure whose null ideal is 2^ω-additive. We also show that there is a countable partition of $[0,1]$ into interior free sets under the m-density topology of any such extension. Sections 2, 3 and 4 contain well known results and are included due to the similarity with the techniques we use in Section 5. We hope this will make the later arguments more accessible. For background on elementary embedding and forcing, we refer the reader to Kanamori's book [3].

2 Induced ideals in ccc extensions

We start with the following question: Suppose κ is a measurable cardinal and I is a witnessing normal prime ideal over κ. Let P be a forcing notion and G a P-generic filter over V. Let $\hat{I}$ be the ideal generated by I over κ in $V[G]$. Describe forcing with $\hat{I}$; i.e., $\mathcal{P}(\kappa)/\hat{I}$. For the purpose of this note, the specific forcings that we consider are all ccc. In this case, the induced ideal is ω_1-saturated and normal.

Proposition 2.1 (Prikry [6]). *Let I be a κ-additive ω_1-saturated ideal over an uncountable cardinal κ. Let P be a ccc forcing notion and G a P-generic filter over V. Let $\hat{I}$ be the ideal generated by I over κ in $V[G]$. Then $\hat{I}$ is a κ-additive ω_1-saturated ideal over κ. Furthermore, if I is normal so is $\hat{I}$.*

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Proof: Let $\langle B_i : i < \theta\rangle \subset \hat{I}$ where $\theta < \kappa$. Let $\langle A_i : i < \theta\rangle \subset I$ be such that $B_i \subseteq A_i$ for each $i < \theta$. Let $p \in P$ force this. In V, for each $i < \theta$, get a maximal antichain $\langle p_{i,n} : n < \omega\rangle$ below p deciding $\mathring{A}_i$. Say $\langle C_{i,n} : n < \omega\rangle \subset I$ is such that $p_{i,n} \Vdash \mathring{A}_i = C_{i,n}$. Let $C_i = \bigcup\{C_{i,n} : n < \omega\}$. Then $p \Vdash \mathring{B}_i \subseteq C_i$ for each $i < \theta$ and hence $\bigcup\{B_i : i < \theta\} \subseteq \bigcup\{C_i : i < \theta\} \in I$. It follows that $\hat{I}$ is κ-additive. Next suppose $\hat{I}$ is not ω_1-saturated and let p force that $\langle X_i : i < \omega_1\rangle$ is a collection of pairwise disjoint $\hat{I}$-positive sets. Work in V. Let $Y_i = \{\alpha < \kappa : \exists q \leq p(q \Vdash \alpha \in \mathring{X}_i)\}$. Then $Y_i \in I^+$ for each $i < \omega_1$. Now observe that there must exist some $A \in I^+$ such that every I-positive subset of A has I-positive intersections with uncountably many Y_i's. Otherwise we can extract an I-positive disjoint refinement of an uncountable subsequence of $\langle Y_i : i < \omega_1\rangle$ which is impossible as I is ω_1-saturated. By thinning down we can also assume that $\bigcup\{Y_i : i < \omega_1\} = A$. It follows that for each $j < \omega_1$, $\bigcup\{Y_i : j < i < \omega_1\}$ contains A modulo I. Hence $\limsup\langle Y_i : i < \omega_1\rangle = A$ modulo I. In particular, some $\alpha < \kappa$ belongs to uncountably many Y_i's and the witnessing conditions q_i's must form an antichain contradicting the ccc-ness of P. Now suppose that I is normal. We'll show that $\hat{I}$ is closed under diagonal unions in $V[G]$. So let $\langle A_\alpha : \alpha < \kappa\rangle \subset I$ and $A_\alpha \subset (\alpha, \kappa) = \{i : \alpha < i < \kappa\}$. Let $p \in P$ force this. Working in V, for each $\alpha < \kappa$, get a maximal antichain $\langle p_{\alpha,n} : n < \omega\rangle$ below p deciding $\mathring{A}_\alpha$. Say $\langle B_{\alpha,n} : n < \omega\rangle \subset I$ is such that $p_{\alpha,n} \Vdash \mathring{A}_\alpha = B_{\alpha,n}$. Let $B_\alpha = \bigcup\{B_{\alpha,n} : n < \omega\}$. Then $B_\alpha \in I$ and $p \Vdash A_\alpha \subseteq B_\alpha \subset (\alpha, \kappa)$ for each $\alpha < \kappa$. By normality of I, we get $\bigcup\{B_\alpha : \alpha < \kappa\} \in I$. Hence $\bigcup\{A_\alpha : \alpha < \kappa\} \in \hat{I}$.

3 Prikry's model

Now let κ be a measurable cardinal with a witnessing normal ideal I. Let $j : V \to M$ be the corresponding ultrapower embedding with critical point κ. We'll repeatedly use $M^\kappa \subset M$. Denote by C_λ, the Cohen algebra for adding λ many Cohen reals; so C_λ is the regular open algebra of 2^λ. Let G be C_λ-generic over V. We attempt to describe, in $V[G]$, the algebra $\mathcal{P}(\kappa)/\hat{I}$ where $\hat{I}$ is the ideal generated by I. If $\lambda < \kappa$, this algebra is trivial, so assume $\lambda \geq \kappa$.

By elementarity plus the fact that M is countably closed, $j(C_\lambda) = C_{j(\lambda)}$ is the Cohen algebra for adding $j(\lambda)$ many Cohen reals. Suppose, H is $C_{j(\lambda)}$-generic over V. Consider, $G = \{p \in C_\lambda : j(p) \in H\}$.

Lemma 3.1. *G is C_λ-generic over V.*

Proof: Clearly, G is a filter over C_λ. Suppose, $\langle p_n : n \in \omega\rangle \subseteq C_\lambda$ is a maximal antichain. Then, $j(\langle p_n : n \in \omega\rangle) = \langle j(p_n) : n \in \omega\rangle$ is a maximal antichain in $C_{j(\lambda)}$. So some $j(p_n) \in H$. Hence $p_n \in G$.

The embedding $j : V \to M$ extends to $j^\star : V[G] \to M[H]$ satisfying $j^\star(G) = H$ by defining $j^\star(X) = val_H(j(\mathring{X}))$. The inclusion $j[G] \subseteq H$ ensures that $j^\star(X)$ does not depend on the choice of the name for X.

Working in $V[G]$, consider the function $\phi : \mathcal{P}(\kappa)/\hat{I} \to C_{j(\lambda)}/j[G]$ defined by:

$$\phi([X]) = [[\kappa \in j(\mathring{X})]]_{C_{j(\lambda)}}/j[G]$$

Lemma 3.2. *ϕ is a Boolean isomoprhism.*

Proof: First note that ϕ is well defined since if $p \Vdash \mathring{X} \Delta \mathring{Y} \subseteq A$ for some $A \in I$, then $[[\kappa \in j(\mathring{X})]]_{C_{j(\lambda)}} \wedge j(p) = [[\kappa \in j(\mathring{Y})]]_{C_{j(\lambda)}} \wedge j(p)$ so that $\phi(X) = \phi(Y)$. It is clear that ϕ preserves boolean operations. To see that it is injective, note that if $\phi(X) = 0$, then for some $p \in G$, $[[\kappa \in j(\mathring{X})]]_{C_{j(\lambda)}} \wedge j(p) = 0$. Then, $p \Vdash \mathring{X} \in \hat{I}$. Finally, if $q \in j(C_\lambda)/j[G]$, then for some $p_\alpha \in C_\lambda$, for $\alpha < \kappa$, $q = [\langle p_\alpha : \alpha < \kappa \rangle]$. Let $\mathring{X}$ be such that $[[\alpha \in \mathring{X}]]_{C_\lambda} = p_\alpha$, for $\alpha < \kappa$. Then, $[[\kappa \in j(\mathring{X})]]_{C_{j(\lambda)}} = j(\mathring{X})([id]) = [\langle p_\alpha : \alpha < \kappa \rangle] = q$. Hence ϕ is surjective.

Corollary 3.3 (Prikry [6]). *In $V[G]$, forcing with $\mathcal{P}(\kappa)/\hat{I}$ is same as adding $|j(\lambda) \backslash j[\lambda]|$ Cohen reals. In particular, when $\kappa \leq \lambda \leq 2^\kappa$, $\mathcal{P}(\kappa)/\hat{I}$ adds 2^κ Cohen reals. If $\lambda = 2^\kappa$, $\mathcal{P}(\kappa)/\hat{I}$ is σ-centered.*

Proof: The first statement is clear. The second follows from the fact that whenever $\kappa \leq \lambda \leq (2^\kappa)^V$, we have $2^\kappa < j(\lambda) < (2^\kappa)^+$. For the third statement use the fact that $2^{(2^\omega)}$ is separable.

4 Solovay's model

Again κ is a measurable cardinal with a witnessing normal ideal I and $j : V \to M$ is the corresponding ultrapower embedding with critical point κ. Let R_λ be the measure algebra on 2^λ for adding λ many random reals with $\lambda \geq \kappa$. So $j(R_\lambda) = R_{j(\lambda)}$ as above. Let H be $R_{j(\lambda)}$-generic over V. Set $G = \{p \in R_\lambda : j(p) \in H\}$. As in Lemma 3.1, we get

Lemma 4.1. *G is R_λ-generic over V.*

The embedding $j : V \to M$ extends to $j^\star : V[G] \to M[H]$ satisfying $j^\star(G) = H$ by defining $j^\star(X) = val_H(j(\mathring{X}))$. In $V[G]$, consider the function $\phi : \mathcal{P}(\kappa)/\hat{I} \to R_{j(\lambda)}/j[G]$ defined by:

$$\phi([X]) = [[\kappa \in j(\mathring{X})]]_{R_{j(\lambda)}}/j[G]$$

Lemma 4.2. *ϕ is a Boolean isomoprhism.*

Corollary 4.3 (Solovay [7]). *In $V[G]$, forcing with $\mathcal{P}(\kappa)/\hat{I}$ is same as adding $|j(\lambda) \backslash j[\lambda]|$ random reals. In particular, in $V[G]$, continuum is real valued measurable.*

5 Around a question of Juhász

Juhász recently asked the following question (communicated by Arnold Miller):

Question 5.1. *Is there a ccc Hausdorff space X without isolated points such that for every partition $X = \bigcup\{Y_n : n \in \omega\}$ there is some Y_n with non empty interior?*

If X is such a space then by passing to some open subset of X we can assume every open subset of X is such a space. So the family of interior free subsets of X forms a σ-ideal that is also ω_1-saturated, as X is ccc. So we need the consistency of a measurable cardinal. Starting with a real valued measurable cardinal below continuum, Kunen, Szymański and Tall have constructed such a space; see Corollary 3.6 in [4]. The next theorem describes another construction that even makes the space T_4.

Theorem 5.2. *Suppose κ is measurable in V. Let C_κ be Cohen forcing for adding κ Cohen reals. Let G be C_κ-generic over V. Then, in $V[G]$, there is a ccc T_4 space X without isolated points such that whenever $X = \bigcup\{Y_n : n \in \omega\}$, some Y_n has non empty interior.*

Proof: Let I be a witnessing normal ideal over κ and $\hat{I}$, the ideal generated by I in $V[G]$. By Corollary 3.3, $\mathcal{P}(\kappa)/\hat{I}$ is isomorphic to C_{2^κ}. For $\alpha < 2^\kappa$, let $E_\alpha = \{f : 2^{2^\kappa} \to 2 : f(\alpha) = 1\}$. Then $\{[E_\alpha] : \alpha < 2^\kappa\}$ is an independent family that completely generates C_{2^κ}. Let $\{[A_\alpha] : \alpha < 2^\kappa\}$ be the corresponding family in $\mathcal{P}(\kappa)/\hat{I}$. We'll define a topology $\mathcal{T}$ on κ by choosing a member A_α from each equivalence class $[A_\alpha]$ and declaring it to be clopen. We do it in such a way that for any two disjoint sets $X, Y \in \hat{I}$, there is some A_α, $\alpha < 2^\kappa$ separating them - i.e., $X \subset A_\alpha$ and $Y \subset \kappa \backslash A_\alpha$. Since there are only 2^κ many such pairs, this can clearly be done. Thus $\mathcal{T}$ is Hausdorff. Also, every set in $\hat{I}$ is $\mathcal{T}$-closed since for any $X \in \hat{I}$, the union of A_α's disjoint with X is $\kappa \backslash X$. We claim that for any $B \subseteq \kappa$, $B \in \hat{I}$ iff the $\mathcal{T}$-interior of B is empty. Notice that the family $\mathcal{F}$ of finite boolean combinations of A_α's is a basis for $\mathcal{T}$. Since each member of this basis is $\hat{I}$-positive, every member of $\hat{I}$ has empty $\mathcal{T}$-interior. Conversely, if B is $\hat{I}$-positive, then for some $X \in \hat{I}$ and $A \in \mathcal{F}$, $A \backslash X \subseteq B$. This is because $\{[A] : A \in \mathcal{F}\}$ is dense in $\mathcal{P}(\kappa)/\hat{I}$. As X is closed, $\mathcal{T}$-interior of B is empty. $\mathcal{T}$ is ccc because $\mathcal{P}(\kappa)/\hat{I}$ is ccc. Since $\hat{I}$ is κ-additive, every partition of κ into fewer than κ many sets contains one set with non empty $\mathcal{T}$-interior. It remains to show that $\mathcal{T}$ is normal. Fix $C, D \subseteq \kappa$ disjoint and $\mathcal{T}$-closed. Let C', D' be $\mathcal{T}$-interiors of C and D. Let $C_1 = C \backslash C'$, $D_1 = D \backslash D'$. Then $C_1, D_1 \in \hat{I}$ since their $\mathcal{T}$-interiors are empty. Let A_α separate C_1, D_1. Then $(A_\alpha \bigcup C') \backslash D$ and $((\kappa \backslash A_\alpha) \bigcup D') \backslash C$ are $\mathcal{T}$-open sets separating C and D. This finishes the proof.

Juhász wondered if the density topology of some countably additive total extension m of Lebesgue measure could give an example of such a space. Recall that for a finitely additive total measure $m : \mathcal{P}([0,1]) \to [0,1]$, a set $A \subseteq [0,1]$ is open in the m-density topology if each point of A is an m-density one point of A. But we noted that using a result of Maharam [5], together with the fact that the measure algebra of m is everywhere inseparable (by a theorem of Gitik and Shelah [1]), one can deduce the existence of some $X \subset [0,1]$ with $m(X) = 1/2$ that divides every Borel set into two pieces of equal measure; so, such an example, with a countably additive measure, is impossible. But, it may still be possible to have a *finitely* additive total extension of Lebesgue measure whose density topology gives an example of such a space. This led us to consider the following question:

Question 5.3. *Does there exist a finitely additive extension $m : \mathcal{P}([0,1]) \to [0,1]$ of Lebesgue measure whose null ideal is countably additive but m is nowhere countably additive?*

This question is answered in the model described by the following theorem:

Theorem 5.4. *Let κ be a measurable cardinal in V. Let G be a generic filter for a finite support iteration of random forcing of length κ. Then in $V[G]$, there is no real valued measurable cardinal below continuum and there is a finitely additive extension $m : \mathcal{P}([0,1]) \to [0,1]$ of Lebesgue measure whose null ideal is countably additive.*

We now sketch a proof of this.

Definition 5.5. *A strictly positive finitely additive probability measure (SPFAM) on a Boolean algebra B is a function $m : B \to [0,1]$ satisfying the following:*

- *For every $b \in B$, $m(b) = 0 \Leftrightarrow b = 0_B$ and $m(1_B) = 1$*
- *For every $a, b \in B$, if $a \cap b = 0$ then $m(a \cup b) = m(a) + m(b)$*

Lemma 5.6. *Let B be a complete Boolean algebra (cBa) with a SPFAM m and let μ, $\mathring{C}$ be B-names such that $[[\mathring{C}$ is a cBa and μ is a SPFAM on $\mathring{C}]]_B = 1$. Then $B \star \mathring{C}$ admits a SPFAM extending m, identifying B with a complete subalgebra of $B \star \mathring{C}$.*

Proof: We begin by reviewing $B \star \mathring{C}$ - See [2] for details. Let $S = \{\tau \in V^B : [[\tau \in \mathring{C}]]_B = 1\}$. Define an equivalence relation on S: $\sigma \sim \tau$ iff $[[\sigma = \tau]]_B = 1$. Let D be a complete set of representatives. Then D is a cBa under Boolean operations induced from $\mathring{C}$. In particular, for any $c_1, c_2 \in D$, $c_1 \leq_D c_2$ iff $[[c_1 \leq_C c_2]]_B = 1$. We let $B \star \mathring{C} = D$. The map $e : B \to D$ defined by setting $e(b)$ to be the unique $\tau \in D$ such that $[[\tau = 1_C]]_B = b$ and $[[\tau = 0_C]]_B = -b$ is a complete embedding of B into D and we identify the image $e[B]$ with B.

Now define $\phi : D \to [0, 1]$ as follows: Let $\tau \in D$. For each $n \geq 1$, let $\langle I_0, I_1, \ldots, I_{2^n-1}\rangle$ be the dyadic partition of $[0, 1]$ into intervals of length $1/2^n$. Let

$$\phi_n(\tau) = \sum_{k=0}^{2^n-1} m([[\mu(\tau) \in I_k]]_B)k/2^n$$

Then $0 \leq \phi_n(\tau) \leq \phi_{n+1}(\tau) \leq 1$ for every $n \geq 1$. Let $\phi(\tau) = \sup_n \phi_n(\tau)$.

Claim 5.7. *ϕ is a SPFAM on D, extending m.*

Clearly, $\phi(0_D) = 0$, $\phi(1_D) = 1$. If $\sigma, \tau \in D$ are disjoint then $[[\sigma \cap \tau = 0]]_B = 1$. Hence $[[\mu(\sigma \cup \tau) = \mu(\sigma) + \mu(\tau)]]_B = 1$ and it follows that $\phi(\sigma \cup \tau) = \phi(\sigma) + \phi(\tau)$. ϕ is strictly positive because $[[\mu$ is strictly positive$]]_B = 1$. Finally if $b \in B$, then $[[e(b) = \{\langle 1_C, b\rangle, \langle 0_C, -b\rangle\}]]_B = 1$. Hence, $\phi(b) = 1 \cdot m(b) + 0 \cdot m(-b) = m(b)$.

Theorem 5.8. *Suppose κ is measurable and I is a witnessing normal ideal. Let G be a generic filter for finite support iteration of random forcing of length κ. Let $\hat{I}$ be the induced ideal in $V[G]$. Then $\mathcal{P}(\kappa)/\hat{I}$ is a cBa that admits a SPFAM m. Furthermore one can identify the random algebra R_ω with a complete subalgebra of $\mathcal{P}(\kappa)/\hat{I}$ on which m agrees with the Lebegsue measure.*

Proof: Let $j : V \to M$ be the ultrapower embedding. Let $\langle B_\alpha, \mathring{C}_\alpha : \alpha < \kappa\rangle$ be the finite support iteration of random forcing; i.e.,

- $B_0 = R_\omega$
- $B_{\alpha+1} = B_\alpha \star \mathring{C}_\alpha$ where $[[\mathring{C}_\alpha = R_\omega]]_{B_\alpha} = 1$
- When λ is limit, B_λ is the Boolean completion of the direct limit of $\langle B_\alpha : \alpha < \lambda\rangle$.

So we have $B_0 \lessdot B_1 \lessdot \cdots \lessdot B = B_\kappa$ where B_κ is the completion of the direct limit of this iteration.

By elementarity plus the fact that M is countably closed, $j(B)$ is the finite support iteration of random forcings of length $j(\kappa)$ in M. Notice that $B \lessdot j(B)$ since $j(B)$ extends

B through a longer iteration. Let G be B-generic over V. Then $j \upharpoonright G$ is identity and hence $j[G] = G$. Let $\hat{I}$ be the ideal generated by I in $V[G]$. Consider the map $\phi : \mathcal{P}(\kappa)/\hat{I} \to j(B)/G$ defined by $\phi([X]) = [[\kappa \in j(\mathring{X})]]_{j(B)}/G$. Then ϕ is a Boolean isomorphism. Now in $M[G]$, $j(B)/G$ is a finite support iteration of random forcing indexed by $j(\kappa)\backslash\kappa$. By previous lemma, we can construct an increasing sequence of SPFAMs $\langle m_\alpha : \kappa+1 \leq \alpha < j(\kappa)\rangle$ on this iteration. Let $m : j(B)/G \to [0,1]$ be their union. Since $M[G]$ and $V[G]$ have same reals, we can also assume that the measure algebra of $m_{\kappa+1}$ is the random algebra in $V[G]$. Now in $V[G]$, we can lift m to a SPFAM on $\mathcal{P}(\kappa)/\hat{I}$ via the isomorphism ϕ and this finishes the proof.

To lift m to an extension of Lebesgue measure on 2^ω, create a tree $\langle X_\sigma : \sigma \in 2^{<\omega}\rangle$ of subsets of κ such that

- $X_\phi = \kappa$
- For every $\sigma \in 2^{<\omega}$, X_σ is a disjoint union of $X_{\sigma 0}$ and $X_{\sigma 1}$
- $m(X_\sigma) = 2^{-|\sigma|}$, where $|\sigma|$ is the length of σ

Furthermore, m restricted to the sigma algebra generated by $\{X_\sigma : \sigma \in 2^{<\omega}\}$ is isomorphic to Lebesgue measure on R_ω under an isomorphism that takes X_σ to $[\sigma] \in 2^\omega$. Let $f : \kappa \to 2^\omega$ be such that $f(\alpha) = y$ iff $\forall n(\alpha \in X_{y\upharpoonright n})$. Define $\nu : \mathcal{P}(2^\omega) \to [0,1]$ by $\nu(Y) = m(f^{-1}[Y])$. Then, ν is a finitely additive total extension of Lebesgue measure whose null ideal is 2^ω-additive. To finish note that since Cohen reals are added cofinally often, every set of reals of size less than κ is Lebesgue null. It is well known, by the Gitik–Shelah theorem [1], that if there a real valued measurable cardinal below continuum then there is a Sierpiński set of size ω_1. Hence there is no real valued measurable in this model and in particular ν is nowhere countably additive.

We now address the question:

Question 5.9. *Let $m : \mathcal{P}([0,1]) \to [0,1]$ be a finitely additive total extension of Lebesgue measure whose null ideal is countably additive. Can the density topology of m on $[0,1]$ provide an example of a ccc Hausdorff space without isolated points such that every partition of $[0,1]$ into countably many sets contains a set with non empty interior?*

It turns out that the answer is no. In fact, we'll show the following:

Theorem 5.10. *Let $m : \mathcal{P}([0,1]) \to [0,1]$ be a finitely additive total extension of Lebesgue measure. Denote the m-density topology by $\mathcal{T}$. Then there is a countable partition of $[0,1]$ into $\mathcal{T}$-interior free sets.*

Proof: Call $S \subseteq [0,1]$ *small* if S can be covered by countably many m-null sets. Let $\{S_n : n \in \omega\}$ be a maximal collection of pairwise disjoint small sets of positive measure. Then $A = [0,1]\backslash \bigcup_{n\in\omega} S_n$ does not contain any small set of positive measure. Hence the null ideal of m restricted to A is countably additive. It is now enough to split A into countably many $\mathcal{T}$-interior free sets.

Lemma 5.11. *Let m be as above. Suppose $X \subset [0,1]$ is open in the m-density topology $\mathcal{T}$. Then there is a G_δ set G (in the usual topology) such that $m(G) = m(X)$ and $X\backslash G$ is Lebesgue null.*

Proof: Fix an arbitrary $\varepsilon > 0$. Let $\mathcal{V}$ be the collection of all closed intervals in which the fractional measure of X is more than $1-\varepsilon$. Since every point of X is a density one point of X, $\mathcal{V}$ is a Vitali covering of X; i.e., for each $\delta > 0$, and $y \in X$ there is an interval $I \in \mathcal{V}$ that contains y and has length less than δ. By the Vitali covering theorem, there is a disjoint subcollection $\{I_n : n \in \omega\} \subset \mathcal{F}$ that covers all but a Lebesgue null part of X. Setting $U_\varepsilon = \bigcup\{I_n : n \in \omega\}$, we get $m(X) \geq \sum_{n\in\omega} m(X \cap I_n) \geq (1-\varepsilon)\sum_{n\in\omega} m(I_n) = (1-\varepsilon)m(U_\varepsilon)$. Let G be the intersection of U_ε's where ε runs over positive rationals.

Lemma 5.12. *Let m, A be as above. Then there is a partition of A into countably many $\mathcal{T}$-interior free sets.*

We first show that there is no positive measure $X \subseteq A$ all of whose positive measure subsets have non empty interior in the m-density topology. Suppose otherwise. Let $\{Y_n : n \in \omega\}$ be a maximal collection of pairwise disjoint $\mathcal{T}$-open subsets of X. Then $Y = \bigcup\{Y_n : n \in \omega\}$ covers all but an m-null part of X. Hence there is a G_δ set G such that $X\Delta G$ is m-null. The same holds of any positive measure subset of X. It follows that $m \restriction \mathcal{P}(X)$ is countably additive but its measure algebra is separable. But this is impossible by the Gitik–Shelah theorem.

Now let $\{W_n : n \in \omega\}$ be a maximal family of pairwise disjoint, $\mathcal{T}$-interior free, positive measure subsets of A. Let $X = A \setminus \bigcup_{n\in\omega} W_n$. Then every positive measure subset of X has non empty interior and hence X must be m-null and we are done.

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