

# Inverse Limits and Function Algebras\*

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## Abstract

Assuming Jensen's principle  $\diamond$ , there is a compact Hausdorff space  $X$  which is hereditarily Lindelöf, hereditarily separable, and connected, such that no perfect subspace of  $X$  is totally disconnected. The Proper Forcing Axiom implies that there is no such space. The  $\diamond$  example also fails to satisfy the CSWP (the complex version of the Stone-Weierstrass Theorem). This space cannot contain the two earlier examples of failure of the CSWP, which were totally disconnected — specifically, the Cantor set (W. Rudin) and  $\beta\mathbb{N}$  (Hoffman and Singer).

## 1 Introduction

All topologies discussed in this paper are assumed to be Hausdorff. It is well-known that if  $X$  is compact and second countable and not scattered, then  $X$  has a subspace homeomorphic to the usual Cantor set,  $2^\omega$ . This is not true of non-second countable spaces. For example, the double arrow space of Alexandroff and Urysohn ([1], p. 76) is compact and not scattered, but is only first countable and does not contain a Cantor subset.

The double arrow space is also HS (hereditarily separable) and HL (hereditarily Lindelöf); that is, all subspaces are both separable and Lindelöf (see [4] Exercise 3.10.C). It is also a LOTS; that is, a totally ordered set with its order topology. The double arrow space is also totally disconnected, and it is natural to ask whether there is a connected version of it. This turns out to be independent of ZFC. Under the Proper Forcing Axiom (PFA), there is no such space:

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**Theorem 1.1** *Assuming PFA, every compact HL space is either totally disconnected or contains a copy of the Cantor set.*

On the other hand, by Theorem 1.3, there will be such a space assuming Jensen's principle  $\diamond$ , which is true in Gödel's universe of constructible sets.

**Definition 1.2** *A space  $X$  is weird iff  $X$  is compact and not scattered, and there is no  $P \subseteq X$  such that  $P$  is perfect and totally disconnected.*

As usual,  $P$  is perfect iff  $P$  is closed and nonempty and has no isolated points. A weird space cannot be second countable. However,

**Theorem 1.3** *Assuming  $\diamond$ , there is a weird  $X$  such that  $X$  is HS and HL.*

Note that a compact  $X$  is HL iff every closed set is a  $G_\delta$  (see [4] Exercise 3.8.A(c)). Applying this to the points, we see that  $X$  must be first countable. Examples of weird spaces which are not first countable occur already in the literature; see Section 6. A weird space cannot be a LOTS; see Corollary 2.7.

We can also get our space in Theorem 1.3 to fail the complex version of the Stone-Weierstrass Theorem. The usual version of this theorem involves subalgebras of  $C(X, \mathbb{R})$ , and is true for all compact  $X$ . If one replaces the real numbers  $\mathbb{R}$  by the complex numbers  $\mathbb{C}$ , the "theorem" is true for some  $X$  and false for others, so it becomes a *property* of  $X$ :

**Definition 1.4** *If  $X$  is compact, then  $C(X) = C(X, \mathbb{C})$  is the algebra of continuous complex-valued functions on  $X$ , with the usual supremum norm.  $\mathcal{A} \sqsubseteq C(X)$  means that  $\mathcal{A}$  is a subalgebra of  $C(X)$  which separates points and contains the constant functions.  $\mathcal{A} \sqsubseteq\!\!\sqsubseteq C(X)$  means that  $\mathcal{A} \sqsubseteq C(X)$  and  $\mathcal{A}$  is closed in  $C(X)$ .  $X$  has the Complex Stone-Weierstrass Property (CSWP) iff every  $\mathcal{A} \sqsubseteq C(X)$  is dense in  $C(X)$ .*

Classical examples from the 1800s show that the CSWP is false for many  $X$ . In particular:

**Definition 1.5**  *$D$  denotes the open unit disc in  $\mathbb{C}$  and  $\mathbb{T}$  denotes the unit circle. The disc algebra  $\mathfrak{D} \sqsubseteq\!\!\sqsubseteq C(\overline{D})$  is the set of  $f \in C(\overline{D})$  which are holomorphic on  $D$ .*

Then  $\mathfrak{D}$  refutes the CSWP of  $\overline{D}$ , and  $\mathfrak{D}|_{\mathbb{T}} = \{f|_{\mathbb{T}} : f \in \mathfrak{D}\}$  refutes the CSWP of  $\mathbb{T}$ . Further negative results were obtained in 1956 by Rudin [17] and in 1960 by Hoffman and Singer [14] (see also [13]):

1. [17] Every compact  $X$  containing a copy of the Cantor set fails the CSWP.
2. [14] Every compact  $X$  containing a copy of  $\beta\mathbb{N}$  fails the CSWP.

Actually, [14] does not mention  $\beta\mathbb{N}$ , and used instead  $S =$  the Stone space of a separable measure algebra, but that is equivalent, since each of  $S$  and  $\beta\mathbb{N}$  contains a copy of the other. The first non-trivial positive result is due to Rudin [18], and some more recent positive results are contained in [12, 15]. In particular,

3. [18] Every compact scattered space satisfies the CSWP.
4. [15] Every compact LOTS which does not contain a copy of the Cantor set satisfies the CSWP.

By (4), the double arrow space is an example of a non-scattered space which has the CSWP. Results (1) through (4) might suggest the (highly unlikely) conjecture that a compact  $X$  has the CSWP whenever it contains neither  $\beta\mathbb{N}$  nor a Cantor set. Under  $\diamond$ , this is refuted by:

**Theorem 1.6** *Assuming  $\diamond$ , there is a weird  $X$  such that  $X$  is HS and HL and  $X$  fails the CSWP.*

As Rudin pointed out, (1)(3) imply that for  $X$  compact metric,  $X$  has the CSWP iff  $X$  does not contain a Cantor subset. By (1)(4), the same “iff” holds when  $X$  is a compact LOTS. By (2), the “iff” does not hold for arbitrary compact spaces, but one might hope to prove it for some other spaces which are small in some way. Theorem 1.6 puts some bounds on this hope.

Obviously, Theorem 1.6 implies Theorem 1.3, but we shall prove Theorem 1.3 first. We then explain what needs to be added to the construction to obtain Theorem 1.6. Both proofs are essentially inverse limit constructions. For Theorem 1.3, we obtain  $X \subset [0, 1]^{\omega_1}$  by an inductive construction; at stage  $\alpha < \omega_1$ , we determine the projection,  $X_\alpha$ , of  $X$  on  $[0, 1]^\alpha$ . Then,  $X$  may be viewed as the inverse limit of  $\langle X_\alpha : \alpha < \omega_1 \rangle$ . For Theorem 1.6, we replace  $[0, 1]$  by  $\overline{D}$ .

Theorem 1.3 is proved in Section 2, which also gives some more information about weird spaces. Theorem 1.6 is proved in Section 4, using a fact about peak points proved in Section 3. Theorem 1.1 is proved in Section 5, which may be read immediately after Section 2.

## 2 Weird Spaces

We list some easy properties of weird spaces:

**Definition 2.1**  $\text{comp}(x, X)$  denotes the connected component of the point  $x$  in the space  $X$ .

**Lemma 2.2** *If  $X$  is weird then:*

1. *If  $Y \subseteq X$  and  $Y$  is closed, then  $Y$  is either scattered or weird.*
2. *For some  $x \in X$ :  $\text{comp}(x, X)$  is not a singleton, so that  $\text{comp}(x, X)$  is weird and connected.*
3.  *$X$  is not second countable.*
4.  *$X$  is not a LOTS.*

**Proof.** For (4), if  $X$  is a LOTS, let  $S \subset X$  be countable and order-isomorphic to the rationals. Since  $\overline{S}$  cannot be totally disconnected, it contains an interval isomorphic to the closed unit interval in  $\mathbb{R}$ , contradicting (1) and (3). ☺

We shall see that no subspace of a countable product of LOTS can be weird either. First:

**Lemma 2.3** *If  $X$  is weird and  $f$  maps  $X$  continuously onto  $Y$ , then either  $Y$  is weird or some  $f^{-1}\{y\}$  is weird.*

**Proof.** Assume no  $f^{-1}\{y\}$  is weird. Then each  $f^{-1}\{y\}$  is scattered.

Note that  $Y$  cannot be scattered. To see this, let  $K$  be the perfect kernel of  $X$ . If  $y$  is an isolated point of  $f(K)$ , then  $K \cap f^{-1}\{y\}$  is scattered and clopen in  $K$ , a contradiction.

If  $Y$  is not weird, fix  $P \subseteq Y$  such that  $P$  is perfect and totally disconnected. Then for  $x \in f^{-1}(P)$ ,  $\text{comp}(x, f^{-1}(P)) \subseteq f^{-1}\{f(x)\}$ , which is scattered, so  $\text{comp}(x, f^{-1}(P)) = \{x\}$ . Thus,  $f^{-1}(P)$  is totally disconnected, and hence scattered (since  $X$  is weird), which is a contradiction, since  $P = f(f^{-1}(P))$  is not scattered. ☺

**Corollary 2.4** *Suppose that  $X$  is weird and  $X \subseteq \prod_{j < n} Z_j$ , where  $n$  is finite and each  $Z_j$  is compact. Then some  $Z_j$  has a weird subspace.*

**Proof.** Induct on  $n$ , using Lemma 2.3. ☺

We now prove the same result for countable products. First, we introduce some notation for products and projections:

**Definition 2.5** *If  $Z_\xi$  are spaces for  $\xi < \beta$  then  $\pi_\alpha^\beta : \prod_{\xi < \beta} Z_\xi \rightarrow \prod_{\xi < \alpha} Z_\xi$  (for  $\alpha \leq \beta$ ) and then  $\varphi_\alpha^\beta : \prod_{\xi < \beta} Z_\xi \rightarrow Z_\alpha$  (for  $\alpha < \beta$ ) and are the natural projections. If  $\vec{z} = \langle z_\xi : \xi < \beta \rangle \in \prod_{\xi < \beta} Z_\xi$ , then  $\varphi_\alpha^\beta(\vec{z}) = z_\alpha$  and  $\pi_\alpha^\beta(\vec{z}) = \langle z_\xi : \xi < \alpha \rangle$ . We sometimes write  $\varphi_\alpha$  for  $\varphi_\alpha^\beta$  when  $\beta$  is clear from context.*

**Lemma 2.6** *Suppose that  $X$  is weird and  $X \subseteq \prod_{j < \omega} Z_j$ , where each  $Z_j$  is compact. Then some  $Z_j$  has a weird subspace.*

**Proof.** Assume that no  $Z_j$  has a weird subspace; we shall derive a contradiction.

Let  $X_n = \pi_n^\omega(X) \subseteq \prod_{j < n} Z_j$ . By Corollary 2.4, no  $X_n$  has a weird subspace.

View  $\bigcup_n X_n$  as a tree, where  $X_n$  is the  $n^{\text{th}}$  level, and the tree order  $<$  satisfies  $y < z$  iff  $y = \pi_m^n(z)$  whenever  $m < n$ ,  $y \in X_m$  and  $z \in X_n$ . Let  $W_n$  be the set of all  $y \in X_n$  such that  $X \cap (\pi_n^\omega)^{-1}\{y\}$  is weird (equivalently, non-scattered). Note that  $\bigcup_n W_n$  is a subtree of  $\bigcup_n X_n$ ; equivalently,  $\pi_m^n(W_n) \subseteq W_m$  whenever  $m < n$ .

First, note that if  $P \subseteq X$  is closed and not scattered, then  $W_n \cap \pi_n^\omega(P) \neq \emptyset$  for each  $n$ . To see this, use the fact that  $P$  is weird and  $\pi_n^\omega(P)$  is not weird, and apply Lemma 2.3 to  $\pi_n^\omega \upharpoonright P$ .

It follows that  $\bigcup_n W_n$  is a perfect tree; that is, if  $y \in W_m$ , then for some  $n > m$ , there are more than one  $z \in W_n$  such that  $\pi_m^n(z) = y$ . To see this, let  $P_0, P_1$  be disjoint perfect subsets of  $X \cap (\pi_m^\omega)^{-1}\{y\}$ , and choose  $n$  such that  $\pi_n^\omega(P_0) \cap \pi_n^\omega(P_1) = \emptyset$ . If  $z_\ell \in W_n \cap \pi_n^\omega(P_\ell)$  (for  $\ell = 0, 1$ ), then  $z_\ell \in W_n$  and  $\pi_m^n(z_\ell) = y$  and  $z_0 \neq z_1$ .

But now we can choose a Cantor subtree. That is, we can choose finite nonempty  $F_n \subseteq W_n$  so that  $m < n \rightarrow \pi_m^n(F_n) = F_m$  and for each  $m$ , there is an  $n > m$  such that  $|F_n \cap (\pi_m^n)^{-1}\{y\}| \geq 2$  for all  $y \in F_m$ . Then  $\{x \in X : \forall n [\pi_n^\omega(x) \in F_n]\}$  is homeomorphic to the Cantor set, a contradiction. ☺

In particular, by Lemma 2.2, and the observation that every closed subspace of a compact LOTS is a compact LOTS:

**Corollary 2.7** *Suppose that  $X \subseteq \prod_{j < \omega} Z_j$ , where each  $Z_j$  is compact and is either second countable or a LOTS. Then  $X$  is not weird.*

We now turn to a proof of Theorem 1.3, which obtains a weird subspace of an uncountable product,  $[0, 1]^{\omega_1}$ . There are many such constructions in the literature; we follow the specific approach in [2]§4, which uses irreducible projections (see [4] Exercise 3.1.C) to ensure that the space is HS and HL.

## 2.1 The Construction

We shall get  $X = X_{\omega_1} \subseteq [0, 1]^{\omega_1}$  with  $X_\alpha = \pi_\alpha^{\omega_1}(X) \subseteq [0, 1]^\alpha$  satisfying:

0.  $X_1 = [0, 1]$ .
1.  $X_\alpha$  is connected whenever  $1 \leq \alpha \leq \omega_1$ .
2.  $\pi_\alpha^\beta : X_\beta \rightarrow X_\alpha$  is irreducible whenever  $1 \leq \alpha \leq \beta \leq \omega_1$ .

In particular,  $\pi_1^{\omega_1} : X_{\omega_1} \rightarrow X_1$  will be irreducible, so  $X$  will be separable and have no isolated points. To make  $X$  HS, we get  $P_\alpha$  and  $\mathcal{P}_\alpha$  for  $1 \leq \alpha < \omega_1$  so that:

3.  $\mathcal{P}_\alpha$  is a countable family of closed subsets of  $X_\alpha$  and  $P_\alpha \in \mathcal{P}_\alpha$ .
4. For all  $P \in \mathcal{P}_\alpha$ :
  - a.  $\pi_\alpha^{\alpha+1} : (\pi_\alpha^{\alpha+1})^{-1}(P) \rightarrow P$  is irreducible, and
  - b.  $(\pi_\alpha^\beta)^{-1}(P) \in \mathcal{P}_\beta$  whenever  $\alpha < \beta < \omega_1$ .

**Lemma 2.8** *Requirement (4) implies that  $\pi_\alpha^\beta : (\pi_\alpha^\beta)^{-1}(P) \rightarrow P$  is irreducible whenever  $P \in \mathcal{P}_\alpha$  and  $\alpha \leq \beta \leq \omega_1$ .*

**Proof.** Induct on  $\beta$ . ☺

To get  $X$  to be HL and HS, we add the next requirement:

5. If  $F \subseteq X$  is closed, then  $\pi_\alpha^{\omega_1}(F) = P_\alpha$  for some  $\alpha < \omega_1$ .

**Lemma 2.9** *Requirements (4)(5) imply that  $X$  is HL and HS.*

**Proof.** To see that  $X$  is HL, use (5) and (4) to see that every closed  $F \subseteq X$  is a  $G_\delta$ : For every closed subset  $F$  of  $X$ , we have  $\pi_\alpha^{\omega_1}(F) = P \in \mathcal{P}_\alpha$ . Then by irreducibility,  $F = (\pi_\alpha^{\omega_1})^{-1}(P)$ , so that  $F$  is a  $G_\delta$ . Also by (5), all closed  $F \subseteq X$  are separable, so  $X$  is HS (since it is HL and hence first countable). ☺

Conditions (0)–(5) are consistent with all  $\pi_\alpha^\beta$  being homeomorphisms, which would make  $X$  homeomorphic to  $[0, 1]$ . To make  $X$  weird, we also choose  $h_\alpha, p_\alpha$ , and  $q_\alpha^n$  for  $n < \omega$  and  $0 < \alpha < \omega_1$  so that:

6.  $p_\alpha \in X_\alpha$  and  $h_\alpha \in C(X_\alpha \setminus \{p_\alpha\}, [0, 1])$  and  $X_{\alpha+1} = \overline{h_\alpha}$ .
7.  $q_\alpha^n \in X_\alpha \setminus \{p_\alpha\}$ , and  $\langle q_\alpha^n : n \in \omega \rangle \rightarrow p_\alpha$ , and all points of  $[0, 1]$  are limit points of  $\langle h_\alpha(q_\alpha^n) : n \in \omega \rangle$ , and  $\{p_\alpha\} \times [0, 1] \in \mathcal{P}_{\alpha+1}$ .
8. For each  $P \in \mathcal{P}_\alpha$ , either  $p_\alpha \notin P$ , or  $p_\alpha \in P$  and  $q_\alpha^n \in P$  for all but finitely many  $n$ .

As usual, we identify  $h_\alpha$  with its graph, which is a subset of  $X_\alpha \times [0, 1]$ ; we also identify  $[0, 1]^\alpha \times [0, 1]$  with  $[0, 1]^{\alpha+1}$ .

**Lemma 2.10** *Requirements (0)(6)(7) imply requirements (1)(2).*

**Proof.** Induct on  $\alpha$ . By (6),  $\pi_\alpha^{\alpha+1} : X_{\alpha+1} \rightarrow X_\alpha$  is one-to-one at all points not in  $(\pi_\alpha^{\alpha+1})^{-1}\{p_\alpha\}$ . The first part of (7) implies that  $\{p_\alpha\} \times [0, 1] \subseteq X_{\alpha+1}$ . ☺

**Lemma 2.11** *Requirements (0) – (8) imply that if  $C \subseteq X_\alpha$  is closed and connected, then  $(\pi_\alpha^{\omega_1})^{-1}(C)$  is connected.*

**Proof.** It is enough to prove that  $(\pi_\alpha^{\alpha+1})^{-1}(C)$  is connected. Using (6), this is clear unless  $p_\alpha \in C$ , in which case apply (7), which implies that  $\{p_\alpha\} \times [0, 1] \subseteq (\pi_\alpha^{\alpha+1})^{-1}(C)$ . ☺

**Lemma 2.12** *Requirement (4a) follows from (6)(7)(8).*

**Proof.** Irreducibility is clear unless  $p_\alpha \in P$ , in which case apply (7)(8). ☺

To help make  $X$  weird we add the requirement:

9. If  $F \subseteq X$  is closed and not scattered, then for some  $\alpha < \omega_1$ ,  $\pi_\alpha^{\omega_1}(F) = P_\alpha$  and  $P_\alpha$  is not scattered and  $p_\alpha \in P_\alpha$ .

Note that we cannot simply omit (5) in favor of (9), since Lemma 2.9 uses (5) for all closed  $F$ , including singletons.

**Lemma 2.13** *Requirements (0) – (9) imply that  $X$  is weird.*

**Proof.** By (9), every closed non-scattered  $F \subseteq X$  satisfies  $\pi_\alpha^{\omega_1}(F) = P_\alpha$ , for some  $\alpha < \omega_1$ , with  $P_\alpha$  not scattered and  $p_\alpha \in P_\alpha$ . Such  $F$  therefore contain  $(\pi_{\alpha+1}^{\omega_1})^{-1}(\{p_\alpha\} \times [0, 1])$ . By (7) and Lemma 2.11, each  $(\pi_{\alpha+1}^{\omega_1})^{-1}(\{p_\alpha\} \times [0, 1])$  is a connected subspace of  $X$ . ☺

**Proof of Theorem 1.3.** To get (5) and (9), use  $\diamond$  to capture all closed subsets of  $[0, 1]^{\omega_1}$ . To get (7)(8) for a fixed  $\alpha$ : First, list  $\mathcal{P}_\alpha$  as  $\{Q^n : n \in \omega\}$ , with  $Q^0 = P_\alpha$ . Let  $d$  be a metric on  $X_\alpha$ . Choose perfect  $F^n \subseteq X_\alpha$  for  $n \in \omega$  so that  $\text{diam}(F^n) \leq 2^{-n}$  and each  $F^{n+1} \subsetneq F^n$ . Let  $\{p_\alpha\} = \bigcap_n F^n$  and let  $q_\alpha^n$  be any point in  $F^{n+1} \setminus F^n$ . Make sure that  $F^0 \subseteq Q^0 = P_\alpha$  whenever  $P_\alpha$  is uncountable, so that  $p_\alpha \in P_\alpha$  is as required by (9). Also make sure that for every  $n$ , either  $F^n \subseteq Q^n$  or  $F^n \cap Q^n = \emptyset$ , so that (8) will hold. ☺

### 3 Peak Sets

Fix  $\alpha < \omega_1$ . The function  $h_\alpha$  occurring in the proof of Theorem 1.3 is easy to construct because  $X_\alpha$  is a compact metric space. Note that there are also uniformly bounded  $g_{\alpha,n} \in C(X_\alpha)$  (for  $n \in \omega$ ) with  $g_{\alpha,n}(x) \rightarrow h_\alpha(x)$  whenever  $x \neq p_\alpha$ . In the proof of Theorem 1.6, we shall furthermore require that each

$g_{\alpha,n} \in \mathcal{A}_\alpha$ , where  $\mathcal{A}_\alpha \subseteq C(X_\alpha)$ . This is not always possible. For example, if  $X_\alpha = \overline{D}$  and  $\mathcal{A}_\alpha = \mathfrak{D}$ , the disc algebra, then we could not find such  $g_{\alpha,n}$  and  $h_\alpha$  unless  $p_\alpha \in \mathbb{T}$ , since  $h_\alpha$  is required to be discontinuous at  $p_\alpha$ . For  $\alpha = 1$ , we shall avoid this problem by defining  $X_1$  to be  $\mathbb{T}$ ; then a suitable  $h_1$  can be concocted using standard facts about  $H^\infty$  (see [10, 13, 16, 19]). To obtain suitable  $h_\alpha$  on  $X_\alpha$  for  $\alpha > 1$ , we shall apply Lemma 3.3 and require that all points of  $X_\alpha$  be peak points; the following is easily seen to be equivalent to the usual definition (see, e.g., [9]):

**Definition 3.1** *Assume that  $X$  is compact,  $\mathcal{A} \subseteq C(X)$ , and  $H$  is a closed subset of  $X$ . Then  $H$  is a peak set (with respect to  $\mathcal{A}$ ) iff there is an  $f \in \mathcal{A}$  such that*

1.  $f(x) = 0$  for all  $x \in H$ .
2.  $\Re(f(x)) > 0$  for all  $x \notin H$ .

$\mathcal{PS}_\mathcal{A}(X)$  is the set of all  $H \subseteq X$  which are peak sets with respect to  $\mathcal{A}$ .  $p \in X$  is a peak point iff  $\{p\}$  is a peak set.

Every peak set is a closed  $G_\delta$  set, but not conversely. For example, if  $H$  is clopen and  $\mathcal{A} \subseteq C(X)$ , then by Runge's Theorem,  $H$  is a peak set iff  $\chi_H \in \mathcal{A}$ . Also, for the disc algebra,  $p \in \overline{D}$  is a peak point iff  $|p| = 1$ .

Our primary interest here is in the peak *points*. However, we mention peak *sets* because these will be used to prove that  $\mathcal{PS}_\mathcal{A}(X)$  contains singletons by applying the following well-known fact:

**Lemma 3.2** *If  $\mathcal{A} \subseteq C(X)$ , then  $\mathcal{PS}_\mathcal{A}(X)$  is closed under countable intersections and finite unions.*

**Proof.** For intersections, fix  $H_n \in \mathcal{PS}_\mathcal{A}(X)$  for  $n \in \omega$ , and let  $H = \bigcap_n H_n$ . Let  $f_n$  satisfy (1)(2) of Definition 3.1 for  $H_n$ , and assume that  $\|f_n\| \leq 2^{-n}$ . Let  $f = \sum_n f_n$ . Then  $f \in \mathcal{A}$  because  $\mathcal{A}$  is closed, and  $f$  satisfies (1)(2) for  $H$ .

For unions, let  $H = H_0 \cup H_1$ , and let  $f_0, f_1$  satisfy (1)(2) of Definition 3.1 for  $H_0, H_1$  respectively. Define  $f(x) = \sqrt{f_0(x)}\sqrt{f_1(x)}$ . Again,  $f \in \mathcal{A}$  because  $\mathcal{A}$  is closed, since  $\sqrt{z}$  can be uniformly approximated by polynomials on any compact subset of  $\{z \in \mathbb{C} : \Re(z) \geq 0\}$ , and  $f$  satisfies (1)(2) for  $H$ . ☺

**Lemma 3.3** *Assume that  $X$  is compact,  $\mathcal{A} \subseteq C(X)$ , and  $p \in X$  is a peak point. Let  $\langle q_n : n \in \omega \rangle$  be a sequence of points in  $X \setminus \{p\}$  converging to  $p$ . Then there are functions  $h$  and  $g_n$  for  $n \in \omega$  such that:*



1. Each  $g_n \in \mathcal{A}$ .
2. Each  $\|g_n\| \leq 1$ .
3.  $h \in C(X \setminus \{p\}, \overline{D})$ .
4. On  $X \setminus \{p\}$ , the  $g_n$  converge to  $h$  uniformly on compact sets.
5.  $|h(x)| \rightarrow 1$  as  $x \rightarrow p$  in  $X \setminus \{p\}$ .
6. Every point in  $\mathbb{T}$  is a limit point of the sequence  $\langle h(q_n) : n \in \omega \rangle$ .

**Proof.** Let  $f_0$  be the function given by Definition 3.1. We plan to obtain  $h$  by composing  $f_0$  with a suitable Blaschke product. The notation will be easier if we define the product in the upper halfplane; see, e.g., [10], §II.2. Let

$$V = \{z \in \mathbb{C} : 0 < -\Re(z) < \Im(z)\} .$$

If  $f(z) = e^{5\pi i/8} \cdot \sqrt[4]{f_0(z)}$ , then  $f \in \mathcal{A}$ ,  $f(p) = 0$ , and  $f(x) \in V$  for all  $x \neq p$ . When  $\Im(\alpha) > 0$ , let

$$B_\alpha(z) = \frac{z - \alpha}{z - \overline{\alpha}} .$$

Then  $|B_\alpha(z)|$  is 1 on the real axis and less than 1 in the upper halfplane. Let  $z_\ell = f(q_\ell) \in V$ ; then  $z_\ell \rightarrow 0$ . We shall choose  $\alpha_n$  in the upper halfplane and form the Blaschke products:

$$B^{(n)}(z) = \prod_{m < n} B_{\alpha_m}(z) \quad B(z) = \prod_{n \in \omega} B_{\alpha_n}(z)$$

They will satisfy:

- a.  $B^{(n)}(z) \rightarrow B(z)$  uniformly on compact subsets of  $V$ .
- b.  $|B(z)| \rightarrow 1$  as  $z \rightarrow 0$  in  $V$ .
- c. Every point in  $\mathbb{T}$  is a limit point of the sequence  $\langle B(z_\ell) : \ell \in \omega \rangle$ .

Assuming that this can be done, the lemma is satisfied by letting  $g_n = B^{(n)} \circ f$  and  $h = B \circ f$ .  $g_n \in \mathcal{A}$  because each  $B^{(n)}$  is holomorphic in a convex neighborhood of  $f(X)$ , and hence can be uniformly approximated on  $f(X)$  by polynomials.

To obtain (a)(b)(c), we choose the  $\alpha_n$ , along with a subsequence,  $\langle z_{\ell_n} : n \in \omega \rangle$ , of  $\langle z_\ell : \ell \in \omega \rangle$ , to satisfy:

- d.  $\alpha_n = \xi_n + i\eta_n$  and  $0 < \xi_n = (n+1)\eta_n$ .
- e.  $z_{\ell_n} = x_n + iy_n$  and  $\eta_n = y_n$ .
- f.  $n > m \Rightarrow \xi_n \leq 2^{-n}\eta_m$ .
- g.  $n > m \Rightarrow |\arg(B_{\alpha_m}(z_{\ell_n})) - \arg(B_{\alpha_m}(0))| \leq 2^{-n}$ .

So,  $\alpha_n$  is to the right of  $V$  and  $\xi_0 \geq \eta_0 \geq \xi_1 \geq \eta_1 \geq \xi_2 \geq \eta_2 \geq \dots$ . The  $\alpha_n$  and  $z_{\ell_n}$  can easily be chosen by induction to satisfy (d)(e)(f)(g), using  $z_\ell \rightarrow 0$  and the continuity of  $B_{\alpha_m}$  at 0. We now verify (a)(b)(c). Observe that  $\eta_n/\xi_n \rightarrow 0$  but  $\sum_n \eta_n/\xi_n = \infty$ ; this will allow us to prove (b) without having  $\lim_{z \rightarrow 0} B(z)$  exist, which would contradict (c).

For (a), note that if  $\alpha = \xi + i\eta$  and  $z = x + iy$  then

$$B_\alpha(z) = \frac{x - \xi + iy - i\eta}{x - \xi + iy + i\eta} = 1 - \frac{2i\eta}{x - \xi + iy + i\eta} .$$

Then, as usual with Blaschke products, (a) follows from  $\sum_n \eta_n < \infty$ , which in turn follows from (d)(f).

For (b), we need to estimate  $|B_\alpha(z)|$ , where  $\alpha = \xi + i\eta$ ,  $z = x + iy \in V$ , and  $0 < \eta \leq \xi$ . Now

$$|B_\alpha(z)|^2 = \frac{(\xi - x)^2 + (y - \eta)^2}{(\xi - x)^2 + (y + \eta)^2} = 1 - \frac{4y\eta}{(\xi - x)^2 + (y + \eta)^2} .$$

Clearly,

$$1 > |B_\alpha(z)|^2 \geq 1 - \frac{4\eta}{y} \quad 1 > |B_\alpha(z)|^2 \geq 1 - \frac{4y}{\eta} , \quad (*)$$

and these are useful when  $\eta \ll y$  or  $y \ll \eta$ . But also note that:

$$1 > |B_\alpha(z)|^2 \geq 1 - \frac{4\eta}{\xi} . \quad (\dagger)$$

To prove this: If  $y \geq \xi$  then  $(\dagger)$  follows from  $(*)$ . If  $y \leq \xi$  then, since  $x \leq 0$ ,  $|B_\alpha(z)|^2 \geq 1 - (4y\eta)/(\xi^2) \geq 1 - (4\eta)/\xi$ .

To prove (b), fix  $z = x + iy \in V$  with  $y \leq \eta_1$ . Next fix  $n \geq 1$  such that  $\eta_{n+1} \leq y \leq \eta_n$ . We show that  $|B(z)| = 1 - o(1)$  as  $n \nearrow \infty$  by estimating each  $|B_{\alpha_m}(z)|^2$ . Applying  $(\dagger)$  and (d), we get  $|B_{\alpha_n}(z)|^2 \geq 1 - 4\eta_n/\xi_n = 1 - 4/(n+1)$ , and likewise  $|B_{\alpha_{n+1}}(z)|^2 \geq 1 - 4/(n+2)$ . For  $m < n$  use  $(*)$  and (d)(f) to get

$$1 > |B_{\alpha_m}(z)|^2 \geq 1 - \frac{4y}{\eta_m} \geq 1 - \frac{4\eta_n}{\eta_m} = 1 - \frac{4\xi_n}{(n+1)\eta_m} \geq 1 - \frac{4 \cdot 2^{-n}}{n+1} ,$$

so  $\prod_{m < n} |B_{\alpha_m}(z)|^2 \geq 1 - 4 \cdot 2^{-n}$ . For  $m > n+1$  use  $(*)$  and (d)(f) to get

$$1 > |B_{\alpha_m}(z)|^2 \geq 1 - \frac{4\eta_m}{y} \geq 1 - \frac{4\eta_m}{\eta_{m+1}} \geq 1 - 4 \cdot 2^{-m} ,$$

so  $\prod_{m > n+1} |B_{\alpha_m}(z)|^2 \geq 1 - 2^{-n+1}$ . Putting these estimates together, we get  $|B(z)| = 1 - o(1)$ .

To verify (c), note that (b) implies that (c) is equivalent to the assertion that  $\{\arg(B(z_{\ell_n})) \bmod 2\pi : n \in \omega\}$  is dense in  $\mathbb{T}$ . We estimate  $\arg(B(z))$ , using:

$$\arg(B_\alpha(z)) = \arctan \frac{\eta - y}{\xi - x} + \arctan \frac{\eta + y}{\xi - x} = \arctan \frac{2\eta(\xi - x)}{(\xi - x)^2 + y^2 - \eta^2} .$$

We are using  $\arctan(u) + \arctan(v) = \arctan((u+v)/(1-uv))$ ; this applies here because all three of  $\arg(\alpha - z)$ ,  $\arg(\bar{\alpha} - z)$ , and  $\arg(B_\alpha(z))$  are in the range  $(-\pi/2, \pi/2)$ . Let  $\theta_n^m = \arg(B_{\alpha_m}(z_{\ell_n}))$ . Then  $\arg(B(z_{\ell_n})) \equiv \sum_m \theta_n^m \bmod 2\pi$ . Define:

$$\sigma^m := \arg(B_{\alpha_m}(0)) = 2 \arctan \frac{\eta_m}{\xi_m} = 2 \arctan \frac{1}{(m+1)} ,$$

and observe that we have:

1.  $0 < \sigma^m \rightarrow 0$  and  $\sum_m \sigma^m = \infty$ .
2.  $\theta_n^n \rightarrow 0$ .
3.  $\sum_{m>n} |\theta_n^m| \leq 2^{-n+2}$ .
4.  $m < n \Rightarrow |\theta_n^m - \sigma^m| \leq 2^{-n}$ .

(1) holds because  $\sigma^m \approx 2/m$ . (2) follows from (e) and  $x_n \leq 0$ , which yields

$$\theta_n^n = \arctan \frac{2\eta_n}{\xi_n - x_n} = \arctan \frac{2\eta_n}{(n+1)\eta_n - x_n} \leq \arctan \frac{2\eta_n}{(n+1)\eta_n} \rightarrow 0 .$$

For (3), use (d)(f) to get, for  $m > n$ :

$$|\theta_n^m| = \arctan \frac{2\eta_m |\xi_m - x_n|}{(\xi_m - x_n)^2 + \eta_n^2 - \eta_m^2} \leq \arctan \frac{4\eta_m \eta_n}{\eta_n^2} \leq \frac{4\xi_m}{(m+1)\eta_n} \leq 4 \cdot 2^{-m} ,$$

so that  $\sum_{m>n} |\theta_n^m| \leq 4 \cdot 2^{-n}$ . (4) is immediate from (g).

Finally, (1) implies that the values  $\sum_{m<n} \sigma^m \bmod 2\pi$  (for  $n \in \omega$ ) are dense in  $\mathbb{T}$ , and (2)(3)(4) imply that as  $n \rightarrow \infty$ , these values get close to  $\arg(B(z_{\ell_n}))$ . ☺

We remark that there are well-known interpolation theorems of Pick, Nevanlinna, Carleson, and others (see [10, 13, 16]) which involve constructing Blaschke products to have given values on a given sequence of points. However, because of our requirement (b) in the above proof, we do not see how to obtain our Blaschke product simply by quoting one of these theorems.

## 4 Subspaces of Polydiscs

We now return to the construction of §2.1, and show how to modify the space so that it also fails the CSWP. To get a function algebra witnessing this failure, it is easier to construct the space in  $\overline{D}^{\omega_1}$  rather than  $[0, 1]^{\omega_1}$ , so we start by replacing  $[0, 1]$  with  $\overline{D}$  in the requirements of §2.1.

We shall get  $X = X_{\omega_1} \subseteq \overline{D}^{\omega_1}$ , with  $X_\alpha = \pi_\alpha^{\omega_1}(X) \subseteq \overline{D}^\alpha$ . Let  $\text{REQ}^-$  denote the requirements consisting of conditions (1)–(5) and (8)–(9) of §2.1 plus:

- $\tilde{0}$ .  $X_1 = \mathbb{T}$ .
- $\tilde{6}$ .  $p_\alpha \in X_\alpha$  and  $h_\alpha \in C(X_\alpha \setminus \{p_\alpha\}, \overline{D})$  and  $X_{\alpha+1} = \overline{h_\alpha}$ .
- $\tilde{7}$ .  $q_\alpha^n \in X_\alpha \setminus \{p_\alpha\}$ , and  $\langle q_\alpha^n : n \in \omega \rangle \rightarrow p_\alpha$ , and all points of  $\mathbb{T}$  are limit points of  $\langle h_\alpha(q_\alpha^n) : n \in \omega \rangle$ , and  $|h_\alpha(x)| \rightarrow 1$  as  $x \rightarrow p_\alpha$  in  $X_\alpha \setminus \{p_\alpha\}$ , and  $\{p_\alpha\} \times \mathbb{T} \in \mathcal{P}_{\alpha+1}$ .

Note that we have the slice  $\{z \in \overline{D} : (p_\alpha, z) \in X_{\alpha+1}\}$  equal to  $\mathbb{T}$ , not  $\overline{D}$ , as one might expect. This will enable us to prove that all points in each  $X_\beta$  are peak points; see Lemma 4.5. Since  $\mathbb{T}$  is connected, the argument is essentially unchanged, and we get a weird HL space as before, using  $\diamond$ .

Along with the  $X_\alpha$ , we need a function algebra on  $X_\alpha$  refuting the CSWP. We use the obvious analog of the disc algebra:

**Definition 4.1**  $\mathfrak{P}_\alpha \sqsubseteq C(\overline{D}^\alpha)$  is the algebra generated by the projections  $\{\varphi_\xi : \xi < \alpha\}$  (see Definition 2.5), and  $\mathfrak{D}_\alpha \sqsubseteq C(\overline{D}^\alpha)$  is the uniform closure of  $\mathfrak{P}_\alpha$ . Let  $\mathcal{A}_\alpha$  be the uniform closure of  $\mathfrak{P}_\alpha \upharpoonright X_\alpha = \{f \upharpoonright X_\alpha : f \in \mathfrak{P}_\alpha\}$ .

For finite  $\alpha$ ,  $\mathfrak{P}_\alpha$  is the algebra of polynomials in  $\alpha$  complex variables on the polydisc  $\overline{D}^\alpha$ , and  $\mathfrak{D}_\alpha$  is the algebra of continuous functions which are holomorphic in the interior of the polydisc. For all  $\alpha > 0$ ,  $\mathfrak{D}_\alpha \neq C(\overline{D}^\alpha)$ . In constructing the  $X_\alpha$ , we also make sure that  $\mathcal{A}_\alpha \neq C(X_\alpha)$ . To do this, we choose all  $h_\alpha$  in  $H^\infty$ . More precisely, we use the following definition to transfer  $H^\infty(\mathbb{T})$  to  $X_\alpha$ :

**Definition 4.2** Let  $\lambda = \lambda_1$  be the Haar probability measure on  $X_1 = \mathbb{T}$ . For  $1 \leq \alpha < \omega_1$ , let  $\lambda_\alpha$  be the unique Borel probability measure on  $X_\alpha$  such that  $\lambda_1$  is the induced measure  $\lambda_\alpha(\pi_1^\alpha)^{-1}$ . For  $1 \leq \alpha \leq \beta < \omega_1$ , define the map  $(\pi_\alpha^\beta)^* : L^\infty(X_\alpha, \lambda_\alpha) \rightarrow L^\infty(X_\beta, \lambda_\beta)$  by  $(\pi_\alpha^\beta)^*([f]) = [f \circ \pi_\alpha^\beta]$ , where  $[g] \in L^\infty$  denotes the equivalence class of  $g$ .

Here and in the following, we frequently use  $\pi_\alpha^\beta, \varphi_\alpha^\beta$  to denote their restrictions,  $\pi_\alpha^\beta \upharpoonright X_\beta, \varphi_\alpha^\beta \upharpoonright X_\beta$ . Note that each  $\lambda_\alpha$  is unique because all points in  $X_1$  outside the countable  $\{\pi_1^\xi(p_\xi) : 1 \leq \xi < \alpha\}$  have a unique preimage under  $\pi_1^\alpha$ . Likewise,  $(\pi_\alpha^\beta)^*$  is a Banach algebra isomorphism.

Let  $\text{REQ}$  consist of the requirements of  $\text{REQ}^-$ , along with this requirement on the  $h_\alpha$ :

$\tilde{10}$ . For  $1 \leq \alpha < \omega_1$ ,  $[h_\alpha] \in (\pi_1^\alpha)^*(H^\infty(\mathbb{T}))$ .

This makes  $X$  fail the CSWP. Requirement  $(\tilde{10})$  is used explicitly in the proof of the next lemma. Lemma 4.4 follows, and produces a continuous function not in  $\mathcal{A}_{\omega_1}$ .

**Lemma 4.3** *Fix  $\beta$  with  $1 \leq \beta < \omega_1$ . Suppose requirement  $(\tilde{10})$  holds for all  $\alpha < \beta$ . Then  $[k] \in (\pi_1^\beta)^*(H^\infty(\mathbb{T}))$  for each  $k \in \mathcal{A}_\beta$ .*

**Proof.** Since  $\mathfrak{P}_\beta \upharpoonright X_\beta$  is generated by  $\{\varphi_\alpha : \alpha < \beta\}$ , it suffices to prove that each  $[\varphi_\alpha] \in (\pi_1^\beta)^*(H^\infty(\mathbb{T}))$ . Now,  $[\varphi_0] = (\pi_1^\beta)^*([I])$ , where  $I(z) = z$ . For  $1 \leq \alpha < \beta$ ,  $[\varphi_\alpha] = (\pi_\alpha^\beta)^*([h_\alpha]) = [h_\alpha \circ \pi_\alpha^\beta]$ . By (10) for  $\alpha < \beta$ ,  $[h_\alpha] = [h \circ \pi_1^\alpha]$  for some  $h \in H^\infty(\mathbb{T})$ . So  $[\varphi_\alpha] = [h \circ (\pi_1^\alpha \circ \pi_\alpha^\beta)] \in (\pi_1^\beta)^*(H^\infty(\mathbb{T}))$ .  $\textcircled{\smile}$

**Lemma 4.4** *Suppose requirements REQ hold. Then  $\mathcal{A}_{\omega_1} \neq C(X)$ .*

**Proof.** Let  $\bar{I} \in C(X_1)$  denote the usual complex conjugation given by  $\bar{I}(z) = \bar{z}$ . Then  $\bar{I} \circ \pi_1^{\omega_1}$  (i.e.,  $\bar{z} \mapsto \bar{z}_0$ ) is not in  $\mathcal{A}_{\omega_1}$ . To see this: it suffices to show that  $\bar{I} \circ \pi_1^\beta \notin \mathcal{A}_\beta$  for all  $\beta < \omega_1$ . Since  $\bar{I} \notin H^\infty(\mathbb{T})$ ,  $(\pi_1^\beta)^*(\bar{I}) \notin (\pi_1^\beta)^*(H^\infty(\mathbb{T}))$  for all  $\beta < \omega_1$ . So the result follows from Lemma 4.3.  $\textcircled{\smile}$

**Lemma 4.5** *Fix  $\beta$  with  $1 \leq \beta < \omega_1$ . Suppose requirement  $(\tilde{10})$  holds for all  $\alpha < \beta$ . Then each  $y \in X_\beta$  is a peak point with respect to  $\mathcal{A}_\beta$ .*

**Proof.** We induct on  $\beta$ . For  $\beta = 1$ , this is clear, since  $X_1 = \mathbb{T}$ .

If  $\beta$  is a limit, then  $\{y\} = \bigcap_{\alpha < \beta} (\pi_\alpha^\beta)^{-1}\{(\pi_\alpha^\beta)(y)\}$ . Applying the lemma inductively, each  $(\pi_\alpha^\beta)(y)$  is a peak point in  $X_\alpha$  with respect to  $\mathcal{A}_\alpha$ , which implies that each  $(\pi_\alpha^\beta)^{-1}\{(\pi_\alpha^\beta)(y)\}$  is a peak set in  $X_\beta$  with respect to  $\mathcal{A}_\beta$ . The result now follows using Lemma 3.2.

Now, say  $\beta = \alpha + 1$ , let  $v = \pi_\alpha^\beta(y)$  and let  $H = (\pi_\alpha^\beta)^{-1}\{v\}$ , which, as above, is a peak set in  $X_\beta$ . If  $v \neq p_\alpha$ , then  $H = \{y\}$ . If  $v = p_\alpha$ , then  $y \in H = \{v\} \times \mathbb{T}$  (using condition  $(\tilde{7})$ ). If  $y = (v, e^{i\theta})$ , then  $K = \{x \in X_\beta : \varphi_\alpha(x) = e^{i\theta}\}$  is also a peak set, and  $\{y\} = H \cap K$ .  $\textcircled{\smile}$

**Proof of Theorem 1.6.** We need to show inductively that requirements REQ can indeed be met. Suppose that we have constructed  $X_\beta$  so that they hold for all  $\alpha < \beta$ . Get  $p_\beta \in X_\beta$  and  $\langle q_\beta^n : n \in \omega \rangle$  converging to  $p_\beta$  as in the proof of Theorem 1.3. By Lemma 4.5,  $p_\beta$  is a peak point. Now get  $h \in C(X_\beta \setminus \{p_\beta\}, \bar{D})$  and  $g_n \in \mathcal{A}_\beta$  as in Lemma 3.3. Then each  $[g_n] \in (\pi_1^\beta)^*(H^\infty(\mathbb{T}))$  by Lemma 4.3, so  $[h] \in (\pi_1^\beta)^*(H^\infty(\mathbb{T}))$  since  $g_n \rightarrow h$  on  $X_\beta \setminus \{p_\beta\}$ . Thus, taking  $h_\beta = h$  satisfies  $(\tilde{10})$  for  $\beta$ . Lemma 3.3 also guarantees that this choice of  $h_\beta$  will satisfy the rest of  $(\tilde{7})$ . The remaining requirements are satisfied as for Theorem 1.3.  $\textcircled{\smile}$

## 5 Some Forcing Orders

**Definition 5.1** Order  $2^{<\omega_1}$  by:  $p \leq q$  iff  $p \supseteq q$ . Let  $\mathbb{1} = \emptyset$ , the empty sequence.

So,  $2^{<\omega_1}$  is a tree, with the root  $\mathbb{1}$  at the top. Viewed as a forcing order, it is equivalent to countable partial functions from  $\omega_1$  to 2. This forcing is countably closed, and thus preserves all  $\diamond$  sequences, and thus preserves the weird space constructed in the proof of Theorem 1.3. To kill such spaces, we shall force with subtrees of  $2^{<\omega_1}$  which satisfy a weakening of countable closure.

**Definition 5.2** A Cantor tree of sequences is a subset  $\{p_s : s \in 2^{<\omega}\} \subseteq 2^{<\omega_1}$  such that each  $p_{s \smallfrown \mu} < p_s$  for  $\mu = 0, 1$ , and each  $p_{s \smallfrown 0} \perp p_{s \smallfrown 1}$ .

That is,  $p_{s \smallfrown 0}$  and  $p_{s \smallfrown 1}$  are incompatible extensions of  $p_s$ .

**Definition 5.3**  $\mathbb{P} \subseteq 2^{<\omega_1}$  has the Cantor tree property iff:

1.  $\mathbb{1} \in \mathbb{P}$  and  $\mathbb{P}$  is a subtree:  $q \geq p \in \mathbb{P} \rightarrow q \in \mathbb{P}$ .
2. If  $p \in \mathbb{P}$  then  $p \smallfrown 0, p \smallfrown 1 \in \mathbb{P}$ .
3. Whenever  $\{p_s : s \in 2^{<\omega}\} \subseteq \mathbb{P}$  is a Cantor tree of sequences, there is at least one  $f \in 2^\omega$  such that  $\bigcup \{p_{f \upharpoonright n} : n \in \omega\} \in \mathbb{P}$ .

Of course, then in (3) there must be uncountably many such  $f$ ; in fact the set of  $f$  satisfying (3) must meet every perfect subset of the Cantor set  $2^\omega$ , since otherwise we could find a subtree of the given Cantor tree which contradicts the Cantor tree property. It is also easily seen by induction that  $\mathbb{P}$  is a normal subtree; i.e.:

**Lemma 5.4** If  $\mathbb{P} \subseteq 2^{<\omega_1}$  has the Cantor tree property, then whenever  $p \in \mathbb{P}$  and  $\text{dom}(p) < \alpha < \omega_1$ , there is a  $q \in \mathbb{P} \cap 2^\alpha$  such that  $q < p$ .

If  $\mathbb{P}$  has the Cantor tree property, then it is proper and forcing with it adds no  $\omega$ -sequences. Such orders are called *totally proper*; see Eisworth and Roitman [3], which gives a number of equivalents, which we use in:

**Lemma 5.5** If  $\mathbb{P} \subseteq 2^{<\omega_1}$  has the Cantor tree property, then  $\mathbb{P}$  is totally proper.

**Proof.** Fix a suitably large regular cardinal  $\theta$ , and let  $M \prec H(\theta)$  be countable and fix  $p \in \mathbb{P} \cap M$ . Following [3], it is sufficient to find a  $q \leq p$  such that whenever  $A \subseteq \mathbb{P}$  is a maximal antichain and  $A \in M$ , there is an  $r \in A \cap M$  with  $q \leq r$ .

To get  $q$ , let  $\{A_n : n \in \omega\}$  list all the maximal antichains which are in  $M$ , and build a Cantor tree  $\{p_s : s \in 2^{<\omega}\} \subseteq \mathbb{P} \cap M$  such that  $p_\emptyset \leq p$  and  $p_s$  extends an element of  $A_n \cap M$  for each  $s \in 2^n$ . Then choose  $f \in 2^\omega$  such that  $q := \bigcup \{p_{f \upharpoonright n} : n \in \omega\} \in \mathbb{P}$ . 😊

Thus, assuming PFA, this  $\mathbb{P}$  will have an uncountable chain. By Lemma 5.7, a weird space will yield such a  $\mathbb{P}$ , and hence cannot be HL under PFA.

**Lemma 5.6** *If  $X$  is compact, connected, and infinite, and  $U \subseteq X$  is a nonempty open set, then there is a closed  $K \subseteq U$  such that  $K$  is connected and infinite.*

**Proof.** Let  $V$  be open and nonempty with  $\overline{V} \subseteq U$ , fix  $p \in V$ , and let  $K = \text{comp}(p, \overline{V})$ . If  $K = \{p\}$ , then there is an  $H$  which is relatively clopen in  $\overline{V}$  such that  $p \in H \subsetneq V$ . But then  $H$  would be clopen in  $X$ , a contradiction. ☺

**Lemma 5.7** *Assume that  $X$  is compact, HL, and not totally disconnected, and assume that  $X$  has no subspace homeomorphic to the Cantor set  $2^\omega$ . Then there exists a  $\mathbb{P}$  with the Cantor tree property which has no uncountable chains.*

**Proof.** Along with  $\mathbb{P}$ , we shall choose sets  $H_p$  for  $p \in \mathbb{P}$  with the following properties:

1.  $H_p$  is an infinite closed connected subset of  $X$ .
2. If  $p \in \mathbb{P}$ , then  $p \frown 0, p \frown 1 \in \mathbb{P}$  and  $H_{p \frown 0}, H_{p \frown 1}$  are disjoint subsets of  $H_p$ .
3. If  $p \in 2^\gamma$ , where  $\gamma$  is a countable limit ordinal and  $p \upharpoonright \alpha \in \mathbb{P}$  for all  $\alpha < \gamma$ , then  $H_p = \bigcap \{H_{p \upharpoonright \alpha} : \alpha < \gamma\}$ , and  $p \in \mathbb{P}$  iff  $H_p$  is infinite.

$H_{\mathbb{1}}$  can be chosen because  $X$  is not totally disconnected. Given  $p \in \mathbb{P}$ , we can choose  $H_{p \frown 0}, H_{p \frown 1}$  by applying Lemma 5.6 to  $H_p$ . To verify the Cantor tree property, let  $\{p_s : s \in 2^{<\omega}\} \subseteq \mathbb{P}$  be a Cantor tree of sequences. For  $f \in 2^\omega$ , let  $p_f = \bigcup \{p_{f \upharpoonright n} : n \in \omega\}$ . If none of these  $p_f$  are in  $\mathbb{P}$ , then each  $H_{p_f}$  would be a singleton,  $\{x_f\}$ . But then  $\{x_f : f \in 2^\omega\}$  is homeomorphic to the Cantor set. ☺

**Proof of Theorem 1.1.** By Lemmas 5.5 and 5.7. ☺

It is well-known that PFA entails large cardinals. PFA implies the existence of inner models with many measurable cardinals, and in the usual construction of a model of PFA, the  $\omega_2$  of the extension is supercompact in the ground model. However, to get the result of Theorem 1.1, it is sufficient to shoot paths through various  $\mathbb{P}$  with the Cantor tree property, and this can easily be done by a finite support iteration of Suslin forcing over any model of  $V = L$ . We first note that under  $\diamond$ , we can put a Suslin tree inside  $\mathbb{P}$ :

**Lemma 5.8** *If  $\mathbb{P} \subseteq 2^{<\omega_1}$  has the Cantor tree property and  $\diamond$  holds, then there is a subtree  $\mathbb{S} \subseteq \mathbb{P}$  such that  $\mathbb{S}$  is a Suslin tree.*

**Proof.** Call  $A \subseteq 2^{<\omega_1}$  *thin* iff  $A \cap 2^{<\alpha}$  is countable for all  $\alpha < \omega_1$ . By  $\diamond$ , choose countable  $A_\alpha \subseteq 2^{<\alpha}$  for  $\alpha < \omega_1$  such that  $\{\alpha : A \cap 2^{<\alpha} = A_\alpha\}$  is stationary for all thin  $A \subseteq 2^{<\omega_1}$ . Now, build  $\mathbb{S}$  in the usual way by constructing inductively

$\mathbb{S} \cap 2^{\leq \alpha}$ . We assume inductively that these are normal trees; that is, whenever  $p \in \mathbb{S}$  and  $\text{dom}(p) < \alpha$ , there is a  $q \in \mathbb{S} \cap 2^\alpha$  such that  $q < p$ .

If  $p \in \mathbb{S}$ , then  $p \frown 0 \in \mathbb{S}$  and  $p \frown 1 \in \mathbb{S}$ ; this handles the successor stage.

If  $\gamma < \omega_1$  is a limit and we have decided on  $\mathbb{S} \cap 2^{< \gamma}$ , then we form  $\mathbb{S} \cap 2^{\leq \gamma}$  by putting countably many nodes into  $\mathbb{S} \cap 2^\gamma$  so that normality is preserved. In addition, if  $A_\gamma \subseteq \mathbb{S}$  and  $A_\gamma$  is a maximal antichain in  $\mathbb{S} \cap 2^{< \gamma}$ , then we make sure that every node in  $\mathbb{S} \cap 2^\gamma$  is below some element of  $A_\gamma$ . This is all possible, staying within  $\mathbb{P}$ , because of the Cantor tree property. ☺

**Theorem 5.9** *Assume that in the ground model  $V$ , we have  $\diamond$  on  $\omega_1$ , together with  $\diamond$  on the  $\omega_1$ -limits of  $\omega_2$ . Then, in  $V$ , there is a ccc forcing order  $\mathbb{M}$  such that whenever  $G$  is  $\mathbb{M}$ -generic over  $V$ ,  $V[G]$  will satisfy MA plus  $2^{\aleph_0} = \aleph_2$  plus the statement that every compact HL space is either totally disconnected or contains a copy of the Cantor set.*

**Proof.** Do the usual finite support iteration to make MA true, with  $\mathbb{M} = \bigcup \{\mathbb{M}_\delta : \delta < \omega_2\}$ , where each  $|\mathbb{M}_\delta| = \aleph_1$ , so that the  $\mathbb{M}_\delta$  extension will satisfy  $\diamond$ . Then the “usual bookkeeping” plus Lemma 5.8 will let us ensure that in  $V[G]$ , every subtree of  $\mathbb{P} \subseteq 2^{< \omega_1}$  with the Cantor tree property has an uncountable chain, so that the result holds by Lemma 5.7.

Observe that  $\mathbb{P} \subseteq 2^{< \omega_1}$  will have size  $\aleph_2$ , and will be described in  $V$  by an  $\mathbb{M}$ -name  $\dot{\mathbb{P}}$  which is forced to have the Cantor tree property. Then, in  $V$ , there will be a club  $C$  of  $\omega_1$ -limits in  $\omega_2$  such that for  $\delta \in C$ , the  $\mathbb{M}_\delta$ -name  $\dot{\mathbb{P}} \upharpoonright \mathbb{M}_\delta$  is also forced to have the Cantor tree property in  $V[G \cap \mathbb{M}_\delta]$ . The “usual bookkeeping” uses  $\diamond$  on the  $\omega_1$ -limits to ensure that we eventually handle  $\mathbb{P}$ . ☺

## 6 Remarks and Questions

We do not know if the result of Theorem 1.1 is consistent with CH: is there a model of ZFC + CH in which every compact HL space is either totally disconnected or contains a copy of the Cantor set? Clearly, by Theorem 1.3,  $\diamond$  would be false in this model. Furthermore, one cannot form this model by a naive iteration of the totally proper orders from Lemma 5.5, since the argument in Gregory [11] shows that CH (or even  $2^{\aleph_0} < 2^{\aleph_1}$ ) implies that there is a tree with the Cantor tree property which has no uncountable chains. We also do not know if one could modify Gregory’s argument to replace  $\diamond$  by CH in the proof of Theorem 1.3.

CH is sufficient to produce a weird space; examples in the literature are not first countable, but have other interesting properties; see Fedorchuk, Ivanov, and van Mill in [8], and the earlier papers of Fedorchuk [6, 7]. The space in [8],



produced from CH, has the property that every infinite closed subspace is strongly infinite dimensional. In particular, there are no convergent  $\omega$ -sequences, so such a space cannot be first countable.

Our weird space construction of Theorem 1.3, in contrast to that of [8], can (using  $\diamond$ ) generate a one dimensional space. Specifically, we can get our space  $X$  to have small inductive dimension one ( $\text{ind}(X) = 1$ ). This implies that  $X$  also has large inductive dimension and covering (or Čech – Lebesgue) dimension one; see [5], Theorems 2.4.3, 3.1.29, and 3.1.30. Recall (see [5], Definition 1.1.1) that  $\text{ind}(X) = 1$  iff  $X$  is not zero dimensional and the family of all open sets with zero dimensional boundaries forms a base for  $X$ . In a weird space, these boundaries must be scattered, and hence countable if the space is HL.

**Theorem 6.1** *Assuming  $\diamond$ , there is a weird space  $X$  such that  $X$  is HS and HL and  $\text{ind}(X) = 1$ .*

We shall prove this by refining the inductive construction of Subsection 2.1. Let  $\mathcal{B}(X)$  be the family of all open subsets of  $X$  with countable boundaries. Note that  $\partial(U \cup V) \subseteq \partial U \cup \partial V$  and  $\partial(U \cap V) \subseteq \partial U \cup \partial V$ , so  $\mathcal{B}(X)$  is closed under finite unions and intersections. We shall make sure inductively that each  $\mathcal{B}(X_\alpha)$  is a base for  $X_\alpha$ . To guarantee that there are sufficiently many sets in  $\mathcal{B}(X_{\alpha+1})$ , supplement requirement (6) by:

6<sup>+</sup>.  $\{x \in X_\alpha \setminus \{p_\alpha\} : h_\alpha(x) = a\}$  is countable for each  $a \in \mathbb{Q}$ .

To ensure that boundaries of basic open sets stay countable at successive stages of the inverse limit construction, supplement requirement (3) by:

3<sup>+</sup>. For  $1 \leq \alpha < \omega_1$ ,  $X_\alpha$  has a countable base  $\mathcal{E}_\alpha \subseteq \mathcal{B}(X_\alpha)$  such that for all  $U \in \mathcal{E}_\alpha$ ,  $\partial U \in \mathcal{P}_\alpha$  and  $\partial U \subseteq \mathcal{P}_\alpha$ .

First, to meet requirements (6) and (6<sup>+</sup>) along with requirement (7), we need two lemmas on Urysohn functions:

**Lemma 6.2** *Suppose that  $Y$  is a compact metric space such that  $\mathcal{B}(Y)$  is a base for  $Y$ . Let  $H, K$  be disjoint closed subsets of  $Y$ . Fix reals  $r < s$  and a countable dense  $D \subseteq (r, s)$ . Then there is an  $h \in C(Y, [r, s])$  such that  $H \subseteq h^{-1}(\{r\})$ ,  $K \subseteq h^{-1}(\{s\})$ , and  $h^{-1}(\{a\})$  is countable for every  $a \in D$ .*

**Proof.** Following the usual Urysohn construction, we choose  $U_a \in \mathcal{B}(Y)$  for  $a \in D \cup \{r, s\}$ , so that  $a < b \rightarrow \overline{U_a} \subseteq U_b$ . Start with  $U_r, U_s$  satisfying  $H \subseteq U_r$  and  $K \subseteq Y \setminus U_s$ . List  $D$  in type  $\omega$ , and choose the other  $U_a$  inductively. Each  $U_a$  may be chosen in  $\mathcal{B}(Y)$  because  $Y$  is compact and the base  $\mathcal{B}(Y)$  is closed under finite

unions. Define the Urysohn function  $h : Y \rightarrow [r, s]$  by  $h(x) = \inf\{a : x \in U_a\}$  for  $x \in U_s$ , and  $h(x) = s$  for  $x \in Y \setminus U_s$ .

To ensure that the  $h^{-1}(\{a\})$  are countable, construct the  $U_a$  so that for each  $a \in D$  :

$$U_a = \bigcup \{U_b : b \in D \cap (r, a)\} \quad \text{and} \quad \overline{U_a} = \bigcap \{U_b : b \in D \cap (a, s)\}. \quad (*)$$

This can easily be done by the usual bookkeeping, using the fact that each  $U_a$  is an  $F_\sigma$  and each  $\overline{U_a}$  is a  $G_\delta$ . Now, suppose  $h(x) = a \in D$ . Then  $x \in \bigcap_{b>a} U_b$  (by the definition of  $h$ ), so  $x \in \overline{U_a}$  by the second half of (\*). But also  $x \notin U_a$  by the first part of (\*). Therefore,  $h^{-1}(\{a\}) \subseteq \partial U_a$ , which is countable.  $\odot$

**Lemma 6.3** *Suppose that  $X$  is a compact metric space such that  $\mathcal{B}(X)$  is a base for  $X$ . Fix  $p \in X$  and  $q_n \in X \setminus \{p\}$  with  $\langle q_n : n \in \omega \rangle \rightarrow p$ . Then there is an  $h \in C(X \setminus \{p\})$  such that:*

1. all points of  $[0, 1]$  are limit points of  $\{h(q_n) : n \in \omega\}$ , and
2.  $h^{-1}(a)$  is countable for every  $a \in \mathbb{Q}$ .

**Proof.** Replacing  $\langle q_n : n \in \omega \rangle$  by a subsequence, we may assume that we have a local base  $\{V_n : n \in \omega\} \subseteq \mathcal{B}$  at  $p$  with  $V_0 = X$  and  $\overline{V_{n+1}} \subseteq V_n$  for each  $n \in \omega$ , and each  $q_n \in V_{2n} \setminus \overline{V_{2n+1}}$ . To get (1): let  $\{r_n : n \in \omega\}$  list a set of irrationals dense in  $[0, 1]$ , and for each  $n \in \omega$  define  $h \equiv r_n$  on the closed set  $\overline{V_{2n}} \setminus V_{2n+1}$ . To get (2): apply Lemma 6.2 to interpolate  $h$  between each pair  $\overline{V_{2n}} \setminus V_{2n+1}$  and  $\overline{V_{2n+2}} \setminus V_{2n+3}$ .  $\odot$

Applying Lemma 6.3, we construct  $X$  so that  $\mathcal{B}(X)$  is a base for  $X$ .

**Proof of Theorem 6.1.** We follow the inductive construction of Subsection 2.1, verifying at each stage  $\beta \leq \omega_1$  that  $\mathcal{B}(X_\beta)$  is a base for  $X_\beta$ .

Fix  $\beta \leq \omega_1$ , and assume that for all  $\alpha < \beta$ ,  $\mathcal{B}(X_\alpha)$  is a base for  $X_\alpha$ . For limit  $\beta$ ,  $\mathcal{B}(X_\beta)$  is a base because it contains all sets of the form  $(\pi_\alpha^\beta)^{-1}(U)$  whenever  $\alpha < \beta$  and  $U \in \mathcal{E}_\alpha$ , for the  $\mathcal{E}_\alpha$  of (3<sup>+</sup>). If  $\beta = \alpha + 1$  for some  $\alpha$ , apply Lemma 6.3 to choose  $h_\alpha$  so that it meets requirements (6), (6<sup>+</sup>), and (7). Then by (6<sup>+</sup>),  $\mathcal{B}(X_{\alpha+1})$  contains all open sets of the form  $X_{\alpha+1} \cap \varphi_\alpha^{-1}((a, 1])$  and  $X_{\alpha+1} \cap \varphi_\alpha^{-1}([0, a))$  for  $a \in \mathbb{Q} \cap (0, 1)$ , where  $\varphi_\alpha$  is projection (see Definition 2.5). Note that  $\mathcal{B}(X_{\alpha+1})$  also contains sets of the form  $(\pi_\alpha^{\alpha+1})^{-1}(U)$  whenever  $U \in \mathcal{B}(X_\alpha)$  and  $p_\alpha \notin \partial U$ . Then,  $\mathcal{B}(X_{\alpha+1})$  is a base because it is closed under finite intersections. To get (3<sup>+</sup>), for all  $\beta < \omega_1$ , use the fact that  $X_\beta$  is second countable to choose a countable base  $\mathcal{E}_\beta \subseteq \mathcal{B}(X_\beta)$ .  $\odot$

We do not know whether there is a one dimensional version of our weird space that also fails the CSWP.

We also do not know if, under any set-theoretic axioms, there is a weird space satisfying the CSWP. The known methods [18, 15, 12] for establishing the CSWP focus on totally disconnected spaces, and would fail for weird spaces.

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