Locally Connected HL Compacta*

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Abstract

It is consistent with $\mathrm{MA} + \neg\mathrm{CH}$ that there is a locally connected hereditarily Lindelöf compact space which is not metrizable.

1 Introduction

All spaces discussed in this paper are assumed to be Hausdorff. A question attributed in 1982 by Nyikos [8] to M. E. Rudin asks whether $\mathrm{MA}+\neg\mathrm{CH}$ implies that every locally connected hereditarily Lindelöf (HL) compact space is metrizable (equivalently, second countable); see Gruenhage [5] for further discussion. Filippov [4] had constructed such a space in 1969 under CH, and his space is also hereditarily separable (HS). Since Filippov used a Luzin set in his construction, and $\mathrm{MA} + \neg\mathrm{CH}$ implies that there are no Luzin sets, it might have been hoped that $\mathrm{MA} + \neg\mathrm{CH}$ refutes the existence of such a space, but that turns out to be false; we shall show in Section 3:

Theorem 1.1 *It is consistent with* $\mathrm{MA}+2^{\aleph_0} = \aleph_2$ *that there is a non-metrizable locally connected compactum which is both HS and HL.*

Our proof shows in ZFC that the Filippov construction succeeds provided that there is a *weakly Luzin set*; details are in Section 2. Weakly Luzin sets are related to entangled sets, and our proof of Theorem 1.1 shows that weakly Luzin sets are consistent with $\mathrm{MA} + 2^{\aleph_0} = \aleph_2$. We can show that PFA refutes spaces which are "like" the Filippov space (see Theorem 4.3), but we do not know whether PFA refutes all non-metrizable locally connected HL compacta.

The Filippov space may be viewed as a connected version of the double arrow space D, which was described in 1929 by Alexandroff and Urysohn [2]. This is a ZFC

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example of a non-metrizable compactum which is both HS and HL, but it is totally disconnected. The cone over D yields a connected example, but this is not locally connected.

D is constructed from $[0,1]$ by replacing the points of $(0,1)$ by neighboring pairs of points. To construct the Filippov space, start with $[0,1]^2$, choose a set $E \subseteq (0,1)^2$, and replace the points of E by circles, obtaining a space Φ_E. This Φ_E is compact and locally connected. Φ_E is metrizable iff E is countable. Furthermore, if E is a Luzin set, then, as Filippov showed, Φ_E is HL, and a similar proof shows that Φ_E is HS as well.

Actually, by Juhász [7] and Szentmiklóssy [9], HS and HL are equivalent for compacta under $\mathrm{MA}(\aleph_1)$, but that result is not needed here. We shall show in ZFC (Theorem 2.5) that Φ_E is HS iff Φ_E is HL iff E is weakly Luzin.

2 Weakly Luzin Sets

We begin by describing Filippov's example [4]. We start with $[0,1]^n$ (where $1 \le n < \omega$), rather than $[0,1]^2$, to show that the construction does not depend on accidental features of two-dimensional geometry. As usual, $S^{n-1} \subset \mathbb{R}^n$ denotes the unit sphere, and $\|x\|$ denotes the length of $x \in \mathbb{R}^n$, using the standard Pythagorean metric. Given $E \subseteq (0,1)^n$, we shall obtain the space Φ_E by replacing all points in E by $(n-1)$–spheres and leaving the points in $[0,1]^n \backslash E$ alone.

Definition 2.1 *$\rho : \mathbb{R}^n \backslash \{0\} \twoheadrightarrow S^{n-1}$ is the perpendicular retraction: $\rho(x) = x/\|x\|$.*

So, $\rho(y-x)$ may be viewed as the direction from x to y.

Definition 2.2 *Fix $E \subseteq (0,1)^n$ and let $E' = [0,1]^n \backslash E$. The* Filippov space *$\Phi_E$, as a set, is $(E \times S^{n-1}) \cup E'$. Define $\pi = \pi_E : \Phi_E \twoheadrightarrow [0,1]^n$ so that $\pi(x,w) = x$ for $(x,w) \in E \times S^{n-1}$, and $\pi(x) = x$ for $x \in E'$. For $\varepsilon > 0$, define, for $x \in E'$:*

$$B(x,\varepsilon) = \{p \in \Phi_E : \|\pi(p) - x\| < \varepsilon\} \quad ,$$

and define, for $x \in E$ and W an open subset of S^{n-1}:

$$B(x,W,\varepsilon) = \{x\} \times W \ \cup \ \{p \in \Phi_E : 0 < \|\pi(p) - x\| < \varepsilon \ \& \ \rho(\pi(p) - x) \in W\} \quad .$$

Give Φ_E the topology which has all the sets $B(x,\varepsilon)$ and $B(x,W,\varepsilon)$ as a base.

Lemma 2.3 *For each $E \subseteq (0,1)^n$: Φ_E is compact and first countable. π_E is a continuous irreducible map from Φ_E onto $[0,1]^n$. Φ_E is metrizable iff E is countable. If $n \ge 2$, then Φ_E is connected and locally connected, and π_E is monotone.*

The proof of this last sentence uses the connectedness of S^{n-1}. When $n = 1$, $S^0 = \{\pm 1\}$, and Φ_E is just the double arrow space obtained by doubling the points of E, so Φ_E is always HS and HL. When $n > 1$, the argument of Filippov shows that Φ_E is HL if E is a Luzin set, but actually something weaker than Luzin suffices:

Definition 2.4 *For $1 \le n < \omega$:*

- ☛ *If $T \subseteq \mathbb{R}^n$, then $T^* = \{x - y : x, y \in T \ \&\ x \neq y\}$*
- ☛ *$T \subseteq \mathbb{R}^n$ is* skinny *iff* $\mathrm{cl}(\rho(T^*)) \neq S^{n-1}$.
- ☛ *$E \subseteq \mathbb{R}^n$ is a* weakly Luzin *set iff E is uncountable and every skinny subset of E is countable.*

Every subset of a skinny set is skinny, and T is skinny iff $\overline{T}$ is skinny. Each skinny set is nowhere dense, so every Luzin set is weakly Luzin. When $n = 1$, T is skinny iff $|T| \le 1$, every uncountable set is weakly Luzin, and the proof of the following theorem reduces to the usual proof that the double arrow space is HS and HL.

When $n > 1$: Under CH, it is easy to construct a weakly Luzin set which is not Luzin (see Example 4.1). PFA implies that there are no weakly Luzin sets. We shall show in Section 3 that a weakly Luzin set is consistent with MA + $\mathfrak{c} = \aleph_2$. Clearly, if there is a weakly Luzin set in $\mathbb{R}^n$, then there is one in $(0,1)^n$.

Theorem 2.5 *For $n \ge 1$ and uncountable $E \subseteq (0,1)^n$, the following are equivalent:*

1. *E is weakly Luzin.*
2. *Φ_E is HS.*
3. *Φ_E is HL.*
4. *Φ_E has no uncountable discrete subsets.*

Proof. For (4) → (1): If E is not weakly Luzin, fix an uncountable skinny $T \subseteq E$. Let $W = S^{n-1} \backslash \mathrm{cl}(\rho(T^*))$, and fix $w \in W$. Then $\{(x, w) : x \in T\} \subset \Phi_E$ is discrete.

Since (2) → (4) and (3) → (4) are obvious, it is sufficient to prove (1) → (2) and (1) → (3). So, assume (1), and let $\langle p_\alpha : \alpha < \omega_1 \rangle$ be an ω_1–sequence of distinct points from Φ_E; we show that it is neither left separated nor right separated. To do this, fix an open neighborhood N_α of p_α for each α; we find $\alpha < \beta < \gamma$ such that $p_\beta \in N_\alpha$ and $p_\beta \in N_\gamma$. This is trivial if $\aleph_1$ of the $\pi(p_\alpha)$ lie in E', or if $\aleph_1$ of the $\pi(p_\alpha)$ are the same point of E. So, thinning the sequence (discarding some points), and shrinking the neighborhoods (replacing them by smaller ones), we may assume that each $p_\alpha = (x_\alpha, w_\alpha) \in E \times S^{n-1}$ and that $N_\alpha = B(x_\alpha, W, \varepsilon)$, where the x_α are distinct points in E, W is open in S^{n-1}, and each $w_\alpha \in W$. Let $T = \{x_\alpha : \alpha < \omega_1\}$. Thinning further, we may assume that $\mathrm{diam}(T) < \varepsilon$, so that $p_\beta \in N_\alpha$ iff $\rho(x_\beta - x_\alpha) \in W$. Thinning again, we may assume that every point of T is a condensation point of T. Since E is weakly Luzin, T cannot be skinny, so $\rho(T^*)$ is dense in S^{n-1}, so fix $\xi \neq \eta$ such that $\rho(x_\eta - x_\xi) \in W$. There are then open $U \ni x_\xi$ and $V \ni x_\eta$ such that $\rho(z-y) \in W$ for all $y \in U$ and $z \in V$. Since $|U \cap T| = |V \cap T| = \aleph_1$, we may fix $\alpha < \beta < \gamma$ with $x_\alpha, x_\gamma \in U$ and $x_\beta \in V$; then $\rho(x_\beta - x_\alpha) \in W$ and $\rho(x_\beta - x_\gamma) \in W$, so $p_\beta \in N_\alpha$ and $p_\beta \in N_\gamma$. ☕

Entangled subsets of $\mathbb{R}$ were discussed by Avraham and Shelah [3] (see also [1]). The weakly Luzin sets and the entangled sets have a common generalization:

Definition 2.6 *For $1 \le n < \omega$ and $1 \le k < \omega$:*

1. *If $E \subseteq \mathbb{R}^n$, then $\widetilde{E} \subseteq (\mathbb{R}^n)^k$ is* derived *from E iff $\widetilde{E} \subseteq E^k$ and whenever $\vec{x} = \langle x_0, \dots x_{k-1}\rangle \in \widetilde{E}$ and $\vec{y} = \langle y_0, \dots y_{k-1}\rangle \in \widetilde{E}$: $x_i \ne y_j$ unless $i = j$ and $\vec{x} = \vec{y}$.*
2. *E is (n,k)–*entangled *iff $E \subseteq \mathbb{R}^n$ is uncountable and whenever $\widetilde{E} \subseteq (\mathbb{R}^n)^k$ is uncountable and derived from E, and, for $i < k$, W_i is open in S^{n-1} with $W_i \ne \emptyset$: there exist $\vec{x}, \vec{y} \in \widetilde{E}$ with $\vec{x} \ne \vec{y}$ and $\rho(x_i - y_i) \in W_i$ for all i.*

Then "weakly Luzin" is equivalent to "$(n,1)$–entangled", and "k–entangled" is equivalent to "$(1,k)$–entangled". $E \subseteq \mathbb{R}$ is $(1,1)$–entangled iff E is uncountable. If E is (n,k)–entangled and $\widetilde{E}$ and the W_i are as in (2), then there are actually uncountable disjoint $X, Y \subseteq \widetilde{E}$ such that $\forall i\, \rho(x_i - y_i) \in W_i$ whenever $\vec{x} \in X$ and $\vec{y} \in Y$. In (2), when $k = 1$, WLOG we may assume that $W_0 = -W_0$.

3 Preserving Failures of SOCA

The *Semi Open Coloring Axiom* (SOCA) is a well-known consequence of the PFA; see Abraham, Rubin, and Shelah [1]. We shall show that certain classes of failures of SOCA can be preserved in ccc extensions satisfying $\mathrm{MA} + 2^{\aleph_0} = \aleph_2$. This is patterned after the proof (see [1, 3]) that an entangled set is consistent with $\mathrm{MA} + 2^{\aleph_0} = \aleph_2$.

Definition 3.1 *For any set E: Let $E^\dagger = (E \times E) \setminus \{(x,x) : x \in E\}$. Fix $W \subseteq E^\dagger$ with $W = W^{-1}$. Then $T \subseteq E$ is W–*free *iff $T^\dagger \cap W = \emptyset$ and T is W–*connected *iff $T^\dagger \subseteq W$.*

Definition 3.2 *(E, W) is* good *iff E is an uncountable separable metric space, $W = W^{-1}$ is an open subset of $E^\dagger$, and no uncountable subset of E is W–free.*

Then, the SOCA is the assertion that whenever (E, W) is good, there is an uncountable W–connected set. An uncountable $E \subseteq \mathbb{R}^n$ is weakly Luzin iff (E, W) is good for all W of the form $\{(x,y) \in E^\dagger : \rho(x-y) \in A\}$, where $A \subseteq S^{n-1}$ is open and $A = -A \ne \emptyset$. We shall prove:

Theorem 3.3 *Assume that in the ground model $\mathbf{V}$, $\mathrm{CH} + 2^{\aleph_1} = \aleph_2$ holds and E is a separable metric space. Then there is a ccc extension $\mathbf{V}[G]$ satisfying $\mathrm{MA} + 2^{\aleph_0} = \aleph_2$ such that for all $W \in \mathbf{V}$, if (E, W) is good in $\mathbf{V}$ then (E, W) is good in $\mathbf{V}[G]$.*

A good (E, W) does not by itself contradict SOCA, since there may be an uncountable subset of E which is W–connected. But, if (E, U) and (E, W) are both good and $U \cap W = \emptyset$, then SOCA is contradicted, since every W–connected set is U–free. Such E, U, W are provided by a weakly Luzin $E \subseteq \mathbb{R}^n$ (for $n \geq 2$). The following combinatorial lemma will be used in the proof of Theorem 3.3.

Lemma 3.4 *Assume the following:*

1. *CH holds.*
2. *$m \in \omega$; and (E, W_i) is good for each $i \leq m$.*
3. *θ is a suitably large regular cardinal and $\langle M_\xi : \xi < \omega_1 \rangle$ is a continuous chain of countable elementary submodels of $H(\theta)$, with $E \in M_0$ and each $M_\xi \in M_{\xi+1}$.*
4. *For $x \in \bigcup_\xi M_\xi \setminus M_0$: $\mathrm{ht}(x)$ is the ξ such that $x \in M_{\xi+1} \backslash M_\xi$.*
5. *$x^i_\alpha \in E \backslash M_0$ for $\alpha < \omega_1$ and $i \leq m$.*
6. *$\mathrm{ht}(x^i_\alpha) \neq \mathrm{ht}(x^j_\beta)$ unless $\alpha = \beta$ and $i = j$.*

Then there are $\alpha \neq \beta$ such that $(x^i_\alpha, x^i_\beta) \in W_i$ for all i.

We remark that (6) expresses the standard trick of using a set of points spaced by a chain of elementary submodels. In (5), we say $x^i_\alpha \in E \backslash M_0$ so that $\mathrm{ht}(x^i_\alpha)$ is defined; note that by CH, $E \subset \bigcup_\xi M_\xi$.

Proof. Induct on m. When $m = 0$, this is immediate from the fact that (E, W_0) is good. Now, assume the lemma for $m-1$, and we prove it for m. Let $\vec{x}_\alpha = \langle x^0_\alpha, \ldots, x^m_\alpha \rangle \in E^{m+1}$. Let $\xi(\alpha, i) = \mathrm{ht}(x^i_\alpha)$. Thinning the ω_1–sequence and rearranging each $\vec{x}_\alpha$ if necessary, we may assume that $\xi(\alpha, 0) < \xi(\alpha, 1) < \cdots < \xi(\alpha, m)$ and that $\alpha < \beta \rightarrow \xi(\alpha, m) < \xi(\beta, 0)$. Let $F = \mathrm{cl}\{\vec{x}_\alpha : \alpha < \omega_1\} \subseteq E^{m+1}$, and fix $\mu < \omega_1$ such that $F \in M_\mu$; there is such a μ by CH.

For $\alpha \geq \mu$: Let $K_\alpha = \{z \in E : \langle x^0_\alpha, \ldots, x^{m-1}_\alpha, z \rangle \in F\}$. K_α is uncountable because $K_\alpha \in M_{\xi(\alpha,m)}$ but K_α contains the element $x^m_\alpha \notin M_{\xi(\alpha,m)}$. Since (E, W_m) is good, choose $u_\alpha, v_\alpha \in K_\alpha$ with $(u_\alpha, v_\alpha) \in W_m$, and then choose disjoint basic open sets $U_m, V_m \subseteq E$ with $u_\alpha \in U_m$, $v_\alpha \in V_m$, and $(x, y) \in W_m$ for all $x \in U_m$ and $y \in V_m$.

Of course, U_m, V_m depend on α, but there are only $\aleph_0$ possible choices, so fix an uncountable set $I \subseteq \{\alpha : \mu \leq \alpha < \omega_1\}$ such that the U_m, V_m are the same for $\alpha \in I$. By the lemma for $m-1$, fix $\gamma, \delta \in I$ such that $\gamma \neq \delta$ and $(x^i_\gamma, x^i_\delta) \in W_i$ for all $i < m$. Now choose disjoint open neighborhoods U_i of x^i_γ and V_i of x^i_δ for $i < m$ so that $(x, y) \in W_i$ whenever $x \in U_i$ and $y \in V_i$. Note that the two open sets $\prod_{i \leq m} U_i$ and $\prod_{i \leq m} V_i$ both meet F, since $u_\gamma \in K_\gamma$ and $v_\delta \in K_\delta$, so $\langle x^0_\gamma, \ldots, x^{m-1}_\gamma, u_\gamma \rangle \in F \cap \prod_{i \leq m} U_i$ and $\langle x^0_\delta, \ldots, x^{m-1}_\delta, v_\delta \rangle \in F \cap \prod_{i \leq m} V_i$. We may then choose α, β such that $\vec{x}_\alpha \in \prod_{i \leq m} U_i$ and $\vec{x}_\beta \in \prod_{i \leq m} V_i$. But then $(x^i_\alpha, x^i_\beta) \in W_i$ for all i. ☕

Lemma 3.5 *In the ground model* $\mathbf{V}$*: Assume CH, let* (E, W) *be good, and let* $\mathbb{Q}$ *be any forcing poset such that* $q \Vdash_{\mathbb{Q}}$ *"*(E, W) *is not good" for some* $q \in \mathbb{Q}$*.*

Then, in $\mathbf{V}$*: there is a ccc poset* $\mathbb{P}$ *of size* $\aleph_1$ *such that* $\mathbb{Q} \times \mathbb{P}$ *is not ccc and such that for* all $U \in \mathbf{V}$*: If* (E, U) *is good then* $\mathbb{1} \Vdash_{\mathbb{P}}$ *"*(E, U) *is good".*

Proof. Fix a $\mathbb{Q}$–name $\mathring{Z}$ such that $q \Vdash$ "$\mathring{Z} \subseteq E$ is uncountable and W–free". Fix θ and the M_ξ so that (3) of Lemma 3.4 hold; then (4) defines $\mathrm{ht}(x)$.

Now, inductively choose $q_\alpha \le q$ and $x^0_\alpha, x^1_\alpha \in E \backslash M_0$ for $\alpha < \omega_1$ so that $q_\alpha \Vdash x^0_\alpha, x^1_\alpha \in \mathring{Z}$ and such that $\mathrm{ht}(x^0_\alpha) < \mathrm{ht}(x^1_\alpha) < \mathrm{ht}(x^0_\beta)$ whenever $\alpha < \beta < \omega_1$. Let

$$\mathbb{P} = \left\{ p \in [\omega_1]^{<\omega} : \forall \{\alpha, \beta\} \in [p]^2 \ \left[(x^0_\alpha, x^0_\beta) \in W \text{ or } (x^1_\alpha, x^1_\beta) \in W \right] \right\} \quad .$$

$\mathbb{P}$ is ordered by reverse inclusion, with $\mathbb{1} = \emptyset$. Each $\{\alpha\} \in \mathbb{P}$, and the pairs $(q_\alpha, \{\alpha\}) \in \mathbb{Q} \times \mathbb{P}$ are incompatible, so $\mathbb{Q} \times \mathbb{P}$ is not ccc.

Now, suppose that we have some good (E, U) and $p \Vdash_{\mathbb{P}}$ "(E, U) is not good"; we shall derive a contradiction. Fix a $\mathbb{P}$–name $\mathring{T}$ such that $p \Vdash$ "$\mathring{T} \subseteq E$ is uncountable and U–free". Then, inductively choose $p_\mu \le p$ and $t_\mu \in E \backslash M_0$ for $\mu < \omega_1$ so that $p_\mu \Vdash t_\mu \in \mathring{T}$ and such that $\mathrm{ht}(t_\mu) < \mathrm{ht}(t_\nu)$ whenever $\mu < \nu < \omega_1$. Our contradiction will use the observation:

$$\mu \ne \nu \to (t_\mu, t_\nu) \notin U \text{ or } p_\mu \perp p_\nu \quad . \qquad (*)$$

Thinning the sequence and extending p if necessary, we may assume that the p_μ form a Δ system with root p; so $p_\mu = p \cup \{\alpha(0, \mu), \ldots, \alpha(c, \mu)\}$, with $\alpha(0, \mu) < \ldots < \alpha(c, \mu)$. We also assume that $\max(p) < \alpha(0, 0)$ and $\mu < \nu \to \alpha(c, \mu) < \alpha(0, \nu)$. Since $p_\mu \in \mathbb{P}$,

$$i \ne j \to (x^0_{\alpha(i,\mu)}, x^0_{\alpha(j,\mu)}) \in W \text{ or } (x^1_{\alpha(i,\mu)}, x^1_{\alpha(j,\mu)}) \in W$$

for each μ. Let $\vec{x}_\mu = (x^0_{\alpha(0,\mu)}, x^1_{\alpha(0,\mu)} \cdots x^0_{\alpha(c,\mu)}, x^1_{\alpha(c,\mu)}) \in E^{2(c+1)}$. Since W is open, we may thin again and assume that all $\vec{x}_\mu$ are sufficiently close to some condensation point of $\{\vec{x}_\mu : \mu < \omega_1\}$ so that for all μ, ν:

$$i \ne j \to (x^0_{\alpha(i,\mu)}, x^0_{\alpha(j,\nu)}) \in W \text{ or } (x^1_{\alpha(i,\mu)}, x^1_{\alpha(j,\nu)}) \in W \quad .$$

Thus, if $p_\mu \perp p_\nu$ then the incompatibility must come from the same index i, so that $(*)$ becomes

$$\mu \ne \nu \to (t_\mu, t_\nu) \notin U \text{ or } \exists i \le c \ \left[(x^0_{\alpha(i,\mu)}, x^0_{\alpha(i,\nu)}) \notin W \text{ and } (x^1_{\alpha(i,\mu)}, x^1_{\alpha(i,\nu)}) \notin W \right] \quad .$$

This comes close to contradicting Lemma 3.4. With an eye to satisfying hypothesis (6), we thin the sequence again and assume that $\mathrm{ht}(t_\mu) \ne \mathrm{ht}(x^\ell_{\alpha(i,\nu)})$ whenever $\mu \ne \nu$. It is still possible to have $\mathrm{ht}(t_\mu) = \mathrm{ht}(x^\ell_{\alpha(i,\mu)})$, but for each μ, $\mathrm{ht}(t_\mu) = \mathrm{ht}(x^\ell_{\alpha(i,\mu)})$ can hold for at most one pair (ℓ, i). Thinning once more, we can assume WLOG that this ℓ is always 1, so that $\mathrm{ht}(t_\mu) \ne \mathrm{ht}(x^0_{\alpha(i,\nu)})$ for all $\mu < \omega_1$ and all $i \le c$. But now the

$(c+2)$–tuples $(t_\mu, x^0_{\alpha(0,\mu)}, \ldots x^0_{\alpha(c,\mu)})$ (for $\mu < \omega_1$) contradict Lemma 3.4, where $W_0 = U$ and the other $W_i = W$.

We also need to show that $\mathbb{P}$ is ccc. If this fails, then choose the p_μ to enumerate an antichain. Derive a contradiction as before, but replace $(*)$ by the stronger fact $\mu \neq \nu \to p_\mu \perp p_\nu$, and delete all mention of $\mathring{T}$ and the t_μ. ☕

We remark that a simplification of the above proof yields the standard proof that an instance of SOCA can be forced by a ccc poset. Forget about $\mathbb{Q}$ and just assume that (E, W) is good. Choose the $x_\alpha \in E \backslash M_0$ for $\alpha < \omega_1$ so that $\mathrm{ht}(x_\alpha) < \mathrm{ht}(x_\beta)$ whenever $\alpha < \beta < \omega_1$. $\mathbb{P}$ is now $\{p \in [\omega_1]^{<\omega} : \forall\{\alpha, \beta\} \in [p]^2 \, [(x_\alpha, x_\beta) \in W]\}$. Then some $p \in \mathbb{P}$ forces an uncountable W–connected set.

Proof of Theorem 3.3. In the ground model $\mathbf{V}$, we build a normal chain of ccc posets, $\langle \mathbb{F}_\alpha : \alpha \leq \omega_2 \rangle$, where $\alpha < \beta \to \mathbb{F}_\alpha \subseteq_c \mathbb{F}_\beta$ and we take unions at limits. So, our model will be $\mathbf{V}[G]$, where G is $\mathbb{F}_{\omega_2}$–generic. $|\mathbb{F}_\alpha| \leq \aleph_1$ for all $\alpha < \omega_2$, while $|\mathbb{F}_{\omega_2}| = \aleph_2$. Given $\mathbb{F}_\alpha$, we choose $\mathring{\mathbb{P}}_\alpha$, which is an $\mathbb{F}_\alpha$–name forced by $\mathbb{1}$ to be a ccc poset of size $\aleph_1$; then $\mathbb{F}_{\alpha+1} = \mathbb{F}_\alpha * \mathring{\mathbb{P}}_\alpha$.

The standard bookkeeping which is used to guarantee that $\mathbf{V}[G] \models \mathrm{MA} + 2^{\aleph_0} = \aleph_2$ is modified slightly here, since we need to assume inductively that $\mathbb{1} \Vdash_{\mathbb{F}_\alpha}$ "(E, W) is good" for all W such that (E, W) is good in $\mathbf{V}$. This is easily seen (similarly to Theorem 49 of [6]) to be preserved at limit α. For the successor stage, assume that we have $\mathbb{F}_\alpha$ and the standard bookkeeping says that we should use $\mathring{\mathbb{Q}}_\alpha$, which is an $\mathbb{F}_\alpha$–name which is forced by $\mathbb{1}$ to be a ccc poset of size $\aleph_1$. Roughly, we ensure that either MA holds for $\mathring{\mathbb{Q}}_\alpha$ or $\mathring{\mathbb{Q}}_\alpha$ ceases to be ccc. More formally, choose $\mathring{\mathbb{P}}_\alpha$ as follows:

Consider this from the point of view of the $\mathbb{F}_\alpha$–extension $\mathbf{V}[G \cap \mathbb{F}_\alpha]$. In this model, CH holds, and we have a ccc poset $\mathbb{Q}_\alpha$, and we must define another ccc poset $\mathbb{P}_\alpha$. We know (using our inductive assumption) that for all $W \in \mathbf{V}$, if (E, W) was good in $\mathbf{V}$ then it is still good. If for all such W, $\mathbb{1} \Vdash_{\mathbb{Q}_\alpha}$ "(E, W) is good", then let $\mathbb{P}_\alpha = \mathbb{Q}_\alpha$. If not, then fix $W \in \mathbf{V}$ with (E, W) good in $\mathbf{V}$ such that $q \Vdash_{\mathbb{Q}_\alpha}$ "(E, W) is not good" for some $q \in \mathbb{Q}_\alpha$. Still working in $\mathbf{V}[G \cap \mathbb{F}_\alpha]$, we apply Lemma 3.5 and let $\mathbb{P}$ be a ccc poset of size $\aleph_1$ such that $\mathbb{Q}_\alpha \times \mathbb{P}$ is not ccc and such that for all $U \in \mathbf{V}[G \cap \mathbb{F}_\alpha]$ (and hence for all $U \in \mathbf{V}$): If (E, U) is good then $\mathbb{1} \Vdash_{\mathbb{P}}$ "(E, U) is good". Since $\mathbb{Q}_\alpha \times \mathbb{P}$ is not ccc, we may fix $p_0 \in \mathbb{P}$ such that $p_0 \Vdash_{\mathbb{P}}$ "$\mathbb{Q}_\alpha$ is not ccc". Let $\mathbb{P}_\alpha = p_0\!\downarrow = \{p \in \mathbb{P} : p \leq p_0\}$. Then $\mathbb{1}_{\mathbb{P}_\alpha} = p_0$ and $\mathbb{1}_{\mathbb{P}_\alpha} \Vdash_{\mathbb{P}_\alpha}$ "$\mathbb{Q}_\alpha$ is not ccc", and all good (E, U) from $\mathbf{V}$ remain good in the $\mathbb{P}_\alpha$ extension.

Now, in $\mathbf{V}$, let $\mathring{\mathbb{P}}_\alpha$ be the name for this $\mathbb{P}_\alpha$ as chosen above. ☕

Proof of Theorem 1.1. In the ground model $\mathbf{V}$, assume that $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$. By CH, we may fix a (weakly) Luzin set $E \subseteq \mathbb{R}^n$ (where $n \geq 2$). Now, apply Theorem 3.3. ☕

4 Further Remarks

In Theorem 1.1, we may also obtain $\mathrm{MA} + 2^{\aleph_0} > \aleph_2$ in our model. To do this, start, in the ground model, with $\mathrm{MA} + 2^{\aleph_0} = \aleph_2$ plus a weakly Luzin set E. Then, by $\mathrm{MA}(\aleph_1)$, E remains weakly Luzin in all ccc extensions, and, if $\kappa = \kappa^{<\kappa} > \aleph_2$, one may take such an extension satisfying $\mathrm{MA} + 2^{\aleph_0} = \kappa$.

When $n \geq 2$, every C^1 arc A in $\mathbb{R}^n$ is a finite union of skinny sets, so A meets every weakly Luzin set in a countable set. The fact that this is not true for arcs in general provides, under CH, a class of examples of weakly Luzin sets which are not Luzin.

Example 4.1 *For $n \geq 2$, there is an arc $A \subset \mathbb{R}^n$ of finite length such that $\rho(U^*) = S^{n-1}$ whenever U is a non-empty relatively open subset of A. Then, whenever $E \subset A$ is a Luzin set in the relative topology of A, $E \subset \mathbb{R}^n$ is weakly Luzin but not Luzin in $\mathbb{R}^n$.*

It is easily seen directly that a weakly Luzin set contradicts SOCA, so that the Filippov space cannot exist under SOCA. We can prove a somewhat more general result using the following well-known consequence of SOCA: it is a weakening of CSM, and is proved equivalent to SOCA in the same way that CSM is proved equivalent to OCA (see [10]):

Lemma 4.2 *Assume SOCA. Let E be an uncountable separable metric space. Assume that F_y, for $y \in E$, is a closed subset of E. Call $T \subseteq E$* connected *with respect to the mapping $y \mapsto F_y$ iff for all $\{y, z\} \in [T]^2$, either $y \in F_z$ or $z \in F_y$. Call T* free *iff for all $\{y, z\} \in [T]^2$, both $y \notin F_z$ and $z \notin F_y$. Then there is an uncountable $T \subseteq E$ such that T is either connected or free.*

Theorem 4.3 *Assume SOCA. Let X be compact, with a continuous map $\pi : X \twoheadrightarrow Y$, where Y is compact metric. Assume further that there is an uncountable $E \subseteq Y$ such that for $y \in E$, there are three points $x_y^i \in \pi^{-1}\{y\}$ for $i = 0, 1, 2$ and disjoint open neighborhoods U_y^i of x_y^i such that $\pi(U_y^i) \cap \pi(U_y^j) = \{y\}$ whenever $i \neq j$.*

Then X has an uncountable discrete subset.

Note that the double arrow space satisfies these hypotheses with "three" weakened to "two", while the Filippov space satisfies these hypotheses with "three" strengthened to "omega".

Proof. Let $F_y^i = \mathrm{cl}(\pi(U_y^i))$, which is a closed set in Y containing y. Shrinking the U_y^i, we may assume that the three sets $F_y^i \backslash \{y\}$ are pairwise disjoint.

Applying Lemma 4.2 three times, we get an uncountable $T \subseteq E$ such that for each i, T is either connected or free with respect to the mapping $y \mapsto F_y^i$. By the disjointness of the $F_y^i \backslash \{y\}$, T can be connected with respect to at most two of these mappings. Fixing i such that T is free with respect to $y \mapsto F_y^i$, we see that $\{x_y^i : y \in T\}$ is discrete. ☕

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5 Addendum

This is not intended to be part of the published version of this paper, but we verify here the remark made in the proof of Theorem 3.3, that the inductive assumption $\mathbb{1} \Vdash_{\mathbb{F}_\alpha}$ "(E, W) is good" is preserved at limit α. This uses a "standard argument" which was also referred to in [3, 1], but we do not know of a simple exposition of it in the literature, so we append a proof here. The next theorem provides a general statement of the argument.

Definition 5.1 *A* hypergraph *is a set* Y *such that* $[\omega_1]^{\leq 1} \subseteq Y \subseteq [\omega_1]^{<\omega}$. *Then* $\mathrm{homog}(H, Y)$ *abbreviates the assertion that* $H \in [\omega_1]^{\omega_1}$ *and* $[H]^{<\omega} \subseteq Y$.

Theorem 5.2 *In* $\mathbf{V}$*, assume that we have a normal chain of ccc posets,* $\langle \mathbb{F}_\alpha : \alpha \leq \gamma \rangle$*, where* γ *is a limit ordinal and* $\alpha < \beta \rightarrow \mathbb{F}_\alpha \subseteq_c \mathbb{F}_\beta$ *and we take unions at limits. Let* Y *be a hypergraph in* $\mathbf{V}$*, and assume that* $\mathbb{1} \Vdash_{\mathbb{F}_\alpha} \neg \exists H \, \mathrm{homog}(H, Y)$ *for all* $\alpha < \gamma$*. Then* $\mathbb{1} \Vdash_{\mathbb{F}_\gamma} \neg \exists H \, \mathrm{homog}(H, Y)$*.*

In the intended aplication, we have $(E, W) \in \mathbf{V}$ and $|E| = \aleph_1$, so we identify E with ω_1. Then $Y = \{s \in [E]^{<\omega} : s^\dagger \cap W = \emptyset\}$.

The theorem is immediate from the following two lemmas. In forcing, we use $\not\perp(p_1, \ldots, p_n)$ for the assertion that there is some q such that $q \leq p_\ell$ for each ℓ.

Lemma 5.3 *For any ccc* $\mathbb{P}$ *and hypergraph* Y*, the following are equivalent:*

a. *For some* $p \in \mathbb{P}$*:* $p \Vdash \exists H \, \mathrm{homog}(H, Y)$*.*
b. *For some* $J \in [\omega_1]^{\omega_1}$*, there are* $p_\mu \in \mathbb{P}$ *for* $\mu \in J$ *such that for all* n *and all* $\mu_1, \ldots, \mu_n \in J$*:*

$$\not\perp(p_{\mu_1}, \ldots, p_{\mu_n}) \rightarrow \{\mu_1, \ldots, \mu_n\} \in Y \quad .$$

Proof. For $(a) \rightarrow (b)$, say $p \Vdash \mathrm{homog}(\mathring{H}, Y)$. Let $J = \{\mu : \exists q \leq p \, [q \Vdash \mu \in \mathring{H}]\}$. Then, for $\mu \in J$, choose $p_\mu \leq p$ such that $p_\mu \Vdash \mu \in \mathring{H}$.

For $(b) \rightarrow (a)$, we use the name $\mathring{H} = \{\langle \check{\mu}, p_\mu \rangle : \mu \in J\}$. Then $\mathbb{1} \Vdash [\mathring{H}]^{<\omega} \subseteq Y$. By the ccc, some p forces $\mathring{H}$ to be uncountable. ☕

The next lemma also shows that properties such as the ccc, precaliber ω_1, and Knaster's property K, are preserved by unions of normal chains.

Lemma 5.4 *Assume that we have a normal chain of ccc posets,* $\langle \mathbb{F}_\alpha : \alpha \leq \gamma \rangle$*, where* γ *is a limit ordinal, and assume that we have* $p_\mu \in \mathbb{F}_\gamma$ *for* $\mu < \omega_1$*. Then for some* $\alpha < \gamma$ *and uncountable* $I \subseteq \omega_1$*, there are* $q_\mu \in \mathbb{F}_\alpha$ *for* $\mu \in I$ *such that for all* $n \geq 1$ *and all* $\mu_1, \ldots, \mu_n \in I$*:*

$$\not\perp(q_{\mu_1}, \ldots, q_{\mu_n}) \rightarrow \not\perp(p_{\mu_1}, \ldots, p_{\mu_n}) \quad .$$

Proof. WLOG, $\mathbb{F}_0 = \{\mathbb{1}\}$. For $p \in \mathbb{F}_\gamma \backslash \{\mathbb{1}\}$, let $\mathrm{ht}(p)$ be the α such that $p \in \mathbb{F}_{\alpha+1} \backslash \mathbb{F}_\alpha$; if $\alpha = \mathrm{ht}(p)$, choose $p^* \in \mathbb{F}_\alpha$ such that $\forall q \in \mathbb{F}_\alpha\, [q \perp p \to q \perp p^*]$; this p^* exists because $\mathbb{F}_\alpha \subseteq_c \mathbb{F}_{\alpha+1}$. Then $p^* \neq \mathbb{1} \to \mathrm{ht}(p^*) < \mathrm{ht}(p)$. Let $\mathrm{ht}(\mathbb{1}) = -1$.

For each μ: define $m = m_\mu \in \omega$ and p_μ^i so that $p_\mu^0 = p_\mu$, $p_\mu^m = \mathbb{1}$, and, for $i < m$, $p_\mu^i \neq \mathbb{1}$ and $p_\mu^{i+1} = (p_\mu^i)^*$.

WLOG, $\gamma = \omega_1$, since the result is trivial unless $\mathrm{cf}(\gamma) = \omega_1$, in which case we can replace the chain $\langle \mathbb{F}_\alpha : \alpha \leq \gamma \rangle$ by a cofinal ω_1–sequence. Then, applying a Δ system argument, we may choose $\alpha < \omega_1$ and uncountable I and $k \leq m < \omega$ so that:

1. $m = m_\mu$ for all $\mu \in I$.
2. All $\mathrm{ht}(p_\mu^k) < \alpha$.
3. Whenever $i, j < k$ and $\mu, \nu \in I$ and $\mu < \nu$: $\mathrm{ht}(p_\mu^i) < \mathrm{ht}(p_\nu^j)$.

Now, let $q_\mu = p_\mu^k$. To verify $\not\perp (q_{\mu_1}, \ldots, q_{\mu_n}) \to \not\perp (p_{\mu_1}, \ldots, p_{\mu_n})$, assume that we have $\mu_1 < \mu_2 < \cdots < \mu_{n-1} < \mu_n$ and $\not\perp (q_{\mu_1}, q_{\mu_2}, \ldots, q_{\mu_{n-1}}, q_{\mu_n})$, and verify successively:

$$\not\perp (p_{\mu_1}, q_{\mu_2}, \ldots, q_{\mu_{n-1}}, q_{\mu_n}), \quad \not\perp (p_{\mu_1}, p_{\mu_2}, \ldots, q_{\mu_{n-1}}, q_{\mu_n}), \quad \ldots\ldots$$
$$\not\perp (p_{\mu_1}, p_{\mu_2}, \ldots, p_{\mu_{n-1}}, q_{\mu_n}), \quad \not\perp (p_{\mu_1}, p_{\mu_2}, \ldots, p_{\mu_{n-1}}, p_{\mu_n}) \ .$$

For example, to prove $\not\perp (p_{\mu_1}, p_{\mu_2}, q_{\mu_3}, q_{\mu_4}, q_{\mu_5}) \to \not\perp (p_{\mu_1}, p_{\mu_2}, p_{\mu_3}, q_{\mu_4}, q_{\mu_5})$, let $\beta = \mathrm{ht}(p_{\mu_2}) + 1$ so that $p_{\mu_1}, p_{\mu_2}, q_{\mu_4}, q_{\mu_5} \in \mathbb{F}_\beta$. Then, using $\not\perp (p_{\mu_1}, p_{\mu_2}, q_{\mu_3}, q_{\mu_4}, q_{\mu_5})$ plus $\mathbb{F}_\beta \subseteq_c \mathbb{F}_{\omega_1}$, we may fix $x \in \mathbb{F}_\beta$ such that $x \leq p_{\mu_1}, p_{\mu_2}, q_{\mu_4}, q_{\mu_5}$ and $x \not\perp q_{\mu_3}$. But $q_{\mu_3} = p_{\mu_3}^k$ and $\mathrm{ht}(p_{\mu_3}^i) \geq \beta$ for $i < k$, so we get $x \not\perp p_{\mu_3}^i$ for $i = k-1, k-2, \ldots, 0$. For $i = 0$, this yields $x \not\perp p_{\mu_3}$, and hence $\not\perp (p_{\mu_1}, p_{\mu_2}, p_{\mu_3}, q_{\mu_4}, q_{\mu_5})$.