Matrices and Ultrafilters

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1 Introduction

In this article, we survey some recent results which produce points with interesting topological properties in the Stone space of a boolean algebra $\mathcal{B}$. Our primary focus is the case where $\mathcal{B} = \mathcal{P}(\kappa)$, a power set algebra; then the points will be in the Čech compactification, $\beta\kappa$ (where κ is discrete). However, these methods also apply to some other complete boolean algebras.

We also present some new results, and we unify all the results under the one umbrella of "hatpoints". In most cases, especially for the new results, we

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present complete proofs. However, occasionally we refer the reader to the literature when we do not see how to improve on a published proof. In addition, we assume that the reader is already familiar with the Stone Representation Theorem and the theory of Čech compactifications, although we begin by reviewing some basic notation regarding these matters.

If $\mathcal{B}$ is a boolean algebra, we use $\mathrm{st}(\mathcal{B})$ to denote its Stone space. Thus, the elements of $\mathrm{st}(\mathcal{B})$ are the ultrafilters on $\mathcal{B}$, and the clopen sets of $\mathrm{st}(\mathcal{B})$ are all of the form $N_b = \{\mathbf{x} \in \mathrm{st}(\mathcal{B}) : b \in \mathbf{x}\}$, for $b \in \mathcal{B}$. This "st" is a contravariant functor which produces an equivalence between the category of boolean algebras and the category of compact 0-dimensional Hausdorff spaces.

When we work in the category of boolean algebras, the notation $h : \mathcal{B} \to \mathcal{A}$ always implies that h is a homomorphism, and $h : \mathcal{B} \twoheadrightarrow \mathcal{A}$ means that in addition, h is onto. When $h : \mathcal{B} \to \mathcal{A}$, the dual $h^* : \mathrm{st}(\mathcal{A}) \to \mathrm{st}(\mathcal{B})$ (defined by $h^*(\mathbf{x}) = h^{-1}(\mathbf{x})$) will be continuous, and $h : \mathcal{B} \twoheadrightarrow \mathcal{A}$ implies that in addition, $h^* : \mathrm{st}(\mathcal{A}) \hookrightarrow \mathrm{st}(\mathcal{B})$ is 1-1, so that it embeds $\mathrm{st}(\mathcal{A})$ into $\mathrm{st}(\mathcal{B})$.

If $\mathcal{F}$ is a filter on $\mathcal{B}$, with $\mathcal{I}$ its dual ideal, we use both $\mathcal{B}/\mathcal{F}$ and $\mathcal{B}/\mathcal{I}$ to denote the quotient algebra. Then $h^* : \mathrm{st}(\mathcal{B}/\mathcal{F}) \hookrightarrow \mathrm{st}(\mathcal{B})$, where h is the natural surjection. We often identify $\mathrm{st}(\mathcal{B}/\mathcal{F})$ with the subspace of $\mathrm{st}(\mathcal{B})$ consisting of those ultrafilters $\mathbf{x}$ on $\mathcal{B}$ which extend $\mathcal{F}$. We use $a \leq_{\mathcal{I}} b$ or $a \leq_{\mathcal{F}} b$ to mean $a \wedge b' \in \mathcal{I}$.

This is illustrated by our view of $\beta\kappa$:

Definition 1.1 $\mathcal{FR} = \mathcal{FR}(\kappa)$ *is the* Fréchet *filter,* $\{X \subseteq \kappa : |\kappa \backslash X| < \kappa\}$.

If κ is any infinite cardinal, given the discrete topology, we have $u(\kappa) \subseteq \kappa^* \subseteq \beta\kappa$. $\beta\kappa$ is the space of ultrafilters on κ; that is, $\beta\kappa = \mathrm{st}(\mathcal{P}(\kappa))$. Then, $\kappa^* = \beta\kappa\backslash\kappa = \mathrm{st}(\mathcal{P}(\kappa/\mathit{fin}))$ is the the space of nonprincipal ultrafilters on κ, where *fin* denotes the ideal of finite sets. $u(\kappa) = \mathrm{st}(\mathcal{P}(\kappa)/{<}\kappa) = st(\mathcal{P}(\kappa)/\mathcal{FR}(\kappa))$ is the space of uniform ultrafilters; that is, $\mathbf{x} \in u(\kappa)$ iff every set in $\mathbf{x}$ has size κ. Here "$<\kappa$" denotes the ideal of sets of size less than κ. For $A, B \in \mathcal{P}(\kappa)$, $A \subseteq^* B$ usually means $A \leq_{\mathcal{FR}(\kappa)} B$.

Our methods construct points in $\mathrm{st}(\mathcal{B})$ which have properties related to "weak P-point":

Definition 1.2 *If θ is an infinite cardinal and X is a topological space:*

- ☞ $\mathbf{x} \in X$ *is a* P_θ-point *in X iff the intersection of any family of* fewer than θ *neighborhoods of* $\mathbf{x}$ *is also a neighborhood of* $\mathbf{x}$.
- ☞ $\mathbf{x} \in X$ *is a* weak P_θ-point *in X iff* $\mathbf{x}$ *is not a limit point of any subset of* $X\backslash\{\mathbf{x}\}$ *of size* less than θ.

So, a P-point is a P_{ω_1}-point, and a weak P-point is a weak P_{ω_1}-point. Every point is a P_ω-point. In any T_1 space, every P_θ-point is a weak P_θ-point.

We now summarize the results we prove here, and explain how to unify them under one framework.

One specific result is:

Theorem 1.3 *Let $\mathcal{B}$ be a complete boolean algebra and $\mathcal{I}$ an ideal on $\mathcal{B}$. Assume that $|\mathcal{B}| = 2^\kappa$ and that $\mathcal{B}$ has an antichain $\{a_\xi : \xi < \kappa\}$ such that the a_ξ are distinct and each $a_\xi \notin \mathcal{I}$. Then there is an $\mathbf{x} \in \mathrm{st}(\mathcal{B}/\mathcal{I}) \subseteq \mathrm{st}(\mathcal{B})$ such that $\mathbf{x}$ is a weak P_κ-point in $\mathrm{st}(\mathcal{B})$.*

Applying this to $\mathcal{P}(\kappa)$ and $\mathcal{P}(\kappa)/{<}\kappa$, and noting that κ can be partitioned into κ disjoint sets of size κ, we get:

Corollary 1.4 *For every infinite κ, there is an $\mathbf{x} \in u(\kappa)$ which is a weak P_κ-point in $\beta\kappa$.*

In fact, *good* ultrafilters (Definition 2.8) have this property. These were introduced by Keisler [10], who proved that they exist when $2^\kappa = \kappa^+$. A *ZFC* proof was given by Kunen [11] (see also Chang and Keisler [4], Theorem 6.1.4), and it is a folklore result that these are weak P_κ-points (Lemma 2.13). The fact that one might use good ultrafilters in arbitrary boolean algebras is due to Balcar and Franěk [3], who proved the following, obtained by setting $\mathcal{I} = \{0\}$:

Corollary 1.5 *If $\mathcal{B}$ is a complete boolean algebra such that $|\mathcal{B}| = 2^\kappa$ and $\mathcal{B}$ has an antichain of size κ, then $\mathrm{st}(\mathcal{B})$ contains a weak P_κ-point.*

Note that this article emphasizes *weak* P_κ-points. If $\mathcal{B}$ is complete and of size less than the first measurable cardinal, then no non-isolated point in $\mathrm{st}(\mathcal{B})$ can be a P-point. In fact, the methods discussed here can guarantee that regardless of measurable cardinals, all of the points constructed will not be P-points; see Section 6.

The following boolean algebra yields an example of Corollary 1.5:

Definition 1.6 *$\mathcal{D}_\kappa$ denotes the regular open algebra of the metric space κ^ω (where κ is discrete).*

Alternatively, one may view $\mathcal{D}_\kappa$ as the completion of the forcing order $\mathrm{Fn}(\omega, \kappa)$ (finite partial functions from ω to κ).

Corollary 1.7 *There is a dense subset $E \subseteq \mathrm{st}(\mathcal{D}_\kappa)$ such that $|E| = \kappa$ and each point of E is a weak P_κ-point in $\mathrm{st}(\mathcal{D}_\kappa)$.*

Proof. For each $p \in \mathrm{Fn}(\omega, \kappa)$ let $[p] = \{f \in \kappa^\omega : f \supset p\}$. Then $[p]$ is clopen in κ^ω, so that $[p] \in \mathcal{D}_\kappa$. Note that $N_{[p]}$ is a clopen set in $\mathrm{st}(\mathcal{D}_\kappa)$ which is homeomorphic to all of $\mathrm{st}(\mathcal{D}_\kappa)$, so by Corollary 1.5, we can choose a weak P_κ-point $\mathbf{x}_p \in N_{[p]}$. Let $E = \{\mathbf{x}_p : p \in \mathrm{Fn}(\omega, \kappa)\}$. Then $|E| \leq \mathrm{Fn}(\omega, \kappa) = \kappa$. The $\mathbf{x}_p$ may not all be distinct, but $|E| = \kappa$ because it is dense and all its elements are weak P_κ-points. $\square$

This example also shows that in Corollary 1.5, one cannot strengthen weak P_κ-point to weak P_{κ^+}-point. Likewise, in Corollary 1.4, no $\mathbf{x} \in u(\kappa)$ can be a weak P_{κ^+}-point in $\beta\kappa$, since $\mathbf{x}$ is a limit of the isolated points. However,

Theorem 1.8 *For every infinite regular κ, there is an $\mathbf{x} \in u(\kappa)$ which is a weak P_{κ^+}-point in $u(\kappa)$.*

This is proved in Baker and Kunen [2], using *mediocre points* (a modification of "good"). The case $\kappa = \omega$ (a weak P-point in $\omega^* = u(\omega)$) is an older result of Kunen [12], using *OK points* (another modification of "good"). In fact, "good", "mediocre", and "OK" are all special cases of the notion of *hat-point* (see Section 2), and we shall derive Theorems 1.3 and 1.8 by one proof, which produces hatpoints in Stone spaces. The exact flavor of hatpoint we get in a specific $\mathrm{st}(\mathcal{B})$ will depend on what kinds of matrices exists in $\mathcal{B}$ (see Section 3). We do not know if Theorem 1.8 holds for singular κ; it would hold if a suitable matrix (described by Lemma 3.9) exists in $\mathcal{P}(\kappa)/{<}\kappa$.

Besides constructing points in $\mathrm{st}(\mathcal{B})$, we shall, by the same proof, construct subsets of $\mathrm{st}(\mathcal{B})$. By way of introduction, consider the following well-known result:

Theorem 1.9 (Efimov [7]) *Let $\mathcal{A}$ be a complete boolean algebra with $|\mathcal{A}| \leq 2^\kappa$. Then $\mathrm{st}(\mathcal{A})$ can be embedded into $u(\kappa)$.*

Equivalently, there is an $h : \mathcal{P}(\kappa)/{<}\kappa \twoheadrightarrow \mathcal{A}$. Since $\mathcal{A}$ is complete, this is easy to prove (as we do in Section 4) using the Sikorski Extension Theorem.

Now, consider the special case where $\mathcal{A}$ is the 2-element algebra and $|\mathrm{st}(\mathcal{A})| = 1$, in which case h can be identified with an ultrafilter, which defines a point in $u(\kappa)$. Of course, Theorem 1.9 is trivial in this case, since it just says that $u(\kappa)$ contains a point – but now we know that in fact it contains a weak P_κ-point, or even a weak P_{κ^+}-point. Thus, we shall investigate improvements of Theorem 1.9 which embed $\mathrm{st}(\mathcal{A})$ as a weak P_θ-*set*; the definition of this notion is part of a general scheme for lifting properties of points to properties of sets:

Definition 1.10 *If "❁-point" is some property of points in a space, then $F \subseteq X$ is a* ❁-set *in X iff the* point *F is a ❁-point in the quotient space X/F obtained by collapsing F to a point.*

Lemma 1.11 *F is a weak P_θ-set in X iff no point of F is a limit point of any subset of $X \backslash F$ of size less than θ.*

We shall show:

Theorem 1.12 *Let $\mathcal{A}$ be a complete boolean algebra with $|\mathcal{A}| \leq 2^\kappa$. Then:*

1. *There is an $F \subseteq u(\kappa)$ such that F is homeomorphic to* $\mathrm{st}(\mathcal{A})$ *and F is a weak P_κ-set in $\beta\kappa$.*
2. *If κ is regular, then there is an $F \subseteq u(\kappa)$ such that F is homeomorphic to* $\mathrm{st}(\mathcal{A})$ *and F is a weak P_{κ^+}-set in $u(\kappa)$.*

Note that Theorem 1.8 is the special case of (2) when $\mathcal{A} = \{0, 1\}$. In the case $\kappa = \omega$, (2) is due to Simon [16], who produced $\mathrm{st}(\mathcal{A})$ as an OK set in $u(\omega) = \omega^*$. Here, we shall derive Theorems 1.3 and 1.12 as special cases of one result, Theorem 5.6, which produces hat*sets* in Stone spaces.

Now, we can put our results together to obtain dense sets of weak P-points:

Definition 1.13 *$D \subseteq X$ is θ-*fuzzy *in X iff $|D| = \theta$, every element of D is a weak P_θ-point in X, and D is dense in itself.*

If D is θ-fuzzy, then no element of D is a weak P_{θ^+}-point.

Theorem 1.14 *For any infinite κ:*

1. *There is an $E \subseteq u(\kappa)$ which is κ-fuzzy in $\beta\kappa$.*
2. *If κ is regular, then there is an $E \subseteq u(\kappa)$ such that E is κ^+-fuzzy in $u(\kappa)$.*

As before, we do not know if the hypothesis that κ is regular can be removed in (2).

Since every point is a P_ω-point, item (1) for $\kappa = \omega$ is immediate from the well-known fact that every non-scattered compact Hausdorff space contains a countable subset which is dense in itself. However (2) has non-trivial content even for $\kappa = \omega$: there is a set of $\aleph_1$ weak P-points in $\omega^* = u(\omega)$ which is dense in itself. We remark that given Theorem 1.8, (2) is easy when $2^\kappa = \kappa^+$, since we can just choose one weak P_{κ^+}-point from each basic clopen set to get a dense set of such points.

Proof of Theorem 1.14.1. Since $|\mathcal{D}_\kappa| = 2^\kappa$, apply Theorem 1.12.1 to let $F \subseteq u(\kappa)$ be a weak P_κ-set in $\beta\kappa$ homeomorphic to $\mathrm{st}(\mathcal{D}_\kappa)$. By Corollary 1.7, let $E \subseteq F$ be κ-fuzzy in F. But then E is also κ-fuzzy in $\beta\kappa$, since no point of F is a limit of any subset of $\beta\kappa\backslash F$ of size less than κ. $\square$

The exact same proof, now applying Theorem 1.12.2 to $\mathcal{D}_{\kappa^+}$, proves Theorem 1.14.2, *provided* that $2^{(\kappa^+)} = 2^\kappa$ (so that $|\mathcal{D}_{\kappa^+}| = 2^\kappa$). If $2^{(\kappa^+)} > 2^\kappa$, then we certainly cannot embed $\mathrm{st}(\mathcal{D}_{\kappa^+})$ into $u(\kappa)$, so we must apply a somewhat more technical version of Theorem 1.12.2 which replaces $\mathcal{A}$ by $\mathcal{A} \cap N$, where N is a κ-closed elementary submodel of the universe with $|N| = 2^\kappa$. This will be discussed in Section 5, where we prove Theorem 1.14.2.

Now, any argument producing weak P_θ-points in $u(\kappa)$ faces the following obstacle: The natural transfinite induction constructs an ultrafilter $\mathbf{x}$ on κ in 2^κ stages. However, $|u(\kappa)| = 2^{2^\kappa}$ by a well-known theorem of Pospíšil [13] (see Corollary 3.5). Thus, there are too many sequences of ultrafilters to simply diagonalize against them all (e.g., at stage α, make sure that $\mathbf{x}$ is not a limit of the α^{th} sequence). Rather, we define a *base property* of $\mathbf{x}$ which implies weak P_θ-point. Such base properties involve $\mathbf{x}$ and a base for the space $u(\kappa)$. Since the weight of $u(\kappa)$ is only 2^κ, there is a chance of success in building $\mathbf{x}$ in 2^κ steps to have such a property.

The simplest such base property is probably "P-point", and, under CH, W. Rudin [14] constructed a P-point in $u(\omega) = \omega^*$ in 2^ω steps. However, for κ regular and strictly between ω and the first measurable cardinal, there cannot be a P-point in $u(\kappa)$ (since it would then be countably complete), and even for $\kappa = \omega$, Shelah [17] showed that P-points in $u(\omega)$ cannot be proved to exist in ZFC. If we are looking for ZFC results, we must turn to somewhat more complex base properties, such as Keisler's notion of "good", and modifications thereof, such as "OK" and "mediocre". These are discussed in Section 2.

Even given a base property, it is not necessarily obvious (or even true) that there is an $\mathbf{x} \in u(\kappa)$ with that property. Successful constructions of such $\mathbf{x}$ often proceed with the aid a matrix of sets consisting of 2^κ independent rows. Then, each step in the construction eats up finitely many rows, and the remaining rows provide the necessary freedom to allow the construction to proceed. Matrices are taken up in more detail in Section 3, and their use in Sections 4 and 5.

2 Hatpoints and Hatsets

Since we shall frequently be taking finite intersections from a given sequence of sets, the following notation will be useful:

Definition 2.1 *Given e_α in a boolean algebra for $\alpha < \theta$, and $s \in [\theta]^{<\omega}$:*

$$e_{\boxed{s}} = \bigwedge_{\alpha \in s} e_\alpha \qquad ; \qquad e_{\boxed{\emptyset}} = 1 \quad .$$

Of course, this definition applies to sequences of sets, $E_\alpha \in \mathcal{P}(X)$, where $\wedge$ is $\cap$; then $E_{\boxed{\emptyset}} = X$.

Definition 2.2 *When $\kappa \leq \theta$, a (θ, κ)* hatfunction *is a function* $\frown : [\theta]^{<\omega} \to [\kappa]^{<\omega}$.

Definition 2.3 *If $\frown$ is a (θ, κ) hatfunction, then* $\mathbf{x}$ *is a* $\frown$point *in the space X iff, given neighborhoods U_r $(r \in [\kappa]^{<\omega})$ of* $\mathbf{x}$*:*

- ✫ *There are neighborhoods V_α $(\alpha < \theta)$ of* $\mathbf{x}$ *such that $V_{\boxed{s}} \subseteq U_{\widehat{s}}$ for each non-empty $s \in [\theta]^{<\omega}$.*

Since $\kappa \leq \theta$, we may also regard $\frown$ as a (θ, θ) hatfunction; then note that the definition of $\frown$point remains unchanged, since only the U_r for $r \in [\kappa]^{<\omega}$ are used. The importance of considering the possibility $\kappa < \theta$ arises in the actual construction of these points. For example, for a suitable $\frown : [\kappa^+]^{<\omega} \to [\kappa]^{<\omega}$, every $\frown$point is weak P_{κ^+}-point (see Lemma 2.15). In Section 5, we shall construct such points in $u(\kappa)$. The construction works because we construct $\mathbf{x} \in u(\kappa)$ in 2^κ steps and there are only 2^κ possible input sequences $\langle U_r : r \in [\kappa]^{<\omega} \rangle$ of neighborhoods (subsets of κ) to consider, even though $2^{(\kappa^+)}$ may be bigger than 2^κ.

Now, let us consider how the notion of "hatpoint" depends on the hatfunction. First, we point out (Lemma 2.7) that bigger hatfunctions yield stronger hatpoints.

Definition 2.4 *A sequence of sets, $\langle U_r : r \in [\kappa]^{<\omega} \rangle$, is* monotone *iff $r \subseteq p \Rightarrow U_r \supseteq U_p$.*

Lemma 2.5 *In the definition (2.3) of "hatpoint", it is sufficient to verify (✫) only in the case of monotone sequences of neighborhoods, $\langle U_r : r \in [\kappa]^{<\omega} \rangle$.*

Proof. Replace each U_r by $\bigcap \{U_p : p \subseteq r\}$. $\square$

Note that the hatfunctions form a lattice in the obvious way:

Definition 2.6 *Given two (θ, κ) hatfunctions* $\frown$, $\smile$:

- $\widehat{\ } \leq \widetilde{\ }$ *iff* $\widehat{s} \subseteq \widetilde{s}$ *for all* $s \in [\theta]^{<\omega}$.
- $\widehat{\ } \bigvee \widetilde{\ }$ *is the hatfunction* $\overline{\ }$ *defined by* $\overline{s} = \widehat{s} \cup \widetilde{s}$.

Lemma 2.7 *Given two* (θ, κ) *hatfunctions* $\widehat{\ }, \widetilde{\ }$:

- *If* $\widehat{\ } \leq \widetilde{\ }$ *then every* $\widetilde{\ }$*point is a* $\widehat{\ }$*point.*
- *If* $\mathbf{x}$ *is a* $(\widehat{\ } \bigvee \widetilde{\ })$*point, then* $\mathbf{x}$ *is both a* $\widehat{\ }$*point and a* $\widetilde{\ }$*point.*

Proof. Apply Lemma 2.5; the result is clear when applied to monotone $\langle U_r : r \in [\kappa]^{<\omega}\rangle$. $\square$

At the bottom of the lattice is the zero hatfunction; more generally, if $\widehat{\ }$ has finite range, then every point is a $\widehat{\ }$point. There is no top hatfunction, but, in the case $\kappa = \theta$, there is a top hatpoint, obtained by using the the identity hatfunction; this is just Keisler's [10] notion of good:

Definition 2.8 $\mathbf{x}$ *is* θ^+-good *iff* $\mathbf{x}$ *is a* $\widehat{\ }$*point, where* $\widehat{\ } : [\theta]^{<\omega} \to [\theta]^{<\omega}$ *is the identity function.*

Keisler defined λ-good to mean θ^+-good for all $\theta < \lambda$.

Lemma 2.9 *If* $\mathbf{x}$ *is* θ^+*-good, then* $\mathbf{x}$ *is a* $\widehat{\ }$*point for all* (θ, θ) *hatfunctions.*

Proof. Given neighborhoods U_r $(r \in [\theta]^{<\omega})$ of $\mathbf{x}$, let $W_s = U_{\widehat{s}}$. Applying "good", obtain V_α $(\alpha < \theta)$ so that $V_{\boxed{s}} \subseteq W_s$, and hence $V_{\boxed{s}} \subseteq U_{\widehat{s}}$, for each non-empty $s \in [\theta]^{<\omega}$. $\square$

We shall see (Lemma 2.13) that θ^+-good points are weak P_θ-points. Thus, one way to construct an $\mathbf{x} \in u(\kappa)$ which is a weak P_{κ^+}-point would be to make the point κ^{++}-good (i.e., use the identity (κ^+, κ^+) hatfunction). But then we have $2^{(\kappa^+)}$ possible input sequences $\langle U_r : r \in [\kappa^+]^{<\omega}\rangle$ to consider, and it is not clear how to handle them all in only 2^κ steps. It turns out that if κ is regular and $2^\kappa = 2^{(\kappa^+)}$, then there is a κ^{++}-good point in $u(\kappa)$ (see [1], Corollary 4.9). However, if κ is regular and uncountable but below the first measurable cardinal, and $2^\kappa = \kappa^+$, then no point in $u(\kappa)$ is κ^{++}-good (see [2], Theorem 2.8). Thus, if we want a *ZFC* result, we need to use a smaller hatfunction, weakening "good" to "mediocre".

Definition 2.10 $\mathbf{x}$ *is a* κ^+-mediocre *point iff* $\mathbf{x}$ *is a* $\widehat{\ }$*point, where for some* $\langle \varphi_\beta : \beta < \kappa^+\rangle$*:*

- ✍ *$\varphi_\beta : \beta \to \kappa$ is 1-1.*
- ✍ *$\widehat{\ } : [\kappa^+]^{<\omega} \to [\kappa]^{<\omega}$ and $\widehat{\{\alpha,\beta\}} = \{\varphi_\beta(\alpha)\}$ whenever $\alpha < \beta < \kappa^+$.*
- ✍ *$\widehat{s} = \emptyset$ whenever $|s| \neq 2$.*

Lemma 2.11 *$\mathbf{x}$ is a κ^+-*mediocre* point iff for some $\langle \varphi_\beta : \beta < \kappa^+ \rangle$:*

- ✍ *$\varphi_\beta : \beta \to \kappa$ is 1-1 and*
- ✍ *Given any κ neighborhoods of $\mathbf{x}$, $\langle U_\xi : \xi < \kappa \rangle$, there are κ^+ neighborhoods of $\mathbf{x}$, $\langle V_\alpha : \alpha < \kappa^+ \rangle$, such that $V_\alpha \cap V_\beta \subseteq U_{\varphi_\beta(\alpha)}$ whenever $\alpha < \beta < \kappa^+$.*

We do not know whether this property depends on the particular sequence $\langle \varphi_\beta : \beta < \kappa^+ \rangle$ used.

Lemma 2.12 *If $\mathbf{x} \in X$ is a weak P_κ-point which is also κ^+-mediocre, then $\mathbf{x}$ is a weak P_{κ^+}-point.*

Proof. Given $Y = \{y_\xi : \xi \in \kappa\} \subseteq X \backslash \{\mathbf{x}\}$, we wish to show that $\mathbf{x} \notin \overline{Y}$. For $\xi \in \kappa$, let U_ξ be a neighborhood of $\mathbf{x}$ disjoint from $\{y_\eta : \eta \le \xi\}$. Now, fix neighborhoods V_α ($\alpha \in \kappa^+$) of $\mathbf{x}$ such that $V_\alpha \cap V_\beta \subseteq U_{\varphi_\beta(\alpha)}$.

Then for some α, $V_\alpha \cap Y = \emptyset$: If not, then we can fix $\xi \in \kappa$ and $E \subseteq \kappa^+$ with $|E| = \kappa^+$ such that $\forall \alpha \in E$ ($y_\xi \in V_\alpha$). So, if $\alpha < \beta$ and $\alpha, \beta \in E$, then $y_\xi \in V_\alpha \cap V_\beta$, so $\varphi_\beta(\alpha) < \xi$. Now, fixing $\beta \in E$ with $|\beta \cap E| = \kappa$, we contradict the fact that φ_β is 1-1. □

Lemma 2.13 *If $\mathbf{x}$ is κ^+-good in the T_1 space X, then $\mathbf{x}$ is a weak P_κ-point in X.*

Proof. Observe that for all $\lambda < \kappa$: $\mathbf{x}$ is λ^{++}-good, and hence (by Lemma 2.9) λ^+-mediocre. Now, use Lemma 2.12, and show, by induction on $\lambda < \kappa$, that $\mathbf{x}$ is a weak P_{λ^+}-point for all $\lambda < \kappa$. □

By Lemmas 2.12 and 2.13, a point which is both κ^+-good and κ^+-mediocre is a weak P_{κ^+}-point. To combine these into one (κ^+, κ) hatfunction, we first pad the domain of the "good" hatfunction from Definition 2.8:

Lemma 2.14 *$\mathbf{x}$ is κ^+-good iff $\mathbf{x}$ is a $\widehat{\ }$point for the (κ^+, κ)-hatfunction defined by: $\widehat{s} = s \cap \kappa$.*

Now, we can apply Lemma 2.7 directly, taking the join of the "mediocre" and the "good" (κ^+, κ)-hatfunctions, to get:

Lemma 2.15 *For some (κ^+, κ)-hatfunction, $\frown$, every $\frown$point in every T_1 space is a weak P_{κ^+}-point.*

Note that we did not discuss hat*sets* in this section, but we do not have to. In view of Definition 1.10, the notion of $F \subseteq X$ being a hatset in X is already defined, and, collapsing F to a point, Lemmas 2.12, 2.13, 2.14, 2.15 apply to obtain various versions of weak P-set from versions of hatset.

The above discussion has been in arbitrary topological spaces. We now see how it applies to Stone spaces. Given $h : \mathcal{B} \twoheadrightarrow \mathcal{A}$ and $h^* : \mathrm{st}(\mathcal{A}) \hookrightarrow \mathrm{st}(\mathcal{B})$, the clopen neighborhoods of $h^*(\mathrm{st}(\mathcal{A}))$ are of the form N_c where $h(c) = 1_{\mathcal{A}}$. Thus,

Lemma 2.16 *Suppose $h : \mathcal{B} \twoheadrightarrow \mathcal{A}$ and $\frown$ is a (θ, κ) hatfunction. Then $h^*(\mathrm{st}(\mathcal{A}))$ is a $\frown$set in $\mathrm{st}(\mathcal{B})$ iff: Whenever $\langle c_r : r \in [\kappa]^{<\omega} \rangle$ is a sequence of elements of $\mathcal{B}$ with each $h(c_r) = 1_{\mathcal{A}}$, there are $d_\alpha \in \mathcal{B}$ for $\alpha < \theta$ such that:*

- *Each $h(d_\alpha) = 1_{\mathcal{A}}$.*
- *$d_{\boxed{s}} \le c_{\widehat{s}}$ for each nonempty $s \in [\theta]^{<\omega}$.*

When we actually obtain such an h in Section 5, the construction will be easier if we assume that the hatfunction is monotone:

Definition 2.17 *A (θ, κ) hatfunction $\frown$ is* monotone *iff $\widehat{\emptyset} = \emptyset$ and $s \subseteq t \Rightarrow \widehat{s} \subseteq \widehat{t}$.*

Lemma 2.18 *Given a (θ, κ) hatfunction $\frown$, there is another (θ, κ) hatfunction $\smile$ such that the notions of $\frown$point and $\smile$point are equivalent, and such that $\smile$ is* monotone.

Proof. Let $\widetilde{t} = \bigcup\{\widehat{s} : s \subseteq t\}$ for non-empty t, and $\widetilde{\emptyset} = \emptyset$. $\square$

We conclude this section with some remarks on OK points, although we shall not mention them again in this article. For more, see [5, 12, 16].

Definition 2.19 $\mathbf{x}$ *is* θ-OK *in the space X iff* $\mathbf{x}$ *is a* $\frown$point *for the (θ, ω) hatfunction defined by: $\widehat{s} = \{|s|\}$.*

It is easy to see that every point is ω-OK, and that the notion "θ-OK" gets stronger as θ gets bigger.

Lemma 2.20 *If X is a T_1 space and $\mathbf{x} \in X$ is ω_1-OK, then $\mathbf{x}$ is a weak P-point in X.*

Proof. Given $Y = \{y_n : n \in \omega\} \subseteq X \backslash \{\mathbf{x}\}$, we wish to show that $\mathbf{x} \notin \overline{Y}$. For $n \in \omega$, let U_n be a neighborhood of $\mathbf{x}$ disjoint from $\{y_m : m \leq n\}$. Now, fix neighborhoods V_α $(\alpha < \omega_1)$ of $\mathbf{x}$ as in the definition (2.3) of $\frown$point. So, whenever $\alpha_1, \ldots, \alpha_n$ are distinct, $V_{\alpha_1} \cap \cdots \cap V_{\alpha_n} \subseteq U_n$. Then no y_n can be contained in infinitely many V_α, so some V_α is disjoint from Y. □

However, an ω_2-OK point need not be a weak P_{ω_2}-point. For example, by [12] there are always $2^{\aleph_0}$-OK points in ω^*, but if one adds Cohen reals to a model of CH, one obtains a model of ZFC in which $2^{\aleph_0}$ is arbitrarily large but there are no weak P_{ω_2}-points in ω^*.

Lemma 2.21 *Every ω_1-mediocre point is ω_1-OK.*

Proof. Now, we must verify the definition of OK point for a given sequence of neighborhoods of $\mathbf{x}$, $\langle U_n : n \in \omega \rangle$. Thus, we must find neighborhoods V_α $(\alpha < \omega_1)$ of $\mathbf{x}$ so that $V_{\alpha_1} \cap \cdots \cap V_{\alpha_n} \subseteq U_n$ whenever $\alpha_1 < \cdots < \alpha_n$. We may (and shall) assume that $U_0 \supseteq U_1 \supseteq \cdots$.

Applying Lemma 2.11 to $\langle U_{n+2} : n \in \omega \rangle$, fix neighborhoods V_α $(\alpha < \omega_1)$ so that $V_\alpha \cap V_\beta \subseteq U_{\varphi_\beta(\alpha)+2}$ whenever $\alpha < \beta < \kappa^+$. We can also assume that each $V_\alpha \subseteq U_1$, so that we need only consider $n \geq 3$. Then, since φ_{α_n} is 1-1, $j := \max\{\varphi_{\alpha_n}(\alpha_m) : 1 \leq m < n\} \geq n - 2$, so $V_{\alpha_1} \cap \cdots \cap V_{\alpha_n} \subseteq U_{j+2} \subseteq U_n$. □

We do not know if the converse of Lemma 2.21 holds.

3 Matrices

We consider here matrices of subsets of κ, and, more generally, of elements of some boolean algebra.

Definition 3.1 *A* matrix *in a boolean algebra $\mathcal{B}$ is a sequence $\mathbb{M} = \langle \mathcal{M}^i : i \in I \rangle$ such that each $\mathcal{M}^i \subseteq \mathcal{B}$. Then:*

- $\mathbb{M}$ *is* independent *with respect to an element $c \in \mathcal{B}$ iff $b_1 \wedge \cdots \wedge b_k \wedge c > 0$ whenever k is finite, each $b_\ell \in \mathcal{M}^{i_\ell}$ ($\ell = 1, \ldots, k$), and $i_1, \ldots, i_k$ are distinct elements of I.*
- $\mathbb{M}$ *is* independent *iff* $\mathbb{M}$ *is independent with respect to* 1.
- *If $\mathcal{F} \subseteq \mathcal{B}$ is a filter, then* $\mathbb{M}$ *is* independent *with respect to $\mathcal{F}$ iff it is independent with respect to every $c \in \mathcal{F}$.*
- $\mathbb{M}$ *is* independent *with respect to an ideal iff it is independent with respect to the dual filter.*

Informally, we think of the $\mathcal{M}^i$ as the "rows" of the matrix. Usually, each row conforms to some configuration specified in advance. The most well-known configuration is just a disjoint family:

Definition 3.2 *The matrix* $\mathbb{M} = \langle \mathcal{M}^i : i \in I \rangle$ *is a* $|I| \times \kappa$ disjoint matrix *iff each* $\mathcal{M}^i$ *is an antichain in* $\mathcal{B}$ *of size* κ*.*

Thus, a $\theta \times \kappa$ disjoint matrix may be indexed as $\{b^i_\eta : i \in \theta \ \& \ \eta < \kappa\}$, where each $b^i_\eta \wedge b^i_\zeta = 0$ whenever $\eta \neq \zeta$. Independence of the matrix asserts that $b^{i_1}_{\eta_1} \wedge \cdots \wedge b^{i_k}_{\eta_k} > 0$ whenever the $i_1, \ldots, i_k$ are distinct. Independence with respect to a filter $\mathcal{F}$ asserts that in addition, these $b^{i_1}_{\eta_1} \wedge \cdots \wedge b^{i_k}_{\eta_k} \notin \mathcal{I}$, where $\mathcal{I}$ is the dual ideal. When $\theta = 2^\kappa$, $\mathcal{B} = \mathcal{P}(\kappa)$, and $\mathcal{F} = \mathcal{FR}(\kappa)$, the existence of such a matrix is equivalent to the following well-known result on independent functions:

Lemma 3.3 (Engelking and Karłowicz [8]) *For any infinite* κ*, there are functions* $f_i : \kappa \to \kappa$*, for* $i < 2^\kappa$*, which are independent in the sense that whenever* k *is finite,* $i_1, \ldots, i_k < 2^\kappa$ *are distinct, and* $\eta_1, \ldots, \eta_k < \kappa$ *are arbitrary,*

$$|\{\xi < \kappa : f_{i_1}(\xi) = \eta_1 \ \& \ \ldots \ \& \ f_{i_k}(\xi) = \eta_k\}| = \kappa \quad .$$

Proof. Index the functions as $\{f_A : A \subseteq \kappa\}$, and let $f_A : E \to \kappa$, where $E = \{(s,p) : s \in [\kappa]^{<\omega} \ \& \ p : \mathcal{P}(s) \to \kappa\}$; note that $|E| = \kappa$. Define $f_A(s,p) = p(A \cap s)$. Independence is proved by noting that given distinct $A_1, \ldots, A_k \in \mathcal{P}(\kappa)$, there are κ many finite s such that the $A_1 \cap s, \ldots, A_k \cap s$ are distinct, and for these, one can choose p so that each $p(A_\ell \cap s) = \eta_\ell$. □

A direct application of Lemma 3.3, letting $\mathcal{M}^i = \{f_i^{-1}\{\eta\} : \eta < \kappa\}$, yields:

Lemma 3.4 *In* $\mathcal{P}(\kappa)$*, there is a* $2^\kappa \times \kappa$ *disjoint matrix which is independent with respect to* $\mathcal{FR}(\kappa)$*.*

This improves the earlier theorem of Hausdorff [9] on the existence of 2^κ independent sets, which yields a $2^\kappa \times 2$ disjoint independent matrix. We digress to point out a well-known consequence of this matrix:

Corollary 3.5 (Pospíšil [13]) $|u(\kappa)| = 2^{2^\kappa}$*.*

Proof. Let $\{A^i_\eta : i \in 2^\kappa \ \& \ \eta < 2\}$ be a $2^\kappa \times 2$ disjoint matrix independent with respect to $\mathcal{FR}(\kappa)$. For each $f : 2^\kappa \to \{0,1\}$, choose an ultrafilter $\mathbf{x}_f \in u(\kappa)$ such that all $A^i_{f(i)} \in \mathbf{x}_f$, and note that these x_f are all distinct. □

Pospíšil's paper does not explicitly mention matrices or Hausdorff's paper, but the construction of the independent matrix is embedded in his proof.

Lemma 3.4 is best possible, in that there cannot be κ^+ disjoint subsets of κ. However, there are κ^+ almost disjoint subsets, and in fact, there is a $2^\kappa \times \kappa^+$ independent disjoint matrix in the boolean algebra $\mathcal{P}(\kappa)/{<}\kappa$ (Baker [1], Corollary 4.15; for regular κ, this was due earlier to Dow [5]).

Lemma 3.4 generalizes to:

Lemma 3.6 *Let $\mathcal{B}$ be a complete boolean algebra and $\mathcal{I}$ an ideal on $\mathcal{B}$. Assume that $\mathcal{B}$ has an antichain, $\{a_\xi : \xi < \kappa\}$, such that the a_ξ are distinct and each $a_\xi \notin \mathcal{I}$. Then in $\mathcal{B}$ there is a $2^\kappa \times \kappa$ disjoint matrix which is independent with respect to $\mathcal{I}$.*

Proof. With the f_i as in Lemma 3.3, let $b^i_\eta = \bigvee\{a_\xi : f_i(\xi) = \eta\}$. $\square$

Balcar and Franěk [3] show that this matrix is sufficient to construct κ^+-good points in $\mathrm{st}(\mathcal{B})$ when $|\mathcal{B}| \leq 2^\kappa$, proving Theorem 1.3. However, if we wish to construct hatpoints for a general (θ, κ) hatfunction, we need a more complicated matrix, each row of which is described by the following definition:

Definition 3.7 *Let $\widehat{\quad}$ be any monotone (θ, κ) hatfunction. If $\mathcal{B}$ is a complete boolean algebra and $\mathcal{G}$ is a filter on $\mathcal{B}$, with dual ideal $\mathcal{J}$: a* $\widehat{\quad}$step-family *on $(\mathcal{B}, \mathcal{G})$ is a collection of elements of $\mathcal{B}$ of the form*

$$\begin{aligned} \mathcal{M} &= \{e_r : r \in [\kappa]^{<\omega}\} \cup \{a_\alpha : \alpha < \theta\} \\ &\cup \{a_{\boxed{s}} \wedge e_r : s \in [\theta]^{<\omega} \ \&\ r \in [\kappa]^{<\omega} \ \&\ \widehat{s} \subseteq r\} \quad , \end{aligned}$$

where the e_r and a_α satisfy:

S1. $\bigvee\{e_r : r \in [\kappa]^{<\omega}\} = 1$, *and* $e_r \wedge e_p = 0$ *for each distinct* $r, p \in [\kappa]^{<\omega}$.
S2. $a_{\boxed{s}} \wedge \bigvee\{e_r : r \not\supseteq \widehat{s}\} \in \mathcal{J}$ *for each* $s \in [\theta]^{<\omega}$.
S3. $a_{\boxed{s}} \wedge e_r \notin \mathcal{J}$ *for each* $s \in [\theta]^{<\omega}$ *and* $r \in [\kappa]^{<\omega}$ *such that* $\widehat{s} \subseteq r$.

Note that the step-family $\mathcal{M}$ is determined by the e_r and the a_α; we have thrown in the $a_{\boxed{s}} \wedge e_r$ so that independence has the desired meaning. Suppose $\mathbb{M} = \langle \mathcal{M}^i : i \in I \rangle$ is a matrix where each $\mathcal{M}^i$ is a step-family, with corresponding e^i_r and a^i_α. Then independence with respect to c asserts:

$$\left(a^{i_1}_{\boxed{s_1}} \wedge e^{i_1}_{r_1}\right) \wedge \cdots \wedge \left(a^{i_k}_{\boxed{s_k}} \wedge e^{i_k}_{r_k}\right) \wedge c \neq 0 \quad , \qquad (*)$$

whenever k is finite, each $\widehat{s_\ell} \subseteq r_\ell$ ($\ell = 1, \ldots, k$), and $i_1, \ldots, i_k$ are distinct elements of I. It is not necessary to consider the a_α (since $a_{\boxed{s}} \geq a_{\boxed{s}} \wedge e_{\widehat{s}}$) or the e_r (since $e_r \geq a_{\boxed{\emptyset}} \wedge e_r$ and (by monotonicity, Definition 2.17) $\widehat{\emptyset} = \emptyset \subseteq r$).

(S1) says that the e_r, for $r \in [\kappa]^{<\omega}$, form a partition of unity; the $e_r \neq 0$ by (S3) (setting $s = \emptyset$, so $\widehat{s} = \emptyset \subseteq r$). Conversely, if $\theta = \kappa$ and $\widehat{\ }$ is the identity hatfunction, which is what we need for a κ^+-good point, then an antichain yields a matrix of step-families:

Lemma 3.8 *Let $\mathcal{B}$ be a complete boolean algebra with an ideal $\mathcal{I}$ and dual filter $\mathcal{F}$. Assume that $\mathcal{B}$ has an antichain, $\{a_\xi : \xi < \kappa\}$, such that the a_ξ are distinct and each $a_\xi \notin \mathcal{I}$. Let $\widehat{\ }$ be the identity (κ, κ) hatfunction. Then there is a matrix $\mathbb{M} = \langle \mathcal{M}^i : i \in 2^\kappa \rangle$ in $\mathcal{B}$ such that each $\mathcal{M}^i$ is a* $\widehat{\ }$step-family *on $(\mathcal{B}, \{1\})$, and $\mathbb{M}$ is independent with respect to $\mathcal{F}$.*

Proof. By Lemma 3.6, let $\mathbb{N} = \langle \mathcal{N}^i : i \in 2^\kappa \rangle$ be a $2^\kappa \times \kappa$ disjoint matrix which is independent with respect to $\mathcal{I}$. Index each row $\mathcal{N}^i$ as $\{e^i_r : r \in [\kappa]^{<\omega}\}$; so, the e^i_r are disjoint as r varies. Since $\mathcal{B}$ is complete, we can expand the e^i_r and assume that $\bigvee\{e^i_r : r \in [\kappa]^{<\omega}\} = 1$, so that (S1) holds. Then, define $a^i_\alpha = \bigvee\{e^i_r : \alpha \in r\}$. Now, $a^i_{\boxed{s}} = \bigvee\{e^i_r : r \supseteq s\}$ for each $s \in [\kappa]^{<\omega}$, so that (S2) holds with $\mathcal{J} = \{0\}$. If $s \subseteq r$, then $a^i_{\boxed{s}} \wedge e^i_r = e^i_r \neq 0$, so (S3) holds and $(*)$ above reduces independence of $\mathbb{M}$ to independence of $\mathbb{N}$. $\square$

The usual constructions of κ^+-good ultrafilters in the literature [3, 4, 11] work directly from the $2^\kappa \times \kappa$ independent disjoint matrix obtained by Lemmas 3.4 or 3.6, but a step like the above proof is embedded somewhere in the argument. Here, we shall present a general proof which constructs $\widehat{\ }$points in boolean algebras, provided that the correct matrix exists for $\widehat{\ }$; then, Lemma 3.8 yields the correct matrix for the "good" hatfunction whenever $\mathcal{B}$ has an appropriate antichain.

In the most general situation which we shall consider here (see Theorem 5.6), we shall have two filters $\mathcal{G} \subseteq \mathcal{F}$ on the complete boolean algebra $\mathcal{B}$, with a matrix of step-families on $(\mathcal{B}, \mathcal{G})$ which is independent with respect to $\mathcal{F}$. Then $\mathrm{st}(\mathcal{B}/\mathcal{F}) \subseteq \mathrm{st}(\mathcal{B}/\mathcal{G}) \subseteq \mathrm{st}(\mathcal{B})$, and we produce points $\mathbf{x} \in \mathrm{st}(\mathcal{B}/\mathcal{F})$ which are hatpoints in $\mathrm{st}(\mathcal{B}/\mathcal{G})$. For example, with $\mathcal{B} = \mathcal{P}(\kappa)$, $\mathcal{G} = \{1\}$, and $\mathcal{F} = \mathcal{FR}(\kappa)$, Lemma 3.8 will allow us to construct an $\mathbf{x} \in \mathrm{st}(\mathcal{B}/\mathcal{F}) = u(\kappa)$ which is κ^+-good, and hence a weak P_κ-point, in $\mathrm{st}(\mathcal{B}/\mathcal{G}) = \beta\kappa$. $\mathbf{x}$ cannot be a weak P_{κ^+}-point in $\beta\kappa$; if we want $\mathbf{x}$ to be a weak P_{κ^+}-point in $u(\kappa)$, we let $\mathcal{F} = \mathcal{G} = \mathcal{FR}(\kappa)$ and let $\widehat{\ }$ be the appropriate hatfunction (see Lemma 2.15); the same Theorem 5.6 yields a weak $\widehat{\ }$point because the appropriate matrix exists by the following result, which we quote from [2]:

Lemma 3.9 *Assume that* κ *is regular and* $\widehat{\ }$ *is any monotone* (κ^+, κ) *hat-function. Then there is a matrix* $\mathbb{M} = \langle \mathcal{M}^i : i \in 2^\kappa \rangle$ *in* $\mathcal{P}(\kappa)$ *such that each* $\mathcal{M}^i$ *is a* $\widehat{\ }$*step-family on* $(\mathcal{P}(\kappa), \mathcal{FR}(\kappa))$*, and* $\mathbb{M}$ *is independent with respect to* $\mathcal{FR}(\kappa)$*.*

We do not know if this lemma is true for singular κ. If it is, then the results proved in this article for regular κ hold also for singular κ, since regularity is not used in the construction of hatpoints or hatsets in Section 5. The construction will use the following consequence of the definition.

Lemma 3.10 *A step-family as in Definition 3.7 satisfies also:*

S4. $a_{\boxed{s}} \wedge e_r \notin \mathcal{J}$ *iff* $\widehat{s} \subseteq r$*.*
S5. $a_{\boxed{s}} \leq_{\mathcal{J}} \bigvee\{e_r : r \supseteq \widehat{s}\}$ *for each* $s \in [\theta]^{<\omega}$*.*

Note that the monotonicity of $\widehat{\ }$ comes in naturally here: If $s \subseteq t$, then $a_{\boxed{s}} \geq a_{\boxed{t}}$; so, if in addition $\widehat{s} \not\subseteq \widehat{t}$, then setting $r = \widehat{t}$ would contradict (S4).

Also, note that monotonicity does *not* imply that $\widehat{\ }$ preserves $\cup$. We might well have $r := \widehat{\{\alpha\}} \cup \widehat{\{\beta\}} \subsetneqq \widehat{\{\alpha, \beta\}}$. Then, modulo $\mathcal{J}$, a_α and a_β meet e_r but $a_\alpha \wedge a_\beta$ does not. In fact, this does happen with the monotone version of the (κ^+, κ)-hatfunction used for weak P_{κ^+}-points (Lemmas 2.15 and 2.18). Thus, for general step-families, we should not expect the situation occurring in the proof of Lemma 3.8, where each a_α is simply a join of some of the e_r.

In this article, step-families for (θ, κ) hatfunctions will be applied only with $\theta = \kappa$ (using Lemma 3.8) and $\theta = \kappa^+$ (using Lemma 3.9). However, the construction in [12] of $2^{\aleph_0}$-OK points in ω^* using independent linked families could be presented as a construction using an independent matrix of $2^{\aleph_0}$ $\widehat{\ }$step-families, where $\widehat{\ }$ is the $(2^{\aleph_0}, \omega)$ hatfunction from Definition 2.19 (made monotone by Lemma 2.18).

4 The Sikorski Extension Theorem

In this section, we isolate some basic features of the inductive construction of a homomorphism from $\mathcal{B}$ to $\mathcal{A}$. In Section 5, we apply this to the construction of hatpoints and hatsets. Our inductions always follow the pattern:

$$\begin{array}{ccccccccc}
\mathcal{C}_0 & \subseteq & \mathcal{C}_1 & \subseteq \cdots\cdots \subseteq & \mathcal{C}_\mu & \subseteq & \mathcal{C}_{\mu+1} & \subseteq \cdots\cdots \\
\downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_\mu} & & \downarrow{\scriptstyle h_{\mu+1}} & \\
\mathcal{A}_0 & \subseteq & \mathcal{A}_1 & \subseteq \cdots\cdots \subseteq & \mathcal{A}_\mu & \subseteq & \mathcal{A}_{\mu+1} & \subseteq \cdots\cdots
\end{array}$$

Definition 4.1 *If $\mathcal{A}, \mathcal{B}$ are boolean algebras: A* λ-homomorphism sequence *from $\mathcal{B}$ to $\mathcal{A}$ is a sequence $\langle \mathcal{C}_\mu, \mathcal{A}_\mu, h_\mu : \mu < \lambda \rangle$ such that:*

P1. $\mathcal{C}_\mu$ is a subalgebra of $\mathcal{B}$, $\mathcal{A}_\mu$ is a subalgebra of $\mathcal{A}$, and $h_\mu : \mathcal{C}_\mu \twoheadrightarrow \mathcal{A}_\mu$.

P2. If $\mu < \nu$ then $\mathcal{A}_\mu \subseteq \mathcal{A}_\nu$, $\mathcal{C}_\mu \subseteq \mathcal{C}_\nu$, and $h_\mu = h_\nu \upharpoonright \mathcal{C}_\mu$.

P3. For limit ν: $\mathcal{C}_\nu = \bigcup_{\mu<\nu} \mathcal{C}_\mu$ and $\mathcal{A}_\nu = \bigcup_{\mu<\nu} \mathcal{A}_\mu$.

Thus, the construction is determined at limit stages. All of the work takes place at successor stages, where, to avoid excessive subscripts, we always display the extension problem as:

$$\begin{array}{ccccc} \mathcal{C} & \subseteq & \widetilde{\mathcal{C}} & \subseteq & \mathcal{B} \\ \downarrow h & \swarrow \widetilde{h} & & & \\ \mathcal{A} & & & & \end{array}$$

Here, we are given $h = h_\mu$ and $\mathcal{C} = \mathcal{C}_\mu$, and we have to define $\widetilde{\mathcal{C}} = \mathcal{C}_{\mu+1}$ and $\widetilde{h} = h_{\mu+1}$ to accomplish a desired task (so, $\mathcal{A}_\mu = \mathrm{ran}(h)$ and $\mathcal{A}_{\mu+1} = \mathrm{ran}(\widetilde{h})$). The simplest task is just to make sure that at the end, we get a homomorphism defined on all of $\mathcal{B}$. Then, at a successor stage, we are given some $y \in \mathcal{B} \backslash \mathcal{C}$, and we want to extend to get $y \in \widetilde{\mathcal{C}}$. If $\mathcal{A}$ is complete, this can always be done by Lemma 4.4, which is the main lemma behind the Sikorski Extension Theorem (see [15], Theorem 33.1).

Definition 4.2 *If $\mathcal{S}$ is any subset of the boolean algebra $\mathcal{B}$, then $(\!(\mathcal{S})\!)$ is the subalgebra finitely generated by $\mathcal{S}$.*

Note that if $\mathcal{S} = \mathcal{C} \cup \{y\}$, where $\mathcal{C}$ is a subalgebra, then the elements of $(\!(\mathcal{S})\!)$ are all of the form $(y \wedge c_1) \vee (y' \wedge c_2)$, where $c_1, c_2 \in \mathcal{C}$.

Definition 4.3 *Suppose that $\mathcal{C}$ is a subalgebra of $\mathcal{B}$ and h is a homomorphism from $\mathcal{C}$ into the complete boolean algebra $\mathcal{A}$. For $y \in \mathcal{B}$, let*

$$h^+(y) = \bigwedge \{h(c) : c \in \mathcal{C} \text{ and } c \geq y\} \qquad h^-(y) = \bigvee \{h(c) : c \in \mathcal{C} \text{ and } c \leq y\} .$$

Lemma 4.4 (Sikorski) *With $\mathcal{A}, \mathcal{B}, \mathcal{C}, h, y$ as in Definition 4.3, and z any element of $\mathcal{A}$: $h^-(y) \leq z \leq h^+(y)$ iff there is an extension of h to a homomorphism $\widetilde{h}$ from $(\!(\mathcal{C} \cup \{y\})\!)$ into $\mathcal{A}$ with $\widetilde{h}(y) = z$. In this case, $\widetilde{h}$ is unique, and is defined by:*

$$\widetilde{h}((y \wedge c_1) \vee (y' \wedge c_2)) = (z \wedge h(c_1)) \vee (z' \wedge h(c_2))$$

for all $c_1, c_2 \in \mathcal{C}$.

For a proof, see [15], p 142. Applying this inductively:

Theorem 4.5 (Sikorski Extension Theorem) *Suppose that $\mathcal{C}$ is a subalgebra of $\mathcal{B}$ and h is a homomorphism from $\mathcal{C}$ into the complete boolean algebra $\mathcal{A}$. Then h can be extended to some $\overline{h} : \mathcal{B} \to \mathcal{A}$.*

Proof. List $\mathcal{B}$ as $\{b_\mu : \mu < \lambda\}$, and construct $\overline{h}$ using a λ-homomorphism sequence. Here, $\mathcal{C}_0 = \mathcal{C}$, $h_0 = h$, and we make sure that $b_\mu \in \mathcal{C}_{\mu+1}$. $\square$

Now, suppose that we want $\overline{h}$ to be onto, so that $\overline{h}^* : \mathrm{st}(\mathcal{A}) \hookrightarrow \mathrm{st}(\mathcal{B})$. In many cases this can be done with the method of independent matrices, obtaining the following lemma, which is the abstract version of Efimov's Theorem 1.9:

Lemma 4.6 *Suppose that $\mathcal{A}, \mathcal{B}$ are boolean algebras with $\mathcal{A}$ complete and $|\mathcal{A}| \le \theta$, and suppose that in $\mathcal{B}$, there is a $\theta \times 2$ independent disjoint matrix $\langle \mathcal{M}^i : i < \theta \rangle$. Then there is a homomorphism h from $\mathcal{B}$ onto $\mathcal{A}$.*

Proof. Here $\mathcal{M}^i = \{c_0^i, c_1^i\}$, where $c_0^i \wedge c_1^i = 0$. We may also assume, expanding the c_0^i, that $c_0^i \vee c_1^i = 1$. Let $\mathcal{C} = (\!(\bigcup_i \mathcal{M}^i)\!)$. List $\mathcal{A}$ as $\{a_i : i < \theta\}$, and define $h : \mathcal{C} \twoheadrightarrow \mathcal{A}$ by $h(c_0^i) = a_i$. By independence ($\mathcal{C}$ is the free algebra on θ generators; see [15], §14), this really defines a homomorphism. Now, extend h to $\mathcal{B}$ by Theorem 4.5. $\square$

Proof of Theorem 1.9. Apply Lemma 4.6. The appropriate matrix exists by Lemma 3.4. $\square$

Now, if, in Lemma 4.6, we want $\overline{h}^* : \mathrm{st}(\mathcal{A}) \hookrightarrow \mathrm{st}(\mathcal{B})$ to embed $\mathrm{st}(\mathcal{A})$ into $\mathrm{st}(\mathcal{B})$ as some sort of a hatset, then we must work harder. In these constructions, the use of the matrix is interleaved with the inductive definition of the homomorphism. We take this up in Section 5, and conclude this section with a more elementary remark:

In Lemma 4.4, say we replace $y \in \mathcal{B}$ by a set $\mathcal{Y} \subseteq \mathcal{B}$, and we want to extend h to $\widetilde{h}$ on $(\!(\mathcal{C} \cup \mathcal{Y})\!)$. Of course, this is possible, by Theorem 4.5, but it is not so simple to describe the possibilities for $\widetilde{h} \restriction \mathcal{Y}$. However, in the following special case, we can send all of $\mathcal{Y}$ to 1:

Lemma 4.7 *With $\mathcal{A}, \mathcal{B}, \mathcal{C}, h$ as in Lemma 4.4, suppose $\mathcal{Y} \subseteq \mathcal{B}$ is such that $h^+(y_1 \wedge \cdots \wedge y_n) = 1$ for all finite n and all $y_1, \ldots, y_n \in \mathcal{Y}$. Then there is a unique extension of h to a homomorphism $\widetilde{h}$ from $(\!(\mathcal{C} \cup \mathcal{Y})\!)$ into $\mathcal{A}$ such that $\mathrm{ran}\, \widetilde{h} = \mathrm{ran}\, h$ and $\widetilde{h}(y) = 1$ for all $y \in \mathcal{Y}$.*

Proof. We just describe $\widetilde{h}$; it is then easy to verify that it works. Fix a $b \in ((\mathcal{C} \cup \mathcal{Y}))$, and then choose $y_1, \ldots, y_n \in \mathcal{Y}$ such that $b \in ((\mathcal{C} \cup \{y_1, \ldots, y_n\}))$. Write b in disjunctive normal form as $b = \bigvee_f (c_f \wedge y(f))$, where each $c_f \in \mathcal{C}$, f ranges over all functions $f : \{1, \ldots, n\} \to \{0, 1\}$, and $y(f) = y_1^{f(1)} \wedge \cdots \wedge y_n^{f(n)}$, where $y_\ell^0 = y_\ell$ and $y_\ell^1 = (y_\ell)'$. In particular, if $\vec{0}$ denotes the identically 0 function, then $y(\vec{0}) = y_1 \wedge \cdots \wedge y_n$. Let $\widetilde{h}(b) = h(c_{\vec{0}})$. $\square$

5 Hatsets in Stone Spaces

We now take up the problem of embedding $\mathrm{st}(\mathcal{A})$ as a $\frown$set. First, we expand on the general framework outlined in Section 3. We start with complete boolean algebras $\mathcal{A}, \mathcal{B}$, and two filters $\mathcal{G} \subseteq \mathcal{F}$ on $\mathcal{B}$. Then $\mathrm{st}(\mathcal{B}/\mathcal{F}) \subseteq \mathrm{st}(\mathcal{B}/\mathcal{G}) \subseteq \mathrm{st}(\mathcal{B})$, and we wish to embed $\mathrm{st}(\mathcal{A})$ into $\mathrm{st}(\mathcal{B}/\mathcal{F})$ so that it is a hatset in $\mathrm{st}(\mathcal{B}/\mathcal{G})$. This embedding will be via an h^*, where $h : \mathcal{B}/\mathcal{F} \twoheadrightarrow \mathcal{A}$; equivalently, $h : \mathcal{B} \twoheadrightarrow \mathcal{A}$ with $h(\mathcal{F}) = \{1\}$. This will be done using the inductive scheme described in Section 4. Let $\mathcal{I}$ be the ideal dual to $\mathcal{F}$. Now, we start off with $\mathcal{C}_0 = ((\mathcal{F})) = \mathcal{F} \cup \mathcal{I}$, and $h_0 : \mathcal{C}_0 \twoheadrightarrow \mathcal{A}_0 = \{0, 1\}$ defined by $h_0(\mathcal{F}) = \{1\}$ and $h_0(\mathcal{I}) = \{0\}$. We work with the aid of a matrix which is kept "independent" in the following sense, due to Simon [16]:

Definition 5.1 *If $\mathbb{M}$ is a matrix in $\mathcal{B}$, $\mathcal{C}$ is a subalgebra of $\mathcal{B}$, and $h : \mathcal{C} \to \mathcal{A}$, then $\mathbb{M}$ is* strongly independent *with respect to h iff $\mathbb{M}$ is independent with respect to every $c \in \mathcal{C}$ such that $h(c) > 0$.*

At the start of the induction, strong independence with respect to h_0 means that $\mathbb{M}$ is independent with respect to $\mathcal{F}$, and the appropriate $\mathbb{M}$ will be given by Lemma 3.8 or Lemma 3.9.

Now, we consider a homomorphism sequence (Definition 4.1) augmented by the use of a matrix.

Definition 5.2 *Let $\mathbb{M} = \langle \mathcal{M}^i : i \in I \rangle$ be a matrix in $\mathcal{B}$. A* λ-matrix sequence *from $\mathcal{B}$ to $\mathcal{A}$ is a sequence $\langle \mathcal{C}_\mu, \mathcal{A}_\mu, h_\mu, I_\mu : \mu < \lambda \rangle$ such that $\langle \mathcal{C}_\mu, \mathcal{A}_\mu, h_\mu : \mu < \lambda \rangle$ is a homomorphism sequence and:*

P4. $I_0 = I$.
P5. If $\mu < \nu$ *then* $I_\mu \supseteq I_\nu$.
P6. $\mathbb{M} \restriction I_\mu$ *is strongly independent with respect to* h_μ.
P7. For limit ν: $I_\nu = \bigcap_{\mu < \nu} I_\mu$.

Again, there is no problem at limits, and we use the matrix at successor stages to achieve the goals of the construction. We first modify the extension lemmas, 4.4 and 4.7, to include the matrix:

Lemma 5.3 *Suppose that $\mathcal{C}$ is a subalgebra of $\mathcal{B}$ and h is a homomorphism from $\mathcal{C}$ into the complete boolean algebra $\mathcal{A}$. Assume that the matrix $\mathbb{M} = \langle \mathcal{M}^i : i \in I\rangle$, is strongly independent with respect to h. Fix any $j \in I$ and let $J = I \backslash \{j\}$. Then:*

1. *If $b \in \mathcal{M}^j$, then $h^+(b) = 1$.*
2. *Assume that $b, e \in \mathcal{M}^j$ and $e \wedge b = 0$. Fix any $z \in \mathcal{A}$. Then $h^+(b) = 1$, $h^-(b) = 0$, and if $\widetilde{h} : ((\mathcal{C} \cup \{b\})) \to \mathcal{A}$ extends h with $h(b) = z$ (as in Lemma 4.4), then $\mathbb{M} \restriction J$ is strongly independent with respect to $\widetilde{h}$.*
3. *Fix $\mathcal{Y} \subseteq \mathcal{B}$. Assume that for all finite n and all $y_1, \ldots, y_n \in \mathcal{Y}$, there are $b \in \mathcal{M}^j$ and $c \in \mathcal{C}$ such that $h(c) = 1$ and $y_1 \wedge \cdots \wedge y_n \geq b \wedge c$. Then $h^+(y_1 \wedge \cdots \wedge y_n) = 1$ for each $y_1, \ldots, y_n \in \mathcal{Y}$, and if $\widetilde{h} : ((\mathcal{C} \cup \mathcal{Y})) \to \mathcal{A}$ with $\widetilde{h}(\mathcal{Y}) = \{1\}$ (as in Lemma 4.7), then $\mathbb{M} \restriction J$ is strongly independent with respect to $\widetilde{h}$.*

Proof. (1) is clear by strong independence, as is $h^-(b) = 0$ in (2). To prove strong independence of $\mathbb{M} \restriction J$ in (2): Fix distinct $j_1, \ldots, j_k \in J$, and fix $d_\ell \in \mathcal{M}^{j_\ell}$ ($\ell = 1, \ldots, k$). Assume that $\widetilde{h}((b \wedge c_1) \vee (b' \wedge c_2)) > 0$, where $c_1, c_2 \in \mathcal{C}$. We must show that $d_1 \wedge \cdots \wedge d_k \wedge ((b \wedge c_1) \vee (b' \wedge c_2)) > 0$. Since $\widetilde{h}((b \wedge c_1) \vee (b' \wedge c_2)) = (z \wedge h(c_1)) \vee (z' \wedge h(c_2))$, there are two cases:

Case 1. $z \wedge h(c_1) > 0$. Then $h(c_1) > 0$, so $d_1 \wedge \cdots \wedge d_k \wedge b \wedge c_1 > 0$ by strong independence of $\mathbb{M}$ with respect to h.

Case 2. $z' \wedge h(c_2) > 0$. Then $h(c_2) > 0$, so $d_1 \wedge \cdots \wedge d_k \wedge b' \wedge c_2 \geq d_1 \wedge \cdots \wedge d_k \wedge e \wedge c_2 > 0$.

To verify $h^+(y_1 \wedge \cdots \wedge y_n) = 1$ in (3), fix $z \in \mathcal{C}$ with $z \wedge (y_1 \wedge \cdots \wedge y_n) = 0$; we must show that $h(z) = 0$. Fix b, c as in the hypothesis of (3). Then $b \wedge z \wedge c = 0$ and $b \in \mathcal{M}^j$, so $h(z \wedge c) = 0$ by strong independence. Then, $h(c) = 1$ yields $h(z) = 0$.

To prove strong independence of $\mathbb{M} \restriction J$ in (3): Fix $w \in ((\mathcal{C} \cup \mathcal{Y}))$ with $\widetilde{h}(w) > 0$ and fix distinct $j_1, \ldots, j_k \in I \backslash \{j\}$ and $b_\ell \in \mathcal{M}^{j_\ell}$ ($\ell = 1, \ldots, k$); we must show that $b_1 \wedge \cdots \wedge b_k \wedge w > 0$. But, $\widetilde{h}(w) > 0$ implies that we can find some $y_1, \ldots, y_n \in \mathcal{Y}$ and $z \in \mathcal{C}$ such that $w \geq z \wedge y_1 \wedge \cdots \wedge y_n$ and $h(z) = \widetilde{h}(z \wedge y_1 \wedge \cdots \wedge y_n) > 0$ (see the proof of Lemma 4.7). Fix b, c as in the hypothesis of (3). Then $h(z \wedge c) = h(z) > 0$, and $(z \wedge c) \in \mathcal{C}$, so strong independence of $\mathbb{M}$ with respect to h yields $b_1 \wedge \cdots \wedge b_k \wedge b \wedge z \wedge c > 0$. Now, $w \geq z \wedge b \wedge c$, so $b_1 \wedge \cdots \wedge b_k \wedge w > 0$. $\square$

Lemma 5.3.2 tells us how to make $h : \mathcal{B} \to \mathcal{A}$ *onto* in our inductive construction. Assuming that each row of the matrix contains a pair of disjoint elements, we can, at some stage μ, choose an arbitrary $z \in \mathcal{A}$ and put z into the range of $h_{\mu+1}$, sacrificing one row. However, if y is an arbitrary element of $\mathcal{B}$ (not a matrix element), it is a bit tricky to put y into the domain. We cannot simply quote Lemma 4.4, as there is no guarantee that the matrix will stay strongly independent. However, by the following argument, due to Simon [16], we may put y in the domain if we sacrifice $|\operatorname{ran}(h_\mu)|$ rows of the matrix.

Lemma 5.4 *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, h, \mathbb{M}$ be as in Lemma 5.3 and fix $y \in \mathcal{B}$. Then there is an extension of h to $\widetilde{h} : ((\mathcal{C} \cup \{y\})) \to \mathcal{A}$ and a $J \subseteq I$ with $|I \backslash J| \leq \max(|\operatorname{ran}(h)|, \aleph_0)$ such that $\mathbb{M} \restriction J$ is strongly independent with respect to $\widetilde{h}$.*

Proof. Call $d \in \operatorname{ran}(h)$ *bad* iff for some finite k, some $c \in \mathcal{C}$, some distinct $i_1, \ldots, i_k \in I$, and some $b_\ell \in \mathcal{M}^{i_\ell}$ ($\ell = 1, \ldots, k$), we have $h(c) = d$ and $c \wedge y \wedge b_1 \wedge \cdots \wedge b_k = 0$. So, 0 is bad, and the set of bad elements is closed downward in $\operatorname{ran}(h)$. Note that the lemma requires us to set $\widetilde{h}(y) \wedge d = 0$ for each bad d (otherwise, $\widetilde{h}(c \wedge y) = \widetilde{h}(y) \wedge d > 0$ would contradict strong independence).

List all the bad elements as $\{d_\alpha : \alpha < \theta\}$. For each α, choose k_α, c_α, $i_1^\alpha, \ldots, i_{k_\alpha}^\alpha$, and $b_1^\alpha, \ldots, b_{k_\alpha}^\alpha$ as in the definition of "bad". Let $J = I \setminus \{i_\ell^\alpha : \alpha < \theta \;\&\; 1 \leq \ell \leq k_\alpha\}$. Let $a = (\bigvee_{\alpha<\theta} d_\alpha)' = \bigwedge_{\alpha<\theta} (d_\alpha)' \in \mathcal{A}$. We shall set $\widetilde{h}(y) = a$, applying Lemma 4.4, but first we must verify that $h^-(y) \leq a \leq h^+(y)$.

To prove $h^-(y) \leq a$, it is sufficient to fix $z \leq y$ with $z \in \mathcal{C}$ and fix α and show that $h(z) \leq (d_\alpha)'$. From $z \leq y$ we get $c_\alpha \wedge z \wedge b_1^\alpha \wedge \cdots \wedge b_k^\alpha = 0$, where $k = k_\alpha$. Since $(c_\alpha \wedge z) \in \mathcal{C}$, strong independence yields $h(c_\alpha \wedge z) = 0$, so $h(z) \leq (h(c_\alpha))' = (d_\alpha)'$.

To prove $a \leq h^+(y)$, we fix $z \geq y$ with $z \in \mathcal{C}$, and show $h(z) \geq a$. From $z \geq y$ we get $z' \wedge y = 0$, so $h(z')$ is bad (and the corresponding k can be 0), so $h(z') = d_\alpha$ for some α, and hence $h(z) = (d_\alpha)' \geq a$.

Now, to verify strong independence, fix $w \in ((\mathcal{C} \cup \{y\}))$ with $\widetilde{h}(w) > 0$, and fix distinct $j_1, \ldots, j_r \in J$ and elements $b_\ell \in \mathcal{M}^{j_\ell}$. We must show that $w \wedge b_1 \wedge \cdots \wedge b_r > 0$. Say $w = (u \wedge y) \vee (v \wedge y')$, where $u, v \in \mathcal{C}$. There are two cases:

Case 1: $\widetilde{h}(u \wedge y) > 0$. That is, $h(u) \wedge a > 0$. Then $h(u)$ is not one of the d_α, so $h(u)$ is not bad, so $u \wedge y \wedge b_1 \wedge \cdots \wedge b_r > 0$.

Case 2: $\widetilde{h}(v \wedge y') > 0$. That is, $h(v) \wedge \bigvee_\alpha d_\alpha > 0$. Fix α with $h(v) \wedge d_\alpha > 0$, so $h(v \wedge c_\alpha) > 0$. By strong independence of $\mathbb{M}$ with respect to h, we have $v \wedge c_\alpha \wedge b_1^\alpha \wedge \cdots \wedge b_k^\alpha \wedge b_1 \wedge \cdots \wedge b_r > 0$, where $k = k_\alpha$. But also $c_\alpha \wedge y \wedge b_1^\alpha \wedge \cdots \wedge b_k^\alpha = 0$, so $y' \geq c_\alpha \wedge b_1^\alpha \wedge \cdots \wedge b_k^\alpha$, and hence $v \wedge y' \wedge b_1 \wedge \cdots \wedge b_r > 0$. $\square$

Finally, while we are building $h : \mathcal{B} \twoheadrightarrow \mathcal{A}$, we must ensure that $h^*(\mathrm{st}(\mathcal{A}))$ becomes a hatset. To do this, we need to obtain the condition in Lemma 2.16, but, bear in mind that $h^*(\mathrm{st}(\mathcal{A}))$ need only be a hatset in $\mathrm{st}(\mathcal{B}/\mathcal{G})$, not in $\mathrm{st}(\mathcal{B})$. So, we phrase the extension lemma as follows:

Lemma 5.5 *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, h, \mathbb{M}, j, J$ be as in Lemma 5.3. Let $\widehat{\ }$ be any monotone (θ, κ) hatfunction. Assume that $\mathcal{B}$ is complete and $\mathcal{G}$ is a filter on $\mathcal{B}$, with dual ideal $\mathcal{J} \subseteq \ker(h)$. Assume that $\mathcal{M}^j$ is a $\widehat{\ }$step-family on $(\mathcal{B}, \mathcal{G})$, as in Definition 3.7. Let $\langle c_r : r \in [\kappa]^{<\omega} \rangle$ be a sequence of elements of $\mathcal{C}$ with each $h(c_r) = 1_{\mathcal{A}}$. Then there are $d_\alpha \in \mathcal{B}$ for $\alpha < \theta$ and an extension of h to some $\widetilde{h} : \widetilde{\mathcal{C}} = ⟪\mathcal{C} \cup \{d_\alpha : \alpha < \theta\}⟫ \to \mathcal{A}$ so that:*

- ✣ *Each $\widetilde{h}(d_\alpha) = 1_{\mathcal{A}}$,*
- ✣ *$d_{\boxed{s}} \leq_{\mathcal{J}} c_{\widehat{s}}$ for each nonempty $s \in [\theta]^{<\omega}$,*

$\mathbb{M} \restriction J$ is strongly independent with respect to $\widetilde{h}$, and $\mathrm{ran}\, \widetilde{h} = \mathrm{ran}\, h$.

Proof. We may assume that the sequence $\langle c_r : r \in [\kappa]^{<\omega} \rangle$ is monotone by replacing each c_r by $\bigwedge\{c_p : p \subseteq r\}$, as in the proof of Lemma 2.5. Now, define

$$d_\alpha = a_\alpha \wedge \bigvee\{c_r \wedge e_r : r \in [\kappa]^{<\omega}\} \quad .$$

Then, applying S5 (Lemma 3.10), S1, and monotonicity of $\langle c_r : r \in [\kappa]^{<\omega} \rangle$:

$$\begin{aligned} d_{\boxed{s}} &= a_{\boxed{s}} \wedge \bigvee\{c_r \wedge e_r : r \in [\kappa]^{<\omega}\} \\ &\leq_{\mathcal{J}} \bigvee\{e_r : r \supseteq \widehat{s}\} \wedge \bigvee\{c_r \wedge e_r : r \in [\kappa]^{<\omega}\} \\ &= \bigvee\{c_r \wedge e_r : r \supseteq \widehat{s}\} \ \leq \ \bigvee\{c_r : r \supseteq \widehat{s}\} \ = \ c_{\widehat{s}} \quad . \end{aligned}$$

Observe that $d_{\boxed{s}} \geq a_{\boxed{s}} \wedge e_{\widehat{s}} \wedge c_{\widehat{s}}$, and that $a_{\boxed{s}} \wedge e_{\widehat{s}} \in \mathcal{M}^j$. We thus get $\widetilde{h}$ by applying Lemma 5.3.3, with $\mathcal{Y} = \{d_\alpha : \alpha < \theta\}$. As in Lemma 4.7, which was used by Lemma 5.3.3, we get $\mathrm{ran}\, \widetilde{h} = \mathrm{ran}\, h$. $\square$

We can now put this all together:

Theorem 5.6 *Let $\mathcal{B}$ be a complete boolean algebra of size 2^κ, with two filters, $\mathcal{G} \subseteq \mathcal{F}$ on $\mathcal{B}$ (so $\mathrm{st}(\mathcal{B}/\mathcal{F}) \subseteq \mathrm{st}(\mathcal{B}/\mathcal{G}) \subseteq \mathrm{st}(\mathcal{B})$). Let $\widehat{\ }$ be any monotone (θ, κ) hatfunction. Assume that $\mathbb{M} = \langle \mathcal{M}^i : i \in 2^\kappa \rangle$ is a matrix which is independent with respect to $\mathcal{F}$, such that each $\mathcal{M}^i$ is a $\widehat{\ }$step-family on $(\mathcal{B}, \mathcal{G})$. Then for every complete boolean algebra $\mathcal{A}$ with $|\mathcal{A}| \leq 2^\kappa$, there is an $h : \mathcal{B} \twoheadrightarrow \mathcal{A}$ such that $h(\mathcal{F}) = \{1\}$ and such that $h^*(\mathrm{st}(\mathcal{A})) \subseteq \mathrm{st}(\mathcal{B}/\mathcal{F})$ is a $\widehat{\ }$set in $\mathrm{st}(\mathcal{B}/\mathcal{G})$.*

Proof. We follow the pattern described in the beginning of Section 4. We construct h by a 2^κ matrix sequence $\langle \mathcal{C}_\mu, \mathcal{A}_\mu, h_\mu, I_\mu : \mu < 2^\kappa \rangle$. $\mathcal{C}_0 = (\!(\mathcal{F})\!)$, $\mathcal{A}_0 = \{0,1\}$, $h_0(\mathcal{F}) = \{1\}$, and $I_0 = 2^\kappa$.

There are three tasks to be accomplished: making $h^*(\mathrm{st}(\mathcal{A}))$ a hatset, making h onto $\mathcal{A}$, and making h defined on all of $\mathcal{B}$. These are accomplished by Lemmas 5.5, 5.3.2, and 5.4, respectively. When we apply one of these lemmas at step μ, we set $\mathcal{C} = \mathcal{C}_\mu$, $h = h_\mu$, and $I = I_\mu$, and use the appropriate lemma to obtain $\mathcal{C}_{\mu+1} = \widetilde{\mathcal{C}}$, $h_{\mu+1} = \widetilde{h}$, and $I_{\mu+1} = J$. Which one we apply at step μ will depend on $\mu \bmod 3$.

We inductively assume that $|I \backslash I_\mu| \le \max(|\mu|, \aleph_0)$, so that in particular I_μ is never empty, which is important when we apply Lemmas 5.5 and 5.3.2. But, since the use of Lemma 5.4 eliminates $|\mathcal{A}_\mu|$ rows of the matrix, we must also assume inductively that $|\mathcal{A}_\mu| \le \max(|\mu|, \aleph_0)$. We must exercise some caution here; we cannot assume that $|\mathcal{C}_\mu| \le \max(|\mu|, \aleph_0)$, since as soon as we apply Lemma 5.5, $|\mathcal{C}_\mu|$ will grow to size θ, which might even equal 2^κ. Fortunately, our extension lemmas bound the size of the range.

To ensure that $h^*(\mathrm{st}(\mathcal{A}))$ is a hatset: Before beginning the construction, fix $c^\mu_r \in \mathcal{B}$ for $\mu < 2^\kappa$ and $\mu \equiv 0 \pmod 3$, such that $\langle c^\mu_r : r \in [\kappa]^{<\omega} \rangle$ is a sequence of elements of $\mathcal{B}$ and such that each such sequence is listed cofinally often in 2^κ. At stage μ with $\mu \equiv 0 \pmod 3$: If each $c^\mu_r \in \mathcal{C}_\mu$ and $h_\mu(c^\mu_r) = 1$, we apply Lemma 5.5 at this stage to ensure this instance of hatpoint, so $|I_\mu \backslash I_{\mu+1}| = 1$. If not, then let $\mathcal{C}_{\mu+1} = \mathcal{C}_\mu$, $h_{\mu+1} = h_\mu$, and $I_{\mu+1} = I_\mu$. Note that in either case, $\mathcal{A}_{\mu+1} = \mathcal{A}_\mu$.

To ensure that h is onto: Before beginning the construction, list $\mathcal{A}$ as $\{a_\mu : \mu < 2^\kappa \ \&\ \mu \equiv 1 \pmod 3\}$. Then, at stage μ with $\mu \equiv 1 \pmod 3$, apply Lemma 5.3.2 to make sure that $a_\mu \in \mathcal{A}_{\mu+1}$. Again, $|I_\mu \backslash I_{\mu+1}| = 1$. $\mathcal{A}_{\mu+1}$, which is obtained via Lemma 4.4, is generated from $\mathcal{A}_\mu$ and the one element a_μ, so that $|\mathcal{A}_\mu| \le \max(|\mu|, \aleph_0)$ implies that $|\mathcal{A}_{\mu+1}| \le \max(|\mu|, \aleph_0) = \max(|\mu+1|, \aleph_0)$.

To ensure that h is defined on all of $\mathcal{B}$: Before beginning the construction, list $\mathcal{B}$ as $\{b_\mu : \mu < 2^\kappa \ \&\ \mu \equiv 2 \pmod 3\}$. Then, at stage μ with $\mu \equiv 2 \pmod 3$, apply Lemma 5.4 to make sure that $b_\mu \in \mathcal{C}_{\mu+1}$. Assume, inductively, that $|\mathcal{A}_\mu| \le \max(|\mu|, \aleph_0)$ and $|I_\mu| \le \max(|\mu|, \aleph_0)$. Again, $\mathcal{A}_{\mu+1}$ is generated from $\mathcal{A}_\mu$ and one element, so that $|\mathcal{A}_{\mu+1}| \le \max(|\mu+1|, \aleph_0)$. Also, $|I_\mu \setminus I_{\mu+1}| \le \max(|\mathcal{A}_\mu|, \aleph_0)$, so $|I \setminus I_{\mu+1}| \le \max(|\mu+1|, \aleph_0)$. □

In particular, we may apply this to the "good" hatfunction, where the matrix can be constructed directly from a disjoint family (see Lemma 3.8); $\mathcal{G} = \{1\}$ here:

Corollary 5.7 *Let $\mathcal{B}$ be a complete boolean algebra of size 2^κ. Let $\mathcal{I}$ be an ideal on $\mathcal{B}$ with dual filter $\mathcal{F}$, and assume that $\mathcal{B}$ has an antichain, $\{a_\xi : \xi < \kappa\}$, such that the a_ξ are distinct and each $a_\xi \notin \mathcal{I}$. Let $\mathcal{A}$ be any complete boolean algebra such that $|\mathcal{A}| \le 2^\kappa$. Then there is an $h : \mathcal{B} \twoheadrightarrow \mathcal{A}$ such that $h(\mathcal{F}) = \{1\}$ and such that $h^*(\mathrm{st}(\mathcal{A})) \subseteq \mathrm{st}(\mathcal{B}/\mathcal{F})$ is a κ^+-good set (and hence a weak P_κ-set) in $\mathrm{st}(\mathcal{B})$.*

Theorem 1.3 is immediate from this, as is Theorem 1.12.1. To prove Theorem 1.12.2, we apply Theorem 5.6 with $\mathcal{B} = \mathcal{P}(\kappa)$ and $\mathcal{F} = \mathcal{G} = \mathcal{FR}(\kappa)$, where, by Lemma 3.9, we have the correct matrix for every monotone (κ^+, κ) hatfunction. In particular, we can use a hatfunction such that hatsets are weak P_{κ^+}-sets; note that by Lemma 2.18, we can replace every hatfunction by a monotone one. We thus have the following corollary, from which Theorem 1.12.2 follows immediately:

Corollary 5.8 *Let κ be any regular cardinal and $\mathcal{A}$ any complete boolean algebra with $|\mathcal{A}| \le 2^\kappa$. Let $\frown$ be any (κ^+, κ) hatfunction. Then $\mathrm{st}(\mathcal{A})$ can be embedded into $u(\kappa)$ as a $\frown$set.*

As pointed out in the Introduction, we have now proved Theorem 1.14.1. However, we have not yet proved Theorem 1.14.2, except in the case $2^\kappa = 2^{(\kappa^+)}$. Here, we wish to construct $E = \{\mathbf{x}_\delta : \delta < \kappa^+\} \subseteq u(\kappa)$ such that E is κ^+-fuzzy in $u(\kappa)$. We know, by Corollary 1.7, that there is a $D = \{\mathbf{y}_\delta : \delta < \kappa^+\} \subseteq \mathrm{st}(\mathcal{D}_{\kappa^+})$ which is κ^+-fuzzy in $\mathrm{st}(\mathcal{D}_{\kappa^+})$; so each $\mathbf{y}_\delta$ is a weak P_{κ^+}-point, and D is dense in itself. Now, $|\mathcal{D}_{\kappa^+}| = 2^{(\kappa^+)}$, so if $2^\kappa = 2^{(\kappa^+)}$, we can insert $\mathrm{st}(\mathcal{D}_{\kappa^+})$ into $u(\kappa)$ as a weak P_{κ^+}-set. In the general case, we must prove a variant of Theorem 5.6 which applies with $\mathcal{A}$ of arbitrary size.

With any complete $\mathcal{A}$, we can certainly follow all the steps in the proof of Theorem 5.6 and produce a homomorphism $h : \mathcal{B} \to \mathcal{A}$; we just cannot make it onto if $|\mathcal{A}| > 2^\kappa$. If $\widetilde{\mathcal{A}} = \mathrm{ran}(h)$, we have $h : \mathcal{B} \twoheadrightarrow \widetilde{\mathcal{A}}$, and then $h^* : \mathrm{st}(\widetilde{\mathcal{A}}) \hookrightarrow \mathcal{B}$. So, if we have $D = \{\mathbf{y}_\delta : \delta < \kappa^+\} \subseteq \mathrm{st}(\mathcal{A})$, then we will have ultrafilters $\widetilde{\mathbf{y}_\delta} = \mathbf{y}_\delta \cap \widetilde{\mathcal{A}} \in \mathrm{st}(\widetilde{\mathcal{A}})$, and then $\mathbf{x}_\delta = h^*(\widetilde{\mathbf{y}_\delta})$ and $E = \{\mathbf{x}_\delta : \delta < \kappa^+\} \subseteq \mathrm{st}(\mathcal{B})$. Note that $\widetilde{\mathbf{y}_\delta} = i^*(\mathbf{y}_\delta)$, where i is inclusion, $i : \widetilde{\mathcal{A}} \hookrightarrow \mathcal{A}$. If D is dense in itself, then E will be dense in itself by continuity of i^* and h^*, *provided* that the $\mathbf{x}_\delta$ are distinct; that is, $\widetilde{\mathcal{A}}$ must be large enough to distinguish the $\mathbf{y}_\delta$. We can, for $\delta \ne \beta$, fix an element $z_{\delta,\beta} \in \mathbf{y}_\delta \backslash \mathbf{y}_\beta$ (WLOG, the $\mathbf{y}_\delta$ are all distinct). It is easy to modify the proof of Theorem 5.6 to make sure that the κ^+ elements $z_{\delta,\beta}$ all wind up in $\mathrm{ran}\, h$.

Also, the proof of Theorem 5.6 easily makes $h^*(\mathrm{st}(\widetilde{\mathcal{A}}))$ a hatset, and hence a weak P_{κ^+}-set, as in Corollary 5.8, assuming that $\mathcal{B}$ has the correct matrix.

Thus, the $\mathbf{x}_\delta \in h^*(\mathrm{st}(\widetilde{\mathcal{A}}))$ will not be limits of sets of size $\leq \kappa$ coming from outside of $h^*(\mathrm{st}(\widetilde{\mathcal{A}}))$. However, they might be limits of such sets coming from inside $h^*(\mathrm{st}(\widetilde{\mathcal{A}}))$, since the $\widetilde{\mathbf{y}_\delta}$ might fail to be weak P_{κ^+}-points in $\mathrm{st}(\widetilde{\mathcal{A}})$ even if the $\mathbf{y}_\delta$ are weak P_{κ^+}-points in $\mathrm{st}(\mathcal{A})$. To make sure that the $\widetilde{\mathbf{y}_\delta}$ are weak P_{κ^+}-points in $\mathrm{st}(\widetilde{\mathcal{A}})$, we generalize Theorem 5.6 to allow a closure operation:

Theorem 5.9 *Let $\mathcal{B}, \mathcal{G}, \mathcal{F}, \kappa, \frown, \mathbb{M}, \mathcal{A}$ be exactly as in Theorem 5.6,* except *that we do not assume $|\mathcal{A}| \leq 2^\kappa$. Let Φ be a family of 2^κ functions, with $\varphi : \mathcal{A}^\kappa \to \mathcal{A}$ for each $\varphi \in \Phi$. Then there is a subalgebra $\widetilde{\mathcal{A}} \subseteq \mathcal{A}$ and an $h : \mathcal{B} \twoheadrightarrow \widetilde{\mathcal{A}}$ such that $\widetilde{\mathcal{A}}$ is closed under all the $\varphi \in \Phi$, and such that $h(\mathcal{F}) = \{1\}$ and $h^*(\mathrm{st}(\widetilde{\mathcal{A}})) \subseteq \mathrm{st}(\mathcal{B}/\mathcal{F})$ is a $\frown$set in $\mathrm{st}(\mathcal{B}/\mathcal{G})$.*

Proof. Exactly as for Theorem 5.6, except that to ensure that $\mathrm{ran}(h)$ is closed under Φ: Before beginning the construction, list $2^\kappa \times 2^\kappa$ as $\{(f(\mu), g(\mu)) : \mu < 2^\kappa \ \&\ \mu \equiv 1 \pmod 3\}$, with each $g(\mu) \leq \mu$. At each stage $\nu < 2^\kappa$: list the closure of $\mathcal{A}_\nu$ under Φ as $\{a^\nu_\zeta : \zeta < 2^\kappa\}$. Then, at stage μ with $\mu \equiv 1 \pmod 3$, apply Lemma 5.3.2 to make sure that $a^{g(\mu)}_{f(\mu)} \in \mathcal{A}_{\mu+1}$. □

Proof of Theorem 1.14.2. Let the $\mathbf{y}_\delta$ and $z_{\delta,\beta} \in \mathbf{y}_\delta \backslash \mathbf{y}_\beta$ be as described above, with $\mathcal{A} = \mathcal{D}_{\kappa^+}$, $\mathcal{B} = \mathcal{P}(\kappa)$ and $\mathcal{F} = \mathcal{G} = \mathcal{FR}(\kappa)$; $\frown$ is any (κ^+, κ) hatfunction such that all $\frown$sets are weak P_{κ^+}-sets (see Lemma 2.15). Assume that among the functions in Φ are the functions with constant value $z_{\delta,\beta}$; then the $\widetilde{\mathbf{y}_\delta}$ will be distinct.

In addition, we assume that the $\mathbf{y}_\delta$ are weak P_{κ^+}-points in $\mathrm{st}(\mathcal{A})$ *because* they are $\frown$points; in fact, they can be κ^{++}-good (by Corollary 5.7), which implies $\frown$point (by Lemma 2.9). So, given a sequence $\vec{u} = \langle u_r : r \in [\kappa]^{<\omega}\rangle$ of elements of $\mathbf{y}_\delta$, there are elements $v_\alpha = v^\delta_\alpha(\vec{u}) \in \mathbf{y}_\delta$ such that $v_{\boxed{s}} \leq u_{\widehat{s}}$ for each non-empty $s \in [\kappa^+]^{<\omega}$. Now, $\vec{u}$ can be coded as an element of $\mathcal{A}^\kappa$ (since $|[\kappa]^{<\omega}| = \kappa$), so that we may view each v^δ_α as a function from $\mathcal{A}^\kappa \to \mathcal{A}$. Making sure that these functions are all in Φ ensures that the $\widetilde{\mathbf{y}_\delta}$ are also $\frown$points, and hence weak P_{κ^+}-points, in $\mathrm{st}(\widetilde{\mathcal{A}})$. □

We remark that Theorem 5.9 could be viewed in the context of elementary submodels (see Dow [6]). That is, we can get $\widetilde{\mathcal{A}} = \mathcal{A} \cap N$, where N is a κ-closed elementary submodel of the universe with $|N| = 2^\kappa$. Then, in proving Theorem 1.14.2, we just used closure of N under the Skolem functions needed for the argument to work.

6 Avoiding P-points

This article has emphasized *weak* P_κ-points. If κ is a measurable cardinal and $\mathbf{x} \in u(\kappa)$ is a κ-complete ultrafilter, then $\mathbf{x}$ is a P_κ-point in $\beta\kappa$. If $\mathbf{x}$ is a normal ultrafilter, then it is also a P_{κ^+}-point in $u(\kappa)$. Inductive constructions like the ones in Section 5 cannot be guaranteed to generate one of these points. However, they can be guaranteed *not* to generate one of these points. Specific instances of this have been pointed out in the literature. By [12], there are weak P-points in ω^* which are not P-points. By [11], there are good ultrafilters on κ which are countably incomplete, which is important for their use in model theory (see §6.1 of Chang and Keisler [4]). Here, we show that these constructions in [4, 11, 12] are part of a general procedure which always works.

One way to build a *non-P*-point $\mathbf{x} \in \mathrm{st}(\mathcal{B})$ is to fix some decreasing ω-sequence $b_0 > b_1 > \cdots$ and put all the b_n into $\mathbf{x}$, together with the complement of any element which is below all the b_n. The following lemma lets us integrate this remark into our constructions of weak P_κ-points.

Lemma 6.1 *Let $\mathcal{B}$ be a boolean algebra with a filter $\mathcal{F}$ and a matrix $\mathbb{M} = \langle \mathcal{M}^i : i \in I \rangle$ which is independent with respect to $\mathcal{F}$. Assume that $J \subseteq I$ and $d_j, e_j \in \mathcal{M}^j$ for $j \in J$ with $d_j \wedge e_j = 0$. Let $\mathcal{F}^\dagger$ be the filter generated by $\mathcal{F}$, all the d_j, and all the $w \in \mathcal{B}$ such that $w' \leq_{\mathcal{F}} d_j$ for infinitely many $j \in J$. Then $\mathcal{F}^\dagger$ is a proper filter and $\mathbb{M} \restriction (I \backslash J)$ is independent with respect to $\mathcal{F}^\dagger$.*

Proof. We must show that $b_{i_1} \wedge \cdots \wedge b_{i_r} \wedge c > 0$ whenever $i_1, \ldots, i_r$ are distinct elements of $I \backslash J$, each $b_{i_\ell} \in \mathcal{M}^{i_\ell}$ ($\ell = 1, \ldots, r$), and $c \in \mathcal{F}^\dagger$. By definition of $\mathcal{F}^\dagger$, we have $c \geq d_{k_1} \wedge \cdots \wedge d_{k_s} \wedge w_1 \wedge \cdots \wedge w_t$, where $k_1, \ldots, k_s$ are distinct elements of J and each $(w_\ell)' \leq_{\mathcal{F}} d_j$ for infinitely many $j \in J$. Now, choose distinct $j_1, \ldots, j_t \in J \backslash \{k_1, \ldots, k_s\}$ such that each $(w_\ell)' \leq_{\mathcal{F}} d_{j_\ell}$, so that $w_\ell \geq_{\mathcal{F}} e_{k_\ell}$, and then choose $u_\ell \in \mathcal{F}$ such that $w_\ell \geq e_{k_\ell} \wedge u_\ell$. Then $b_{i_1} \wedge \cdots \wedge b_{i_r} \wedge c \geq b_{i_1} \wedge \cdots \wedge b_{i_r} \wedge d_{k_1} \wedge \cdots \wedge d_{k_s} \wedge e_{j_1} \wedge \cdots \wedge e_{j_t} \wedge u_j \wedge \cdots \wedge u_j > 0$ by independence of $\mathbb{M}$ with respect to $\mathcal{F}$. $\square$

So, in Theorems 5.6 and 5.9, where we started with filters $\mathcal{G} \subseteq \mathcal{F}$ on $\mathcal{B}$, we now have $\mathcal{G} \subseteq \mathcal{F} \subseteq \mathcal{F}^\dagger$, and hence $\mathrm{st}(\mathcal{B}/\mathcal{F}^\dagger) \subseteq \mathrm{st}(\mathcal{B}/\mathcal{F}) \subseteq \mathrm{st}(\mathcal{B}/\mathcal{G}) \subseteq \mathrm{st}(\mathcal{B})$. As long as J is infinite in Lemma 6.1, each point of $\mathrm{st}(\mathcal{B}/\mathcal{F}^\dagger)$ will be a non-P-point in the space $\mathrm{st}(\mathcal{B}/\mathcal{F})$, and hence also in the larger spaces $\mathrm{st}(\mathcal{B}/\mathcal{G})$ and $\mathrm{st}(\mathcal{B})$. Here, $I = 2^\kappa$, and as long as $|I \backslash J| = 2^\kappa$, we can replace $\mathcal{F}$ by $\mathcal{F}^\dagger$ in the theorem to get our hatset inside $\mathrm{st}(\mathcal{B}/\mathcal{F}^\dagger)$. Thus, we have:

Theorem 6.2 *In Theorems 5.6 and 5.9, we can obtain h so that no point of the $\frown$set ($h^*(\mathrm{st}(\mathcal{A}))$ in 5.6 and $h^*(\mathrm{st}(\widetilde{\mathcal{A}}))$ in 5.9) is a P-point in $\mathrm{st}(\mathcal{B}/\mathcal{F})$.*

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