Irredundant Sets in Atomic Boolean Algebras*

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Abstract

Assuming GCH, we construct an atomic boolean algebra whose pi-weight is strictly less than the least size of a maximal irredundant family.

1 Introduction

We begin by reviewing some standard notation regarding boolean algebras. Koppelberg [4] and Monk [7, 8] contain much more information.

Notation 1.1 *In this paper, the three calligraphic letters* $\mathcal{A}, \mathcal{B}, \mathcal{C}$ *will always denote boolean algebras; in particular,* $\mathcal{A}$ *will always denote a finite-cofinite algebra. Other calligraphic letters denote subsets of boolean algebras.* b' *denotes the boolean complement of* b. $\mathcal{B} \subseteq \mathcal{C}$ *means that* $\mathcal{B}$ *is a sub-algebra of* $\mathcal{C}$*, and* $\mathcal{B} \subset \mathcal{C}$ *or* $\mathcal{B} \subsetneqq \mathcal{C}$ *means that* $\mathcal{B}$ *is a* proper *sub-algebra of* $\mathcal{C}$*. Also,* $\mathrm{st}(\mathcal{B})$ *denotes the Stone space of* $\mathcal{B}$.

The symbols $\subset$ and $\subsetneqq$ are synonymous, but we shall use $\subset$ when the properness is obvious; e.g., "let $\mathcal{B} \subset \mathcal{P}(\omega)$ be a countable sub-algebra and $\cdots\cdots$".

Some further notation is borrowed either from topology (giving properties of $\mathrm{st}(\mathcal{B})$) or from forcing (regarding $\mathcal{B}\backslash\{\mathbb{0}\}$ as a forcing poset).

Definition 1.2 *If* $\mathcal{S} \subseteq \mathcal{B}$*, then* $\mathcal{S}$ *is* dense in $\mathcal{B}$ *iff* $\forall b \in \mathcal{B}\backslash\{\mathbb{0}\}\ \exists d \in \mathcal{S}\backslash\{\mathbb{0}\}\ [d \le b]$.

In forcing, we would say that $\mathcal{S}\backslash\{\mathbb{0}\}$ is dense in $\mathcal{B}\backslash\{\mathbb{0}\}$.

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Definition 1.3 *The* pi-weight*, $\pi(\mathcal{B})$, is the least size of a dense subset $\mathcal{S} \subseteq \mathcal{B}$.*

This is the same as the topological notion $\pi(\mathrm{st}(\mathcal{B}))$.

Definition 1.4 *If $\mathcal{E} \subseteq \mathcal{B}$, then $\mathrm{sa}(\mathcal{E})$ is the sub-algebra of $\mathcal{B}$ generated by $\mathcal{E}$.*

Note that $\mathrm{sa}(\emptyset) = \{\mathbb{0}, \mathbb{1}\}$. The notation $\langle \mathcal{E} \rangle$ is more common in the literature, but we shall frequently use the angle brackets to denote sequences.

If $\mathcal{E}$ is a set of non-zero vectors in a vector space, then $\mathcal{E}$ is linearly independent iff no $a \in \mathcal{E}$ is generated from the other elements of $\mathcal{E}$; equivalently, iff no non-trivial linear combination from $\mathcal{E}$ is zero. In boolean algebras, these two notions are not equivalent, and are named, respectively, "irredundance" and "independence":

Definition 1.5 *$\mathcal{E} \subseteq \mathcal{B}$ is* irredundant *iff $a \notin \mathrm{sa}(\mathcal{E} \backslash \{a\})$ for all $a \in \mathcal{E}$.*

Definition 1.6 *For $a \in \mathcal{B}$: $a^1 = a'$ and $a^0 = a$. Then, $\mathcal{E} \subseteq \mathcal{B}$ is* independent *iff for all $n \in \omega$ and all distinct $a_0, \ldots, a_{n-1} \in \mathcal{E}$ and all $\epsilon \in {}^n2$, $\bigwedge_{i<n} a_i^{\epsilon(i)} \neq \mathbb{0}$.*

Some remarks that follow easily from the definitions: Every independent set is irredundant. If $\mathcal{E}$ is a chain and $|\mathcal{E}| \geq 2$ and $\mathbb{0}, \mathbb{1} \notin \mathcal{E}$, then $\mathcal{E}$ is irredundant but not independent. No irredundant set can contain $\mathbb{0}$ or $\mathbb{1}$ because $\mathbb{0}, \mathbb{1} \in \mathrm{sa}(\mathcal{G})$ for every $\mathcal{G}$, even if $\mathcal{G} = \emptyset$.

Irredundance and independence are similar in that they treat an element and its complement equivalently:

Lemma 1.7 *Fix $\mathcal{E} \subseteq \mathcal{B} \backslash \{\mathbb{0}, \mathbb{1}\}$. If $b, b' \in \mathcal{E}$, then $\mathcal{E}$ is neither irredundant nor independent. If $b \in \mathcal{E}$ and $b' \notin \mathcal{E}$ and $\tilde{\mathcal{E}}$ is obtained from $\mathcal{E}$ by replacing b by b', then $\mathcal{E}$ is irredundant iff $\tilde{\mathcal{E}}$ is irredundant and $\mathcal{E}$ is independent iff $\tilde{\mathcal{E}}$ is independent.*

Monk [8] defines:

Definition 1.8 $\mathrm{Irr}_{\mathrm{mm}}(\mathcal{B})$ *is the* m*inimum size of a* m*aximal irredundant subset of $\mathcal{B}$.*

The following provides a simple way to prove maximal irredundance:

Lemma 1.9 *If $\mathcal{E} \subseteq \mathcal{B}$ is irredundant and $\mathrm{sa}(\mathcal{E}) = \mathcal{B}$, then $\mathcal{E}$ is maximally irredundant in $\mathcal{B}$.*

The following provides a simple way to refute maximal irredundance. It is attributed to McKenzie in Koppelberg [4] (see Proposition 4.23):

Lemma 1.10 *Assume that* $\mathcal{E} \subseteq \mathcal{B}$*, where* $\mathcal{E}$ *is irredundant. Fix* $d \in \mathcal{B}\backslash\mathrm{sa}(\mathcal{E})$ *such that* $\forall a \in \mathrm{sa}(\mathcal{E})\, [a \le d \to a = \mathbb{0}]$*. Then* $\mathcal{E} \cup \{d\}$ *is irredundant. In particular, if* $\mathcal{E}$ *is maximally irredundant in* $\mathcal{B}$*, then* $\mathrm{sa}(\mathcal{E})$ *is dense in* $\mathcal{B}$*.*

Proof. Assume that $\mathcal{E} \cup \{d\}$ is not irredundant. Then there are distinct $a, a_1, \ldots a_n \in \mathcal{E}$ such that $a \in \mathrm{sa}\{a_1, \ldots a_n, d\}$. Then, fix $u, w \in \mathrm{sa}\{a_1, \ldots a_n\}$ such that $a = (u \wedge d) \vee (w \wedge d')$. Now, $a \wedge d' = w \wedge d'$, so $a \,\Delta\, w \le d$, so $a \,\Delta\, w = \mathbb{0}$. Then $a = w \in \mathrm{sa}\{a_1, \ldots a_n\}$, contradicting irredundance of $\mathcal{E}$. ☕

Corollary 1.11 *When* $\mathcal{B}$ *is infinite,* $\pi(\mathcal{B}) \le \mathrm{Irr_{mm}}(\mathcal{B}) \le |\mathcal{B}| \le 2^{\pi(\mathcal{B})}$*.*

Proof. For the first $\le$: If $\mathcal{E}$ is maximally irredundant in $\mathcal{B}$, then $\mathrm{sa}(\mathcal{E})$ must be infinite (since it is dense in $\mathcal{B}$), so $\pi(\mathcal{B}) \le |\mathrm{sa}(\mathcal{E})| = |\mathcal{E}|$. ☕

Note that Lemma 1.10 does not say that $\mathcal{E}$ must be dense in $\mathcal{B}$, and the first $\le$ can fail for finite $\mathcal{B}$. For example, let $\mathcal{B} = \mathcal{P}(4)$ be the 16 element boolean algebra. If $\mathcal{E} = \{a, b\}$ is an independent set (e.g., $a = \{0, 1\}$ and $b = \{1, 2\}$), then $\mathrm{sa}(\mathcal{E}) = \mathcal{B}$. So, $\mathcal{E}$ is a maximal irredundant set, showing that $\mathrm{Irr_{mm}}(\mathcal{B}) = 2$, although $\pi(\mathcal{B}) = 4$. Also, let $\mathcal{F}$ be the set of the four atoms (singletons). Then $\mathrm{sa}(\mathcal{F}) = \mathcal{B}$. So, $\mathcal{F}$ is a maximal irredundant set.

Since there can be maximal irredundant sets of different sizes in $\mathcal{B}$, there is no simple notion of "dimension" as in vector spaces. This phenomenon can occur in infinite $\mathcal{B}$ as well:

Example 1.12 *Let* $\mathcal{B} = \mathcal{P}(\kappa)$*, where* κ *is any infinite cardinal. Then* $\mathcal{B}$ *has maximal irredundant set of size* 2^κ*, but* $\pi(\mathcal{B}) = \mathrm{Irr_{mm}}(\mathcal{B}) = \kappa$*.*

Proof. Following Hausdorff [2], let $\mathcal{E}$ be an independent set of size 2^κ; then $\mathcal{E}$ is irredundant and is contained in a maximal irredundant set. To prove that $\mathrm{Irr_{mm}}(\mathcal{B}) = \kappa$: As in [8], let $\mathcal{F} = \kappa\backslash\{0\}$; that is, $\mathcal{F}$ is the set of all proper initial segments of κ. $\mathcal{F}$ is a chain, and hence irredundant. To prove maximality, fix $c \in \mathcal{P}(\kappa)\backslash\mathrm{sa}(\mathcal{F})$; we show that $\mathcal{F} \cup \{c\}$ is not irredundant. By Lemma 1.7, WLOG $0 \in c$ (otherwise, replace c by c'). Let δ be the least ordinal not in c. Then $\delta, \delta + 1 \in \mathcal{F}$ and $\delta = c \cap (\delta + 1)$, refuting irredundance. ☕

In view of examples like this, Monk [8] asks (Problem 1):

Question 1.13 *Does* $\mathrm{Irr_{mm}}(\mathcal{B}) = \pi(\mathcal{B})$ *for every infinite* $\mathcal{B}$*?*

Assuming GCH, the answer is "no":

Theorem 1.14 *If $2^{\aleph_1} = \aleph_2$, then there is an atomic boolean algebra $\mathcal{B}$ such that $\pi(\mathcal{B}) = \aleph_1 < \mathrm{Irr}_{\mathrm{mm}}(\mathcal{B})$.*

We do not know whether the hypothesis "$2^{\aleph_1} = \aleph_2$" can be eliminated here, although it can be weakened quite a bit, as we shall see from the proof of Theorem 1.14 in Section 4. This weakening (described in Theorem 3.10) is expressed in terms of some cardinals, such as $\mathfrak{b}_{\omega_1}$, $\mathfrak{d}_{\omega_1}$, etc., that are obtained by replacing ω by ω_1 in the definitions of the standard cardinal characteristics of the continuum, such as $\mathfrak{b}$, $\mathfrak{d}$, etc. These cardinals are discussed further in Section 3, which also uses them to give some lower bounds to the size of a $\mathcal{B}$ that can possibly satisfy Theorem 1.14. Section 2 contains some preliminary observations on atomic boolean algebras.

2 Remarks on Atomic Boolean Algebras

We are trying to find an atomic $\mathcal{B}$ that answers Monk's Question 1.13 in the negative; that is, such that $\pi(\mathcal{B}) < \mathrm{Irr}_{\mathrm{mm}}(\mathcal{B})$. Observe first:

Lemma 2.1 *If $\mathcal{B}$ is infinite and atomic and $\kappa = \pi(\mathcal{B})$, then κ is the number of atoms, and κ is infinite, and $\mathcal{B} \cong \widetilde{\mathcal{B}}$, where $\mathcal{A} \subseteq \widetilde{\mathcal{B}} \subseteq \mathcal{P}(\kappa)$, and $\mathcal{A}$ is the finite-cofinite algebra on κ.*

So, we need only consider $\mathcal{B}$ with $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{P}(\kappa)$. The proof of Example 1.12 generalizes immediately to:

Lemma 2.2 *Assume that $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{P}(\kappa)$, where κ is any infinite cardinal and $\mathcal{A}$ is the finite-cofinite algebra and $\kappa \subseteq \mathcal{B}$ (i.e., $\mathcal{B}$ contains all initial segments of κ). Then $\pi(\mathcal{B}) = \mathrm{Irr}_{\mathrm{mm}}(\mathcal{B}) = \kappa$.*

When $\kappa = \omega$, $\mathcal{B}$ must contain all initial segments, so the two lemmas imply:

Lemma 2.3 *If $\mathcal{B}$ is atomic and $\pi(\mathcal{B}) = \aleph_0$ then $\mathrm{Irr}_{\mathrm{mm}}(\mathcal{B}) = \aleph_0$.*

So, we shall focus here on obtaining our $\mathcal{B}$ with $\kappa = \omega_1$. Then, note that in Lemma 2.2, a club of initial segments suffices:

Lemma 2.4 *Assume that $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{P}(\omega_1)$, where $\mathcal{A}$ is the finite-cofinite algebra and $C \subseteq \mathcal{B}$ for some club $C \subseteq \omega_1$. Then $\pi(\mathcal{B}) = \mathrm{Irr}_{\mathrm{mm}}(\mathcal{B}) = \aleph_1$.*

Proof. Shrinking C and adding in 0 if necessary, we may assume that C, in its increasing enumeration, is $\{\delta_{\omega\cdot\alpha} : \alpha < \omega_1\}$, where $\delta_0 = 0$ and $\delta_{\omega\cdot(\alpha+1)} \geq \delta_{\omega\cdot\alpha} + \omega$ for each α. Then each set-theoretic difference $\delta_{\omega\cdot(\alpha+1)} \backslash \delta_{\omega\cdot\alpha}$ is countably infinite, so we can enumerate this set as $\{\delta_{\omega\cdot\alpha+\ell} : 0 < \ell < \omega\}$. We now have a 1-1 (but not increasing) enumeration of ω_1 as $\{\delta_\xi : \xi < \omega_1\}$, and, using $\mathcal{A} \subseteq \mathcal{B}$, each initial segment $\{\delta_\xi : \xi < \eta\} \in \mathcal{B}$. We can now apply Lemma 2.2 to the isomorphic copy of $\mathcal{B}$ obtained via the bijection $\xi \mapsto \delta_\xi$.

We shall avoid this issue by constructing a $\mathcal{B}$ with $\pi(\mathcal{B}) < \mathrm{Irr}_{\mathrm{mm}}(\mathcal{B})$ so that $\mathcal{B}$ contains no countably infinite sets at all; we shall call such $\mathcal{B}$ *dichotomous*:

Definition 2.5 *$\mathcal{A}$ always denotes the finite-cofinite algebra on ω_1. $\mathcal{B}$ is* dichotomous *iff $\mathcal{B}$ is a sub-algebra of $\mathcal{P}(\omega_1)$ and $\mathcal{A} \subseteq \mathcal{B}$ and $\forall b \in \mathcal{B}\,[b \in \mathcal{A}$ or $|b| = |\omega_1 \backslash b| = \aleph_1]$.*

Note that $\forall b \in \mathcal{B}\,[b \in \mathcal{A}$ or $|b| = |\omega_1 \backslash b| = \aleph_1]$ is equivalent to $\forall b \in \mathcal{B}\,[|b| \neq \aleph_0]$.

To get an easy example of a dichotomous $\mathcal{B}$ of size $2^{\aleph_1}$: Following Hausdorff [2], let the sets $J_\alpha \subset \omega_1$ for $\alpha < 2^{\aleph_1}$ be independent in the sense that all non-trivial boolean combinations are uncountable (not just non-empty). Then $\mathcal{B} = \mathrm{sa}(\mathcal{A} \cup \{J_\alpha : \alpha < 2^{\aleph_1}\})$ is dichotomous. However, it is quite possible that $\mathrm{Irr}_{\mathrm{mm}}(\mathcal{B}) = \aleph_1$ because the following lemma may apply. This goes in the opposite direction from Lemma 2.4:

Lemma 2.6 *Assume that $\mathcal{A} \subseteq \mathcal{B} \subset \mathcal{P}(\omega_1)$, and assume that $\omega_1 = \bigcup\{S_\xi : \xi < \omega_1\}$, where the S_ξ are disjoint countably infinite sets and*

$$\forall b \in \mathcal{B} \backslash \mathcal{A}\ \exists \xi\ [S_\xi \cap b \neq \emptyset\ \&\ S_\xi \backslash b \neq \emptyset]\quad . \tag{$*$}$$

Then $\mathrm{Irr}_{\mathrm{mm}}(\mathcal{B}) = \aleph_1$.

Proof. List each S_ξ as $\{\sigma_\xi^\ell : \ell \in \omega\}$. Then, let $\mathcal{E}$ be the set of all $\{\sigma_\xi^0, \ldots, \sigma_\xi^\ell\}$ such that $\xi < \omega_1$ and $\ell < \omega$. Then $\mathrm{sa}(\mathcal{E}) = \mathcal{A}$, so $\mathcal{E}$ is maximally irredundant in $\mathcal{A}$.

Also, $\mathcal{E}$ remains maximal in $\mathcal{B}$. Proof: fix $b \in \mathcal{B} \backslash \mathcal{A}$ and then fix ξ as in $(*)$. WLOG $\sigma_\xi^0 \in b$ (otherwise, swap b/b'). Then let ℓ be least such that $\sigma_\xi^\ell \notin b$; so, $\{\sigma_\xi^0, \ldots, \sigma_\xi^{\ell-1}\} \subseteq b$. Then $\{\sigma_\xi^0, \ldots, \sigma_\xi^{\ell-1}\} = \{\sigma_\xi^0, \ldots, \sigma_\xi^\ell\} \cap b$, so $\mathcal{E} \cup \{b\}$ is not irredundant.

To see how this lemma might apply to $\mathcal{B} = \mathrm{sa}(\mathcal{A} \cup \{J_\alpha : \alpha < 2^{\aleph_1}\})$: Start with any partition $\{S_\xi : \xi < \omega_1\}$. Choose any T_ξ with $\emptyset \subsetneqq T_\xi \subsetneqq S_\xi$. Then, choose the independent J_α so that each $J_\alpha \cap S_\xi$ is either T_ξ or $\emptyset$.

Assuming that $2^{\aleph_1} = \aleph_2$, our $\mathcal{B}$ satisfying Theorem 1.14 will in fact be dichotomous and of the form $\mathrm{sa}(\mathcal{A} \cup \{J_\alpha : \alpha < \omega_2\})$, where the J_α are independent, but the J_α

will be chosen inductively, in ω_2 steps to avoid situations such as the one described in Lemma 2.6. The next section defines some cardinals below $2^{\aleph_1}$ that will be useful both in describing properties of clubs and in deriving a version of Theorem 1.14 that applies in some models of $2^{\aleph_1} > \aleph_2$.

3 Some Small Cardinals

We begin with some remarks on club subsets of ω_1.

Definition 3.1 *Given a club $C \subseteq \omega_1$, we define the* associated partition *of ω_1 into $\aleph_1$ non-empty countable sets, which we shall call the C–*blocks*, and label them as S_ξ^C (or, just S_ξ) for $\xi < \omega_1$. If $0 \in C$, write $C = \{\gamma_\xi : \xi < \omega_1\}$ in increasing enumeration; then $S_\xi = [\gamma_\xi, \gamma_{\xi+1})$. If $0 \notin C$, let $S_\xi^C = S_\xi^{C \cup \{0\}}$.*

Note that if we are given sets S_ξ satisfying the hypotheses of Lemma 2.6, then there is a club C such that each S_ξ meets only one C–block. Then, $\{S_\xi^C : \xi < \omega_1\}$ also will satisfy the hypotheses of Lemma 2.6.

For our purposes, "thinner" clubs will yield "better" partitions. As usual, for subsets of ω_1, $D \subseteq^* C$ means that $D \backslash C$ is countable. Then observe

Lemma 3.2 *If $D \subseteq^* C \subseteq \omega_1$ and D, C are clubs, then all but countably many D–blocks are unions of C–blocks.*

Given $\aleph_1$ clubs C_α, for $\alpha < \omega_1$, there is always a club D such that $D \subseteq^* C_\alpha$ for all α. Whether this holds for more than $\aleph_1$ clubs depends on the model of set theory one is in. The basic properties here are controlled by the cardinals $\mathfrak{b}_{\omega_1}$ and $\mathfrak{d}_{\omega_1}$.

Cardinal characteristics of the continuum (e.g., $\mathfrak{b}$, $\mathfrak{d}$, etc.) are well-known, and are discussed in set theory texts (e.g., [3, 5]), and in much more detail in the paper of Blass [1]. In analogy with $\mathfrak{b}$ and $\mathfrak{d}$, we use $\mathfrak{b}_{\omega_1}$ to denote the least size of an unbounded family in ${\omega_1}^{\omega_1}$, while $\mathfrak{d}_{\omega_1}$ denotes the least size of a dominating family. Then $\mathfrak{b}_{\omega_1}$ is regular and $\aleph_2 \le \mathfrak{b}_{\omega_1} \le \mathfrak{d}_{\omega_1} \le 2^{\aleph_1}$. Furthermore, statements such as $\mathfrak{b}_{\omega_1} = \aleph_2$ and $\mathfrak{b}_{\omega_1} = 2^{\aleph_1}$ and $\aleph_2 < \mathfrak{b}_{\omega_1} < 2^{\aleph_1}$ are consistent with CH plus $2^{\aleph_1}$ being arbitrarily large; see [5] §V.5 for an exposition of these matters. For our purposes here, it will often be useful to rephrase $\mathfrak{b}_{\omega_1}$ and $\mathfrak{d}_{\omega_1}$ in terms of clubs:

Lemma 3.3 *Let $\mathfrak{C}$ be the set of all club subsets of ω_1. Then $\mathfrak{d}_{\omega_1}$ is the least κ such that (a) holds and $\mathfrak{b}_{\omega_1}$ is the least κ such that (b) holds:*

$$\begin{array}{ll} a. & \exists \mathfrak{D} \subseteq \mathfrak{C} \, [|\mathfrak{D}| = \kappa \ \& \ \forall C \in \mathfrak{C} \, \exists D \in \mathfrak{D} \, [D \subseteq^* C]] \\ b. & \exists \mathfrak{B} \subseteq \mathfrak{C} \, [|\mathfrak{B}| = \kappa \ \& \ \neg \exists C \in \mathfrak{C} \, \forall D \in \mathfrak{B} \, [C \subseteq^* D]] \end{array} .$$

The following definition relates clubs to the proof of Lemma 2.6. As before, $\mathcal{A}$ always denotes the finite-cofinite algebra on ω_1.

Definition 3.4 *A club $C \subset \omega_1$ is* nice *iff all $S_\xi = S_\xi^C$ are infinite. If C is nice, then $\mathcal{E} \subset \mathcal{A}$ is* induced by *C iff $\mathcal{E}$ is obtained from the S_ξ as in the proof of Lemma 2.6. That is, list each S_ξ as $\{\sigma_\xi^\ell : \ell \in \omega\}$; then, $\mathcal{E} = \{\{\sigma_\xi^0, \ldots, \sigma_\xi^\ell\} : \xi < \omega_1 \,\&\, \ell < \omega\}$.*

Of course, $\mathcal{E}$ is not uniquely defined from C, since $\mathcal{E}$ depends on a choice of an enumeration of each S_ξ. Note that $\mathcal{E}$ must be maximally irredundant in $\mathcal{A}$. Whether $\mathcal{E}$ remains maximal in some $\mathcal{B} \supseteq \mathcal{A}$ will depend on $\mathcal{B}$.

For dichotomous $\mathcal{B}$, the hypothesis $(*)$ of Lemma 2.6, when C is nice and $S_\xi = S_\xi^C$, is equivalent to saying that no $b \in \mathcal{B}$ is *blockish* for C:

Definition 3.5 *If $C \subseteq \omega_1$ is a club, then $b \subseteq \omega_1$ is* blockish *for C iff both b and b' are unions of $\aleph_1$ C–blocks.*

We next consider the ω_1 version of the *reaping number* $\mathfrak{r}$. This $\mathfrak{r}$ is less well-known than $\mathfrak{b}$ and $\mathfrak{d}$, but it is discussed in Blass [1].

Definition 3.6 *If $\mathcal{R} \subseteq [\omega_1]^{\aleph_1}$, then $T \subseteq \omega_1$* splits *$\mathcal{R}$ iff $|X \cap T| = |X \setminus T| = \aleph_1$ for all $X \in \mathcal{R}$. Then, $\mathfrak{r}_{\omega_1}$ is the least cardinality of an $\mathcal{R} \subseteq [\omega_1]^{\aleph_1}$ such that no $T \subseteq \omega_1$ splits $\mathcal{R}$.*

Related to this, one might be tempted to define a *strong reaping number*:

Definition 3.7 *If $\mathcal{R} \subseteq [\omega_1]^{\aleph_1}$, then the nice (Definition 3.4) club C* strongly splits *$\mathcal{R}$ iff every set T that is blockish for C splits $\mathcal{R}$. Then, $\hat{\mathfrak{r}}_{\omega_1}$ is the least cardinality of an $\mathcal{R} \subseteq [\omega_1]^{\aleph_1}$ such that no nice club strongly splits $\mathcal{R}$.*

Some simple remarks: The nice club C strongly splits $\mathcal{R}$ iff for each $X \in \mathcal{R}$, all but countably many C–blocks meet X. Also, if $C \subseteq^* D$ and D strongly splits $\mathcal{R}$, then C strongly splits $\mathcal{R}$. Actually, $\hat{\mathfrak{r}}_{\omega_1} = \mathfrak{b}_{\omega_1}$ (although the concept of $\hat{\mathfrak{r}}_{\omega_1}$ will be useful); the cardinals that we have defined are related by the following inequalities:

Lemma 3.8 $\aleph_2 \leq \mathfrak{b}_{\omega_1} = \hat{\mathfrak{r}}_{\omega_1} \leq \mathfrak{r}_{\omega_1} \leq 2^{\aleph_1}$ *and* $\aleph_2 \leq \mathfrak{b}_{\omega_1} = \hat{\mathfrak{r}}_{\omega_1} \leq \mathfrak{d}_{\omega_1} \leq 2^{\aleph_1}$.

Proof. For $\mathfrak{b}_{\omega_1} \leq \hat{\mathfrak{r}}_{\omega_1}$: For each $X \in \mathcal{R}$, choose a nice club C_X such that all C_X–blocks meet X. If $|\mathcal{R}| < \mathfrak{b}_{\omega_1}$, then there is a nice club C such that $C \subseteq^* C_X$ for each $X \in \mathcal{R}$.

For $\hat{\mathfrak{r}}_{\omega_1} \leq \mathfrak{b}_{\omega_1}$: Fix $\kappa < \hat{\mathfrak{r}}_{\omega_1}$; we shall show that $\kappa < \mathfrak{b}_{\omega_1}$. So, let C_α, for $\alpha < \kappa$ be clubs. Then, fix a nice club D that strongly splits $\{C_\alpha : \alpha < \kappa\}$; so, for each α, all but countably many D–blocks meet C_α. But then $\tilde{D} \subseteq^* C_\alpha$ for each α, where $\tilde{D}$ is the set of limit points of D.

It is not clear which of the many independence results involving these cardinals on ω go through for the ω_1 versions. Of course, all these cardinals are $\aleph_2$ if $2^{\aleph_1} = \aleph_2$. Also, the following is easy by standard forcing arguments:

Lemma 3.9 *In V, assume that* GCH *holds and $\kappa > \aleph_2$ is regular. Then there are cardinal-preserving forcing extensions $V[G]$ satisfying each of the following:*

1. $\aleph_2 = \mathfrak{b}_{\omega_1} < \mathfrak{d}_{\omega_1} = \mathfrak{r}_{\omega_1} = 2^{\aleph_1} = \kappa$.
2. $\aleph_2 < \mathfrak{b}_{\omega_1} = \mathfrak{d}_{\omega_1} = \mathfrak{r}_{\omega_1} = 2^{\aleph_1} = \kappa$.
3. $\aleph_2 = \mathfrak{b}_{\omega_1} = \mathfrak{d}_{\omega_1} < \mathfrak{r}_{\omega_1} = 2^{\aleph_1} = \kappa$.

Proof. For (1), use the "countable Cohen" forcing $\mathrm{Fn}_{\aleph_1}(\kappa, 2)$. For (2), use a countable support iteration to get $V[G]$ satisfying Baumgartner's Axiom (see [5], §V.5).

For (3), just use the standard Cohen forcing $\mathrm{Fn}(\kappa, 2)$. Then in $V[G]$, $\mathfrak{r}_{\omega_1} = 2^{\aleph_1} = \kappa$ as in (1), but $\mathfrak{b}_{\omega_1} = \mathfrak{d}_{\omega_1} = \aleph_2$ because ccc forcing doesn't change $\mathfrak{b}_{\omega_1}$ or $\mathfrak{d}_{\omega_1}$.

With this proof, $V[G] \models$ CH in (1)(2), but $V[G] \models 2^{\aleph_0} = \kappa$ in (3).

We do not know whether $\mathfrak{r}_{\omega_1} \geq \mathfrak{d}_{\omega_1}$ is a ZFC theorem. For the standard ω version, $\mathfrak{r} < \mathfrak{d}$ holds in the Miller real model (see [1], §11.9), but it's not clear how to make that construction work on ω_1. The following theorem will be proved in Section 4:

Theorem 3.10 *If $\mathfrak{r}_{\omega_1} \geq \mathfrak{d}_{\omega_1}$ then there is a dichotomous $\mathcal{B}$ such that $\mathcal{A} \subset \mathcal{B} \subset \mathcal{P}(\omega_1)$ and $|\mathcal{B}| = \mathfrak{d}_{\omega_1}$ and $\mathrm{Irr}_{\mathrm{mm}}(\mathcal{B}) \geq \mathfrak{b}_{\omega_1}$.*

Theorem 1.14 is an immediate consequence of this, and if $\mathfrak{r}_{\omega_1} \geq \mathfrak{d}_{\omega_1}$ is a ZFC theorem, then we would have a ZFC example answering Question 1.13 in the negative. The following shows that one cannot get $|\mathcal{B}| < \mathfrak{d}_{\omega_1}$ in Theorem 3.10:

Lemma 3.11 *Assume that $\mathcal{A} \subseteq \mathcal{B} \subset \mathcal{P}(\omega_1)$ and $\mathcal{B}$ is dichotomous and $|\mathcal{B}| < \mathfrak{d}_{\omega_1}$. Then $\mathrm{Irr}_{\mathrm{mm}}(\mathcal{B}) = \aleph_1$.*

Proof. For each $b \in \mathcal{B} \backslash \mathcal{A}$, let $f_b : \omega_1 \to \omega_1$ be such that for all ξ: $\xi < f_b(\xi)$ and $b \cap (\xi, f_b(\xi)) \neq \emptyset$ and $b' \cap (\xi, f_b(\xi)) \neq \emptyset$. Now, using $|\mathcal{B}| < \mathfrak{d}_{\omega_1}$, fix $g : \omega_1 \to \omega_1$ that is not dominated by the f_b; that is, for all $b \in \mathcal{B} \backslash \mathcal{A}$, $Y_b := \{\xi : f_b(\xi) < g(\xi)\}$ is uncountable. Let C be a nice club of fixedpoints of g; that is, each $\delta \in C$ is a limit ordinal and $g(\xi) < \delta$ whenever $\xi < \delta$.

Now, fix $b \in \mathcal{B} \backslash \mathcal{A}$. Then fix $\xi \in Y_b$ with $\xi > \min(C)$. So, we have $\xi < f_b(\xi) < g(\xi)$. Let $\delta = \min\{\mu \in C : \mu \geq g(\xi)\}$. Then $\xi < \delta \to g(\xi) < \delta$, so $\xi < f_b(\xi) < g(\xi) < \delta$.

For each $\gamma < \delta$ such that $\gamma \in C$: $\gamma < g(\xi)$ (by definition of δ), so $\gamma \leq \xi$ (since $\xi < \gamma \to g(\xi) < \gamma$). Fixing $\gamma = \sup(C \cap \delta)$, we have $\gamma \leq \xi < f_b(\xi) < g(\xi) < \delta$. So, ξ and $f_b(\xi)$ are in the same C–block, say $S^C_\eta = [\gamma, \delta)$, so by our choice of f_b, $b \cap S^C_\eta \neq \emptyset$ and $b' \cap S^C_\eta \neq \emptyset$. Then $(*)$ of Lemma 2.6 holds, so $\mathrm{Irr}_{\mathrm{mm}}(\mathcal{B}) = \aleph_1$. ☕

Assuming CH, we can remove the hypothesis that $\mathcal{B}$ is dichotomous:

Lemma 3.12 *Assume* CH*, and let $\mathcal{B}$ be atomic with $\pi(\mathcal{B}) = \aleph_1$ and $|\mathcal{B}| < \mathfrak{d}_{\omega_1}$. Then* $\mathrm{Irr}_{\mathrm{mm}}(\mathcal{B}) = \aleph_1$.

Proof. WLOG, $\mathcal{A} \subseteq \mathcal{A}^* \subseteq \mathcal{B} \subset \mathcal{P}(\omega_1)$, where $\mathcal{A}^*$ is the set of all $b \in \mathcal{B}$ such that b or b' is countable. Then $|\mathcal{A}^*| = \aleph_1$ by CH.

Apply the proof of Lemma 3.11, but just using f_b for $b \in \mathcal{B} \backslash \mathcal{A}^*$. This yields an $\mathcal{E} \subset \mathcal{A}$ such that $\mathcal{E}$ is maximally irredundant in $\mathcal{A}$ and $\mathcal{E} \cup \{b\}$ is not irredundant for all $b \in \mathcal{B} \backslash \mathcal{A}^*$. Let $\mathcal{E}^* \subseteq \mathcal{A}^*$ be maximally irredundant in $\mathcal{A}^*$ with $\mathcal{E}^* \supseteq \mathcal{E}$. Then $\mathcal{E}^*$ is maximally irredundant in $\mathcal{B}$ and $|\mathcal{E}^*| = \aleph_1$. ☕

4 A Very Blockish Boolean Algebra

Here we shall prove Theorem 3.10. Our only use of the assumption $\mathfrak{r}_{\omega_1} \geq \mathfrak{d}_{\omega_1}$ will be to prove Lemma 4.2 below. First, a remark on preserving dichotomicity:

Lemma 4.1 *Assume that $\mathcal{A} \subseteq \mathcal{B} \subset \mathcal{P}(\omega_1)$, $\mathcal{B}$ is dichotomous, and $b \subseteq \omega_1$. Then TFAE:*

1. $\mathrm{sa}(\mathcal{B} \cup \{b\})$ *is dichotomous.*
2. $|u \cap b| \neq \aleph_0$ *and* $|u \cap b'| \neq \aleph_0$ *for all* $u \in \mathcal{B}$.

Proof. (1) $\to$ (2) is immediate from the definition of "dichotomous". Conversely, if (1) is false, fix $a = (u \cap b) \cup (w \cap b') \in \mathrm{sa}(\mathcal{B} \cup \{b\})$ such that $|a| = \aleph_0$, where $u, w \in \mathcal{B}$. Then at least one of $(u \cap b)$ and $(w \cap b')$ has size $\aleph_0$, so (2) is false. ☕

Note that (2) holds whenever $|b| = \aleph_1$ and b splits $\mathcal{B} \cap [\omega_1]^{\aleph_1}$.

Lemma 4.2 *Assume that $\kappa := \mathfrak{d}_{\omega_1} \leq \mathfrak{r}_{\omega_1}$. Then there is a $\mathcal{B}$ such that $\mathcal{A} \subset \mathcal{B} \subset \mathcal{P}(\omega_1)$, and $\mathcal{B}$ is dichotomous, and $|\mathcal{B}| = \kappa$, and*

$$\textit{For all clubs } C \subset \omega_1 \textit{ there is a } b \in \mathcal{B} \textit{ such that } b \textit{ is blockish for } C. \qquad (\dagger)$$

Proof. Let $C_\mu \subset \omega_1$ for $\mu < \kappa$ be nice (Definition 3.4) clubs such that for every club $C \subset \omega_1$ there is a μ with $C_\mu \subseteq C$.

Now, build a chain $\langle \mathcal{B}_\mu : \mu \leq \kappa \rangle$, where $\mathcal{A} \subseteq \mathcal{B}_\mu \subseteq \mathcal{P}(\omega_1)$ and $\mu \leq \nu \to \mathcal{B}_\mu \subseteq \mathcal{B}_\nu$ and all $\mathcal{B}_\mu$ are dichotomous and $|\mathcal{B}_\mu| = \max(|\mu|, \aleph_1)$. Let $\mathcal{B}_0 = \mathcal{A}$, and take unions at limits. Choose $\mathcal{B}_{\mu+1} \supseteq \mathcal{B}_\mu$ so that $\mathcal{B}_{\mu+1} = \mathrm{sa}(\mathcal{B}_\mu \cup \{J_\mu\})$, where J_μ is blockish for C_μ. Assuming that this can be done, setting $\mathcal{B} = \mathcal{B}_\kappa$ satisfies the lemma.

Fix μ; we show that an appropriate $J = J_\mu$ can be chosen: J will be blockish for C_μ and $|u \cap J| = |u \cap J'| = \aleph_1$ for all infinite (= uncountable) $u \in \mathcal{B}_\mu$. Then, we can simply apply Lemma 4.1.

Let $S_\xi = S_\xi^{C_\mu}$; these sets are disjoint and countably infinite. For $u \in \mathcal{B}_\mu \cap [\omega_1]^{\aleph_1}$, let $\hat{u} = \{\xi < \omega_1 : u \cap S_\xi \neq \emptyset\}$. Then $|\hat{u}| = \aleph_1$. Since $|\mathcal{B}_\mu| < \mathfrak{r}_{\omega_1}$, fix $T \subset \omega_1$ such that T splits $\{\hat{u} : u \in \mathcal{B}_\mu \cap [\omega_1]^{\aleph_1}\}$. Then, let $J = \bigcup\{S_\xi : \xi \in T\}$. Then $|u \cap J| = |u \cap J'| = \aleph_1$ for all $u \in \mathcal{B}_\mu \cap [\omega_1]^{\aleph_1}$. ☕

Lemma 4.5 below shows (in ZFC) that any $\mathcal{B}$ satisfying (†) also satisfies Theorem 3.10 — that is, $\mathrm{Irr}_{\mathrm{mm}}(\mathcal{B}) \geq \mathfrak{b}_{\omega_1}$. We remark that (†) implies that for all clubs C, the S_ξ^C fail to satisfy $(*)$ of Lemma 2.6. But that alone proves nothing, since possibly $\mathrm{Irr}_{\mathrm{mm}}(\mathcal{B}) = \aleph_1$ via some $\mathcal{E}$ that is not at all related to families induced by clubs (Definition 3.4). But our argument will in fact show (Lemma 4.4) that such families are all that we need to consider.

We remark that the J_μ used in the proof of Lemma 4.2 are independent in the sense that all non-trivial finite boolean combinations are uncountable; this is easily proved using the fact that $|u \cap J_\mu| = |u \setminus J_\mu| = \aleph_1$ for all infinite $u \in \mathrm{sa}\{J_\nu : \nu < \mu\}$. But, as remarked above (see end of Section 2), independence alone is not enough to prove $\mathrm{Irr}_{\mathrm{mm}}(\mathcal{B}) > \aleph_1$.

Lemma 4.3 *Assume that $\mathcal{A} \subseteq \mathcal{B} \subset \mathcal{P}(\omega_1)$ and $\mathcal{B}$ is dichotomous and $|\mathcal{B}| < \mathfrak{b}_{\omega_1}$. In addition, assume that $\mathcal{E} \subseteq \mathcal{B}$ and $\mathcal{E}$ is irredundant. Then there is a nice club D such that for any c that is blockish for D:*

1. $|c \cap b| = |c' \cap b| = \aleph_1$ *for all infinite* $b \in \mathcal{B}$.
2. $c \notin \mathcal{B}$. 3. $\mathrm{sa}(\mathcal{B} \cup \{c\})$ *is dichotomous.* 4. $\mathcal{E} \cup \{c\}$ *is irredundant.*

Proof. Using $|\mathcal{B}| < \mathfrak{b}_{\omega_1} = \hat{\mathfrak{r}}_{\omega_1}$ (Lemma 3.8), fix a nice club C such that (1) holds for every c that is blockish for C. Then, for such c, (2) holds (setting $b = c$) and (3) holds by Lemma 4.1. Now, we cannot simply let $D = C$, since we have not used $\mathcal{E}$ yet; for example, it is quite possible that $\mathcal{E}$ contains some $\{\alpha\}$ and $\{\alpha, \beta\}$ and there is a c that is blockish for C such that $c \cap \{\alpha, \beta\} = \{\alpha\}$, so that $\mathcal{E} \cup \{c\}$ is not irredundant.

But, our proof will replace C by a thinner club D obtained via a chain of elementary submodels.

We recall some standard terminology on elementary submodels, following the exposition in [5] §III.8: Fix a suitably large regular θ. Then, a *nice chain of elementary submodels of* $H(\theta)$ is a sequence $\langle M_\xi : \xi < \omega_1 \rangle$ such that $M_0 = \emptyset$, and $M_\xi \prec H(\theta)$ for $\xi \neq 0$, and all M_ξ are countable, and $\xi < \eta \to M_\xi \in M_\eta$ & $M_\xi \subset M_\eta$, and $M_\eta = \bigcup_{\xi<\eta} M_\xi$ for limit η. For $x \in \bigcup_\xi M_\xi$, $\mathrm{ht}(x)$ (the *height* of x) denotes the ξ such that $x \in M_{\xi+1} \backslash M_\xi$. Given such a chain, let $\gamma_\xi = M_\xi \cap \omega_1 \in \omega_1$; so $\gamma_0 = 0$. The *associated club* is $D = \{\gamma_\xi : \xi < \omega_1\}$. If $S_\xi = [\gamma_\xi, \gamma_{\xi+1}) = \{\delta : \mathrm{ht}(\delta) = \xi\}$, then these S_ξ are precisely the S^D_ξ described in Definition 3.1.

We shall use such a chain, with $C \in M_1$. This will ensure that $D \subset C \cup \{0\}$. We also assume that $\mathcal{E} \in M_1$.

Let c be blockish for D (and hence for C). Then $c \notin \mathcal{B}$, so $c \notin \mathcal{E}$. WLOG, $\mathcal{E}$ is maximally irredundant in $\mathcal{B}$; if not, we can replace $\mathcal{E}$ by some maximally irredundant $\tilde{\mathcal{E}} \supset \mathcal{E}$ such that $\tilde{\mathcal{E}} \in M_1$.

Before proving irredundance of $\mathcal{E} \cup \{c\}$, we introduce some notation. For each $\delta \in \omega_1$ and each $e \in \mathcal{E}$, let $h(\delta, e)$ be the *smallest finite* $r \in \mathrm{sa}(\mathcal{E}\backslash\{e\})$ such that $\delta \in r$; if there is no such finite r, let $h(\delta, e) = \infty$. Maximality of $\mathcal{E}$ plus Lemma 1.10 implies that $\{\delta\} \in \mathrm{sa}(\mathcal{E})$, and hence $\{\delta\} \in \mathrm{sa}(\mathcal{W})$ for some finite $\mathcal{W} \subset \mathcal{E}$. Then $h(\delta, e) = \{\delta\} \neq \infty$ for all $e \in \mathcal{E}\backslash\mathcal{W}$. Observe that

$$h(\delta, e) \neq \infty \text{ \& } h(\varepsilon, e) \neq \infty \ \to \ h(\delta, e) = h(\varepsilon, e) \text{ or } h(\delta, e) \cap h(\varepsilon, e) = \emptyset \ . \qquad (*)$$

To prove $(*)$, use the definition of $h(\delta, e)$ as "the *smallest* $r \cdots$": If $\delta \notin h(\varepsilon, e)$, then $h(\delta, e) \cap h(\varepsilon, e) = \emptyset$ (otherwise one could replace $h(\delta, e)$ by the smaller $h(\delta, e) \backslash h(\varepsilon, e)$). If $\delta \in h(\varepsilon, e)$ and $\varepsilon \in h(\delta, e)$, then $h(\delta, e) = h(\varepsilon, e)$ (otherwise one could replace both $h(\delta, e)$ and $h(\varepsilon, e)$ by the smaller $h(\delta, e) \cap h(\varepsilon, e)$).

Now, assume that $\mathcal{E} \cup \{c\}$ is not irredundant. Then, fix $a \in \mathcal{E}$ such that $a \in \mathrm{sa}((\mathcal{E} \setminus \{a\}) \cup \{c\})$. Then, fix $u, w \in \mathrm{sa}(\mathcal{E}\backslash\{a\})$ such that $a = (u \cap c) \cup (w \cap c')$. Then $u \cap w \subseteq a \subseteq u \cup w$. Let $s = (u \cup w) \setminus (u \cap w) = u \,\Delta\, w$. Then $s \in \mathrm{sa}(\mathcal{E}\backslash\{a\})$. Note that s is finite. To prove this, use (1) four times, plus the fact that $u, w, a \in \mathcal{B}$:

$(w\backslash a) \cap c' = \emptyset$ so $w\backslash a$ is finite.
$(u\backslash a) \cap c = \emptyset$ so $u\backslash a$ is finite.
$((w\backslash u) \setminus (w\backslash a)) \cap c = \emptyset$ so $(w\backslash u) \setminus (w\backslash a)$ is finite so $(w\backslash u)$ is finite
$((u\backslash w) \setminus (u\backslash a)) \cap c' = \emptyset$ so $(u\backslash w) \setminus (u\backslash a)$ is finite so $(u\backslash w)$ is finite

Then, $s = (w\backslash u) \cup (u\backslash w)$ is finite.

For $\delta \in s$, $h(\delta, a) \neq \infty$ because $h(\delta, a) \subseteq s$. For $\delta \in s$ and $\xi < \omega_1$, $\delta \in M_\xi \leftrightarrow h(\delta, a) \in M_\xi$ (hence $\mathrm{ht}(h(\delta, a)) = \mathrm{ht}(\delta)$). Proof: The $\leftarrow$ direction is clear because

$\delta \in h(\delta, a)$ and $h(\delta, a)$ is finite. For the $\to$ direction, there are two cases: If $a \in M_\xi$, use $M_\xi \prec H(\theta)$. If $a \notin M_\xi$, use $\mathcal{E} \in M_\xi \prec H(\theta)$ to get a finite $\mathcal{W} \in M_\xi$ such that $\mathcal{W} \subset \mathcal{E}$ and $\{\delta\} \in \mathrm{sa}(\mathcal{W})$. Then $a \notin \mathcal{W}$ so $h(\delta, a) = \{\delta\}$.

Let $t = \bigcup\{h(\delta, a) : \delta \in s \cap c\}$. Then $s \cap c \subseteq t \subseteq s$. Also, this is a finite union, so $t \in \mathrm{sa}(\mathcal{E}\backslash\{a\})$. But actually, $t = s \cap c$: If this fails, then fix $\varepsilon \in t\backslash c$. Then $\varepsilon \in h(\delta, a)$ for some $\delta \in s \cap c$. Applying $(*)$, $h(\delta, a) = h(\varepsilon, a)$, and so $\mathrm{ht}(\delta) = \mathrm{ht}(h(\delta, a)) = \mathrm{ht}(h(\varepsilon, a)) = \mathrm{ht}(\varepsilon)$. But $\delta \in c$ and $\varepsilon \notin c$, so this contradicts the fact that c is blockish.

So, $s \cap c \in \mathrm{sa}(\mathcal{E}\backslash\{a\})$, and hence also $s \cap c' \in \mathrm{sa}(\mathcal{E}\backslash\{a\})$. But then $a = (u \cap w) \cup (u \cap s \cap c) \cup (w \cap s \cap c') \in \mathrm{sa}(\mathcal{E}\backslash\{a\})$, contradicting irredundance of $\mathcal{E}$. ☕

For dichotomous $\mathcal{B}$, we have a dichotomy for $\mathrm{Irr}_{\mathrm{mm}}(\mathcal{B})$; either $\mathrm{Irr}_{\mathrm{mm}}(\mathcal{B}) = \aleph_1$ or $\mathrm{Irr}_{\mathrm{mm}}(\mathcal{B}) \geq \mathfrak{b}_{\omega_1}$:

Lemma 4.4 *Assume that $\mathcal{A} \subseteq \mathcal{B} \subset \mathcal{P}(\omega_1)$ and $\mathcal{B}$ is dichotomous and $\mathrm{Irr}_{\mathrm{mm}}(\mathcal{B}) < \mathfrak{b}_{\omega_1}$. Then $\mathrm{Irr}_{\mathrm{mm}}(\mathcal{B}) = \aleph_1$. Furthermore, there is a nice club $D \subset \omega_1$ such that every $\mathcal{E} \subset \mathcal{A}$ that is induced by D is maximally irredundant in $\mathcal{B}$, and no $c \in \mathcal{B}$ is blockish for D.*

Proof. Fix $\mathcal{F} \subset \mathcal{B}$ such that $|\mathcal{F}| < \mathfrak{b}_{\omega_1}$ and $\mathcal{F}$ is maximally irredundant in $\mathcal{B}$. Let $\widetilde{\mathcal{B}} = \mathrm{sa}(\mathcal{F})$. Then $\mathcal{A} \subseteq \widetilde{\mathcal{B}} \subseteq \mathcal{B} \subset \mathcal{P}(\omega_1)$ and $|\widetilde{\mathcal{B}}| < \mathfrak{b}_{\omega_1}$. Apply Lemma 4.3 to $\mathcal{F}$ and $\widetilde{\mathcal{B}}$. This produces a nice club D such that for any c that is blockish for D: $c \notin \widetilde{\mathcal{B}}$ and $\mathcal{F} \cup \{c\}$ is irredundant.

Now, consider any $c \in \mathcal{B}\backslash\mathcal{A}$. By maximality of $\mathcal{F}$ in $\mathcal{B}$, c is not blockish for D. Since $|c| = |c'| = \aleph_1$, there must be some ξ such that $S^D_\xi \cap c \neq \emptyset$ and $S^D_\xi \cap c' \neq \emptyset$; this is condition $(*)$ of Lemma 2.6. Then the proof of that lemma shows that every $\mathcal{E} \subset \mathcal{A}$ that is induced by D is maximally irredundant in $\mathcal{B}$. ☕

It is important here that $\mathcal{B}$ be dichotomous. Otherwise, let $\mathcal{B} = \mathcal{P}(\omega_1)$. Then $\mathrm{Irr}_{\mathrm{mm}}(\mathcal{B}) = \aleph_1$ (Example 1.12), but no $\mathcal{E} \subseteq \mathcal{A}$ is maximally irredundant in $\mathcal{B}$ (Lemma 4.3, applied with $\mathcal{B} = \mathcal{A}$).

Lemma 4.4 implies immediately:

Lemma 4.5 *Assume that $\mathcal{A} \subset \mathcal{B} \subset \mathcal{P}(\omega_1)$, and $\mathcal{B}$ is dichotomous, and for all clubs $C \subset \omega_1$ there is some $b \in \mathcal{B}$ such that b is blockish for C. Then $\mathrm{Irr}_{\mathrm{mm}}(\mathcal{B}) \geq \mathfrak{b}_{\omega_1}$.*

Proof of Theorems 3.10 and 1.14. For Theorem 3.10, apply Lemma 4.5 to the $\mathcal{B}$ obtained in Lemma 4.2. Then Theorem 1.14 is the special case of Theorem 3.10 where $2^{\aleph_1} = \aleph_2$. ☕

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