Bohr Topologies and Partition Theorems for Vector Spaces *

Kenneth Kunen †
University of Wisconsin, Madison, WI 53706, U.S.A.
kunen@math.wisc.edu

June 12, 1997

Abstract

We prove a Ramsey-style theorem for sequences of vectors in an infinite-dimensional vector space over a finite field. As an application of this theorem, we prove that there are countably infinite Abelian groups whose Bohr topologies are not homeomorphic.

1 Introduction

This paper does two things. First, we prove a partition theorem for sequences of vectors in a vector space. Second, we apply this theorem to study the Bohr topologies for these vector spaces.

The partition theorem involves sequences, $X = \langle x_s : s \in [\omega]^n \rangle$, in some vector space, $\mathbb{V}$, over a *finite* field. If $\mathbb{V}$ itself is finite, then Ramsey's Theorem says that for some infinite $A \subseteq \omega$, the sequence $X \restriction A = \langle x_s : s \in [A]^n \rangle$ is constant. Our theorem states that even for infinite $\mathbb{V}$, one can get $X \restriction A$ to be in one of a finite number of possible normal forms. For $n = 1$, $X = \langle x_\alpha :$

*1991 Mathematics Subject Classification: Primary 54H11; Secondary 05C55, 54D35. Key Words and Phrases: Bohr Topology, Ramsey's Theorem.

†Author supported by NSF Grant CCR-9503445. The author is grateful to Steve Watson and the referee for helpful comments on the first draft of this paper.

$\alpha \in \omega\rangle$, and our theorem obtains A such that the sequence $\langle x_\alpha : \alpha \in A\rangle$ is either linearly independent or constant. The proof for the $n = 1$ case is an easy exercise. The statement and proof of the results for $n > 1$ are in Section 3, which may be read without reading either Section 2 or the rest of this Introduction.

These partition results have the following application for Bohr topologies. Let G be an Abelian group. Then, $G^\#$ denotes the set G with the Bohr topology; this is the coarsest topology which makes all characters (homomorphisms into the circle group) continuous. See van Douwen [4] for basic properties of $G^\#$ and references to the earlier literature. It has been an open question, originally asked by van Douwen [3], and stated in Comfort [2], whether $G^\#$ and $H^\#$ must be homeomorphic topological spaces whenever G and H are Abelian groups of the same cardinality. This is certainly true for finite groups, since then the topology is discrete. We show that this is false for infinite groups.

Specifically, for each prime p, let $\mathbb{V}_p$ be the vector space over $\mathbb{Z}_p$ of dimension $\aleph_0$. In computing $\mathbb{V}_p^\#$, we just consider $\mathbb{V}_p$ as an (additive) Abelian group, and ignore the vector space structure. We show (Corollary 4.2) that for distinct primes, p and q, the spaces $\mathbb{V}_p^\#$ and $\mathbb{V}_q^\#$ are not homeomorphic; in fact, there is no 1-1 continuous map from $\mathbb{V}_p$ into $\mathbb{V}_q$. Section 2 gives a more detailed description of the topology of $\mathbb{V}_p^\#$, and of some sequences which occur in $\mathbb{V}_p^\#$. Then, in Section 4, we apply the results of Section 3 to show that these sequences cannot occur in $\mathbb{V}_q^\#$ if q is a different prime. Section 2 might provide one possible motivation for studying the partition results in Section 3.

Our description of sequences in the Bohr topology is similar in spirit to the work of K. P. Hart and J. van Mill [6]. In fact, it is clear from [6] that one should try to distinguish the topologies of various $G^\#$ by studying the convergence properties of sequences in $G^\#$. Independently of us, S. Watson [8] discovered a related non-homeomorphism result, also by following the Hart – van Mill paradigm. Let $\mathbb{V}_p^\kappa$ be the vector space over $\mathbb{Z}_p$ of dimension κ. Watson, using an Erdös – Rado argument, showed that for κ a suitably large cardinal, $\mathbb{V}_p^\kappa$ and $\mathbb{V}_q^\kappa$ are not homeomorphic for distinct primes, p and q.

The partition results of Section 3 might be of interest for vector spaces over various finite fields. However, for applications to Bohr topologies, one only needs to consider the fields $\mathbb{Z}_p$, since the vector space over $GF(p^k)$ of dimension $\aleph_0$, viewed as a group, is isomorphic to $\mathbb{V}_p$. Section 5 contains some remarks on extending our Bohr topology results to groups other than the $\mathbb{V}_p$.

2 The Bohr Topology

In this section, we give a more detailed characterization of the topology of $G^\#$ for the particular G we plan to study. The reader unfamiliar with Bohr topologies can simply take Definition 2.2 and Lemma 2.1 as a definition of the topology.

Definition 2.1 *For a prime p, let $\mathbb{V}_p$ be the vector space over $\mathbb{Z}_p$ of dimension $\aleph_0$. Let e_α for $\alpha < \omega$ be a basis for $\mathbb{V}_p$.*

By definition, the Bohr topology, $\mathbb{V}_p^\#$, is generated by all the group homomorphisms, φ, from $\mathbb{V}_p$ into the unit circle group in the complex plane. However, each $\varphi(v)$ must be a p^{th} root of unity, so we might just as well generate the topology by homomorphisms into $\mathbb{Z}_p$ (the additive group of integers modulo p).

Definition 2.2 *For each $\varphi \in \mathrm{Hom}(\mathbb{V}_p, \mathbb{Z}_p)$ and each $k \in \mathbb{Z}_p$, let $N_\varphi^k = \{v \in \mathbb{V}_p : \varphi(v) = k\}$.*

Lemma 2.1 *The N_φ^k are all clopen sets, and form a sub-base for the topology of $\mathbb{V}_p^\#$.*

Corollary 2.2 *In $\mathbb{V}_p^\#$, the sets of the form $N_{\varphi_1}^0 \cap \cdots \cap N_{\varphi_n}^0$ form a clopen base at 0.*

It is sometimes useful to represent vectors in terms of the basis vectors, e_α, and then to compute the topology in terms of the co-ordinates. Write elements of $\mathbb{V}_p$ as $\vec{c} = \sum_\alpha c_\alpha e_\alpha$, where $c_\alpha \in \mathbb{Z}_p$ (for $\alpha \in \omega$), and $c_\alpha = 0$ for all but finitely many α.

Definition 2.3 *For each $I \subseteq \omega$ and each $k \in \mathbb{Z}_p$, let $U_I^k = \{\vec{c} \in \mathbb{V}_p : \sum_{\alpha \in I} c_\alpha \equiv k \pmod{p}\}$.*

Lemma 2.3 *The U_I^k are all clopen sets, and form a sub-base for the topology of $\mathbb{V}_p^\#$.*

Proof. Each U_I^k is clopen because $U_I^k = N^k_{\chi_I}$, where $\chi_I(e_\alpha)$ is 1 for $\alpha \in I$ and 0 for $\alpha \notin I$. To prove they form a sub-base, fix $\vec{c} \in N^k_\varphi$. For $j \in \mathbb{Z}_p$, let $I_j = \{\alpha : \varphi(e_\alpha) = j\}$, so that $\varphi = \sum\{j\chi_{I_j} : j \in \mathbb{Z}_p\}$. Let $\ell_j = \chi_{I_j}(\vec{c})$. Then $\vec{c} \in \bigcap\{U_{I_j}^{\ell_j} : j \in \mathbb{Z}_p\} \subseteq N^k_\varphi$. ∎

Clearly, then, a clopen base at 0 is given by finite intersections of the U_I^0, but it is useful to point out that the various I can always be taken from a partition. A *partition* of ω into n pieces is map $\mathcal{I} : \omega \to n$; n will always be finite; the i^{th} piece, I_i, is just $\mathcal{I}^{-1}\{i\}$.

Definition 2.4 *For each partition $\mathcal{I}$ of ω into n pieces, let $U(\mathcal{I})$ be the set of all $\vec{c} \in \mathbb{V}_p$ such that $\sum_{\alpha\in I_i} c_\alpha \equiv 0 \pmod p$ for all $i < n$.*

Lemma 2.4 *The $U(\mathcal{I})$, for partitions, $\mathcal{I}$, form a clopen base at 0 in $\mathbb{V}_p^\#$.*

Proof. $U(\mathcal{I})$ is clopen because it is the intersection of the $U_{I_i}^0$. Now, let $K = \bigcap\{U_{J_i}^0 : i < n\}$ be a basic clopen neighborhood of 0. The J_i need not form a partition. But, let $\mathcal{I}$ be the partition of ω into 2^n pieces obtained by taking all possible intersections of the J_i or their complements. Then $0 \in U(\mathcal{I}) \subseteq K$. ∎

Next, we consider the special elements of $\mathbb{V}_p$ corresponding to finite subsets of ω.

Definition 2.5 *For each prime p: For $s \in [\omega]^{<\omega}$, let $e_s = \sum_{\alpha\in s} e_\alpha \in \mathbb{V}_p$; so, $e_\emptyset = 0$. Let $\mathcal{T}_p$ be the induced topology on $[\omega]^{<\omega}$ (so that the map $s \mapsto e_s$ is a homeomorphism onto its range). For each partition $\mathcal{I}$ of ω into n pieces, let $V_p(\mathcal{I})$ be the set of all $s \in [\omega]^{<\omega}$ such that $|s \cap I_i| \equiv 0 \pmod p$ for all $i < n$.*

Lemma 2.5 *The $V_p(\mathcal{I})$, for partitions, $\mathcal{I}$, form a clopen base at $\emptyset$ in $[\omega]^{<\omega}$ under $\mathcal{T}_p$.*

In the case $p = 2$, the map $s \mapsto e_s$ is a group isomorphism from the group $[\omega]^{<\omega}$ (under symmetric difference) onto $\mathbb{V}_2$, so that $\mathcal{T}_2$ is just the topology $([\omega]^{<\omega})^\#$.

Note that for each $s \in [\omega]^{<\omega}$, the set $\{t : s \subseteq t \in [\omega]^{<\omega}\}$ is clopen in each $\mathcal{T}_p$. From this it is easy to see:

Lemma 2.6 *For each prime p and each $k > 0$, $[\omega]^k$ is relatively discrete in the topology $\mathcal{T}_p$. Furthermore, $\emptyset$ is in the closure of $[\omega]^k$ in $\mathcal{T}_p$ iff $p \mid k$.*

This lemma is essentially due to Hart and van Mill [6], who showed that if $k = p$, then, in $\mathbb{V}_p^{\#}$, the only limit point of $\{e_s : s \in [\omega]^p\}$ is 0.

We shall show (Theorem 4.1) that if $p \mid k$ and $p < k$, then $\{\emptyset\} \cup [\omega]^k$ in $\mathcal{T}_p$ is not homeomorphic to any subset of any $\mathbb{V}_q^{\#}$, whenever q is a prime other than p. The proof seems to require a detailed study of all possible sequences in $\mathbb{V}_q$ indexed by various $[\omega]^k$. We take this up in the next section.

We remark here that k cannot simply be taken to be p. For example, $\{\emptyset\} \cup [\omega]^2$ in $\mathcal{T}_2$ is homeomorphic to $\{0\} \cup \{x_\alpha - x_\beta : \alpha < \beta < \omega\}$ in any $\mathbb{V}_q^{\#}$, if the x_α are all linearly independent. Also, $\{\emptyset\} \cup [\omega]^3$ in $\mathcal{T}_3$ is homeomorphic to $\{0\} \cup \{x_\alpha + x_\beta + y_\beta + y_\gamma : \alpha < \beta < \gamma < \omega\}$ in $\mathbb{V}_2^{\#}$ if the x_α and y_β are all independent.

3 Normal Forms.

Throughout this section, K is a fixed *finite* field. Let $\mathbb{V}$ be a vector space over K. If $n \in \omega$ and B is an infinite subset of ω, an *n-ary sequence indexed by B* from $\mathbb{V}$ is a map $X : [B]^n \to \mathbb{V}$; n is the *arity* of X. We shall often display X as a sequence, $X = \langle x_s : s \in [B]^n\rangle$. Note that n could be 0, in which case the sequence is just a singleton, $X = \langle x_\emptyset\rangle$. A *system* of sequences is of the form $\mathcal{X} = \langle X^i : i < k\rangle$, where k is finite and each X^i an n^i-ary sequence indexed by the same B.

In vector spaces, we consider linear independence to be a property of *sequences*, rather than *sets* of vectors; that is, an indexed sequence of vectors, $\langle w_i : i \in I\rangle$, is *independent* iff there is no indexed sequence of scalars, $\langle c_i : i \in I\rangle$ such that $0 < |\{i : c_i \neq 0\}| < \aleph_0$ and $\sum_{i\in I} c_i w_i = 0$; equivalently, the w_i, for $i \in I$, are all distinct, and $\{w_i : i \in I\}$ is independent in the usual sense in linear algebra. An n-ary sequence $X = \langle x_s : s \in [B]^n\rangle$ is *independent* iff the vectors x_s are independent in this (sequence) sense. For the $n = 0$ case, we just have one vector, $x_\emptyset$, and "independent" means "$x_\emptyset \neq 0$". The system $\mathcal{X}$ is *independent* iff the indexed sequence of vectors $\langle x_s^i : i < k, s \in [B]^{n^i}\rangle$ is independent.

If $X = \langle x_s : s \in [B]^n\rangle$ is an n-ary sequence and A is an infinite subset of B, then let $X{\restriction}A = \langle x_s : s \in [A]^n\rangle$. If $\mathcal{X}$ is the system, $\langle X^i : i < k\rangle$, then $\mathcal{X}{\restriction}A = \langle X^i{\restriction}A : i < k\rangle$.

The goal of this section is to prove (Theorem 3.4) that given any such X, B, n, one may always find an A and an *independent* system $\mathcal{W}$ such that

$X \restriction A$ is "derived from" $\mathcal{W}$ in one of a finite number of possible ways. This also shows that there are finitely many possible "normal forms" for such $X \restriction A$. For example, if $X = \langle x_0 \rangle$ is 0-ary, then $X \restriction A = X$, and the two possible forms for X are "zero" and "non-zero". If $X = \langle x_\alpha : \alpha \in B \rangle$ is 1-ary, then, as remarked in the Introduction, we can always find an infinite $A \subseteq B$ such that $X \restriction A$ is either independent or constant.

For 2-ary sequences, $X = \langle x_{\alpha,\beta} : \alpha < \beta \ ; \ \alpha, \beta \in B \rangle$, there are more possible normal forms. For example, we could have $x_{\alpha,\beta} = y_\alpha + c y_\beta$, where Y is an independent 1-ary sequence, and c is a fixed scalar. This is not the same as the form $x_{\alpha,\beta} = y_\alpha + z_\beta$, where the y_α, z_β are all independent; that is, $\mathcal{W} = \langle Y, Z \rangle$ is an independent system.

For $n > 1$, we proceed by induction. However, the induction will be simpler if we get B and A to be in some Ramsey ultrafilter, Ψ, on ω. That way, we can easily prove the result for $n+1$ by using the result for n in the ultrapower of $\mathbb{V}$. Although Ramsey ultrafilters exist under CH, their existence is not provable in ZFC [5][7]. However, by general metamathematical arguments (see the proof of Theorem 3.4), any combinatorial theorem about countable objects which follows from CH is provable without CH. Also, by doing somewhat more work, one can prove Theorem 3.4 directly, without mentioning ultrafilters.

Recall that a *Ramsey ultrafilter* is a non-principal ultrafilter Ψ on ω such that each partition $P : [\omega]^n \to k$ (for n, k finite) has a homogeneous set *in* Ψ. See Booth [1] for basic properties of Ramsey ultrafilters. In particular, we use the following diagonalization property:

Lemma 3.1 *If Ψ is Ramsey, and T is a non-empty subtree of $\omega^{<\omega}$ such that $\forall s \in T[\{\beta : s\beta \in T\} \in \Psi]$, then there is a set $\{\alpha_i : i \in \omega\} \in \Psi$ such that $\alpha_0 < \alpha_1 < \cdots$ and such that $\langle \alpha_i : i < n \rangle \in T$ for each n.*

An example of the use of this lemma is the following one, which is really the $n = 1$ result again, but now phrased in terms of the ultrafilter:

Lemma 3.2 *If Ψ is Ramsey and X is a 1-ary sequence from $\mathbb{V}$ indexed by $B \in \Psi$, then there is an $A \in \Psi$ such that $X \restriction A$ is either independent or constant.*

Proof. Say $X = \langle x_\alpha : \alpha \in B \rangle$. Try to choose, inductively, $\alpha_0, \alpha_1, \ldots$ from B such that the x_{α_i} are all independent; then A will be $\{\alpha_i : i \in \omega\}$.

By Lemma 3.1, to prove that we may get $A \in \Psi$, it is sufficient to assume that we have chosen $\alpha_0 \ldots \alpha_{n-1}$, and prove that there is a Ψ-measure 1 set of possible choices, β, for α_n. If this is not the case, then almost every x_β is a linear combination of $\langle x_{\alpha_i} : i < n \rangle$. Since the field K is finite, this linear combination is actually the same for almost every β. That is, restricted to some set $A \in \Psi$, the sequence is constant. ∎

Actually, we shall never explicitly quote this lemma again, but we have presented it as a simple introduction to the general method. To state the general result, we first need to define "derived from".

If W and X are sequences indexed by A, where W is m-ary and X is n-ary, we say that X is a *simple derived sequence* from W iff $n \geq m$ and for some $i_0 < i_1 < \cdots < i_{m-1} < n$, we have

$$x_{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}} = w_{\alpha_{i_0}, \alpha_{i_1}, \ldots, \alpha_{i_{m-1}}}$$

for all $\{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\} \in [A]^n$. Then, if $\mathcal{W} = \langle W^i : i < k \rangle$ is a system of sequences, we say that X is a *derived sequence* from $\mathcal{W}$ iff $X = \sum_{\ell < L} c_\ell Y^\ell$ for some $L < \omega$, scalars c_ℓ for $\ell < L$, and sequences Y^ℓ for $\ell < L$, where each Y^ℓ is a simple derived sequence from some W^i (where i can depend on ℓ). Note that if $Y = \langle y_s : s \in [A]^n \rangle$, we are using cY for $\langle c y_s : s \in [A]^n \rangle$.

If $\mathcal{V} = \langle V^i : i < k \rangle$ and $\mathcal{W} = \langle W^i : i < \ell \rangle$ are two systems we use $\mathcal{V} \cup \mathcal{W}$ for the *concatenation* of $\mathcal{V}, \mathcal{W}$, which is a system of $k + \ell$ sequences. We consider this to be an *extension* of $\mathcal{V}$ (and, also of $\mathcal{W}$, since the order in which the sequences are listed is never important).

Then the basic extension result is:

Lemma 3.3 *Given an independent system $\mathcal{V}$, an n-ary sequence X, and a Ramsey ultrafilter Ψ, there is an extension of $\mathcal{V}$, of the form $\mathcal{V}' = \mathcal{V} \cup \mathcal{W}$, and an $A \in \Psi$, such that $X \restriction A$ is a derived sequence from $\mathcal{V}' \restriction A$, and $\mathcal{V}' \restriction A$ is independent.*

Actually, we are primarily interested in the case where $\mathcal{V}$ is empty, but the lemma as stated, for arbitrary $\mathcal{V}$, is more suitable to a proof by induction on n. When $\mathcal{V}$ is empty, we get the following theorem as an immediate corollary:

Theorem 3.4 *If X is an n-ary sequence indexed by B from $\mathbb{V}$, then there is an infinite $A \subseteq B$ and an independent system $\mathcal{W}$ such that $X \restriction A$ is a derived sequence from $\mathcal{W} \restriction A$.*

Proof. This is trivial from Lemma 3.3 under CH, since one may get A in any Ramsey ultrafilter containing B. In general, quote Lemma 3.3 in the forcing extension of the universe which makes CH true by collapsing $2^{\aleph_0}$ with countable conditions. Since the theorem involves only countable objects, its truth in the forcing extension implies its truth in the real universe. ∎

We now proceed to prove Lemma 3.3. The proof will be by induction on arity, and the arity 0 case is handled by the following lemma, which allows one to split off 0-ary systems from non-0-ary systems.

Lemma 3.5 *Suppose $\mathcal{W} = \mathcal{V} \cup \mathcal{X}$ is a system, where all the sequences in $\mathcal{X}$ are 0-ary, and none of the sequences in $\mathcal{V}$ are 0-ary. Suppose that $\mathcal{V}$ is independent and $\mathcal{X}$ is independent. Then $\mathcal{W}{\restriction}A$ is independent for some $A \in \Psi$.*

Proof. It is sufficient to choose A so that whenever $x \in span(\mathcal{X})$ is a non-zero vector, $x \notin span(\mathcal{V}{\restriction}A)$. Now, for each such x: if $x \in span(\mathcal{V})$, then (since $\mathcal{V}$ is independent) x is expressed in a unique way as a linear combination of vectors from $\mathcal{V}$, and we may simply choose A to omit one of the indices used in this expression (since none of the sequences in $\mathcal{V}$ are 0-ary). Then, since all the sequences in $\mathcal{X}$ are 0-ary, there are only finitely many such x, so we may in fact choose a co-finite A which works for each x. ∎

The induction step will use the ultrapower to reduce the arity by one. Let $\widehat{\mathbb{V}}$ be the ultrapower, $\mathbb{V}^\omega/\Psi$. If X is an $(n+1)$-ary sequence, we let $\widehat{X}$ be the n-ary sequence in $\widehat{\mathbb{V}}$ such that $\hat{x}_s$ is the equivalence class of $\beta \mapsto x_{s,\beta}$. Note that we're using $x_{s,\beta}$ as shorthand for $x_{s\cup\{\beta\}}$. If X is 0-ary, then $\widehat{X}$ is not defined. If $\mathcal{W}$ is a system, then $\widehat{W}$ is the system obtained by replacing each n^i-ary W^i by the (n^i-1)-ary $\widehat{W}^i$ (when $n^i > 0$), and deleting all the 0-ary W^i.

Since $\mathbb{V} \subset \widehat{\mathbb{V}}$, and these are both vector spaces over the same finite field, we may also form the quotient vector space, $\widehat{\mathbb{V}}/\mathbb{V}$. The following two lemmas relate independence of $\mathcal{W}$ in $\mathbb{V}$ to independence of $\widehat{\mathcal{W}}$ in $\widehat{\mathbb{V}}/\mathbb{V}$.

Lemma 3.6 *If $\mathcal{W}$ is independent in $\mathbb{V}$, then $\widehat{\mathcal{W}}$ is independent in $\widehat{\mathbb{V}}/\mathbb{V}$.*

Proof. If not, we would have, in the ultrapower, an equation of the form

$$c_1\hat{w}_{s_1} + \cdots + c_k\hat{w}_{s_k} = v \in \mathbb{V}$$

Then, for almost every β, we would have

$$c_1 w_{s_1,\beta} + \cdots + c_k w_{s_k,\beta} = v$$

But, now, by varying β (over two distinct values), we contradict the independence of $\mathcal{W}$. ∎

Lemma 3.7 *If $\widehat{\mathcal{W}}$ is independent in $\widehat{\mathbb{V}}/\mathbb{V}$, and $\mathcal{W}$ contains no 0-ary sequences, then $\mathcal{W}\restriction A$ is independent in $\mathbb{V}$ for some $A \in \Psi$.*

Proof. Inductively choose $\alpha_0 < \alpha_1 < \alpha_2 < \cdots$, making sure that each $\mathcal{W}\restriction A_n$ is independent, where $A_n = \{\alpha_m : m < n\}$. Then, A will be $\{\alpha_m : m < \omega\}$. It is enough to check that given A_n, the set of possibilities, β, for α_n is in Ψ. If this fails, then for ae β, we have an identity of the form:

$$c_1 w_{s_1} + \cdots + c_p w_{s_p} \;+\; d_1 w_{t_1,\beta} + \cdots + d_q w_{t_q,\beta} = 0$$

where the s_i and t_j are subsets of A_n; $q \neq 0$ (since $\mathcal{W}\restriction A_n$ is independent). At first, the coefficients and the s_i, t_j could depend on β, but we may assume they are fixed, since there are only finitely many possibilities. Let $v = c_1 w_{s_1} + \cdots + c_p w_{s_p}$. Then in the ultrapower we have

$$d_1 \hat{w}_{t_1} + \cdots + d_q \hat{w}_{t_q} = -v$$

contradicting independence of $\widehat{\mathcal{W}}$ in $\widehat{\mathbb{V}}/\mathbb{V}$. ∎

Proof of Lemma 3.3 We induct on the arity of X.

If X is 0-ary: If X is in the span of the 0-ary sequences from $\mathcal{V}$, we may let $\mathcal{V}' = \mathcal{V}$. If not, we let $\mathcal{V}' = \mathcal{V} \cup \{X\}$; by Lemma 3.5, some $\mathcal{V}'\restriction A$ is independent.

Now, assume the lemma holds for n-ary X, and assume X is $n+1$-ary. Apply the lemma to the n-ary $\widehat{X}$ to extend $\mathcal{V}$ to $\mathcal{V}'$ and get $A \in \Psi$ such that in $\widehat{\mathbb{V}}/\mathbb{V}$, $\widehat{\mathcal{V}'}\restriction A$ is independent and $\widehat{X}\restriction A$ is derived from $\widehat{\mathcal{V}'}\restriction A$. We may assume that $\mathcal{V}'$ is formed from $\mathcal{V}$ by only adding sequences of arity > 0, so that by Lemmas 3.7 and 3.5, we may assume also that $\mathcal{V}'\restriction A$ is independent. Since $\widehat{X}$ is derived in $\widehat{\mathbb{V}}/\mathbb{V}$, there is a Y which is derived from $\mathcal{V}'$ such that $\widehat{X}$ and $\widehat{Y}$ are the same in $\widehat{\mathbb{V}}/\mathbb{V}$. Thus, there is an n-ary Z such that for each $s \in [\omega]^n$, we have $x_{s,\beta} = y_{s,\beta} + z_s$ for almost every β. Applying the lemma again for the n-ary Z, we may extend $\mathcal{V}'$ to a $\mathcal{V}''$ so that for some $B \in \Psi$, $\mathcal{V}''\restriction B$ is independent and $Z \restriction B$ is derived from $\mathcal{V}'' \restriction B$. Diagonalizing (by Lemma 3.1), we may also assume that $x_{s,\beta} = y_{s,\beta} + z_s$ for every $s \cup \{\beta\} \in [B]^{n+1}$, so that $X \restriction B$ is derived from $\mathcal{V}'' \restriction B$. ∎

In Section 4, we shall argue directly from Theorem 3.4, but we remark that one may use this theorem to list, for each n, a finite number of normal forms, such that every n-ary sequence is, restricted to some A, in one of these normal forms. Note that the possibilities for the independent system $\mathcal{V}$ simplify somewhat, since we may merge the components of $\mathcal{V}$ with the same arity, if they are used similarly. More specifically, fix a Ramsey ultrafilter Ψ. If X and Y are two n-ary sequences, say $X \sim Y$ iff there is an automorphism F of $\mathbb{V}$ and an $A \in \Psi$ such that $x_s = F(y_s)$ for all $s \in [A]^n$. Then, for each n, there are only finitely many $\sim$ equivalence classes. For $n = 0$, there are two classes: "non-0" and "0". For $n = 1$, there are three classes: "independent", "non-0 constant" and "constantly 0". Now, suppose X is 2-ary. $\mathcal{V}$ could be empty, in which case X is the constant 0. Or, $\mathcal{V}$ could be non-empty but contain only 0-ary sequences, in which case we could merge them to one, and get X to be a non-zero constant. Continuing in this way, we get the following possibilities:

1. $x_{\alpha,\beta} = 0$.
2. $x_{\alpha,\beta} = v \neq 0$.
3. $x_{\alpha,\beta} = v_\alpha$.
4. $x_{\alpha,\beta} = v_\beta$.
5. $x_{\alpha,\beta} = v_\alpha + w_\beta$.
6. $x_{\alpha,\beta} = v_\alpha + cv_\beta$.
7. $x_{\alpha,\beta} = v_{\alpha,\beta}$.

Here, all the vectors on the right side of the "=" are independent, and c is some non-zero scalar. Thus, if K is the base field, there are $|K| + 5$ equivalence classes of 2-ary sequences.

4 A Non-Homeomorphism

Since $\mathbb{V}_p^\#$ and $\mathbb{V}_q^\#$ are topological groups, any map between them can be translated to a map which takes 0 to 0. Thus, to prove that they are not homeomorphic, it is sufficient to prove:

Theorem 4.1 *Suppose p and q are distinct primes, and suppose $p \mid k$ and $p < k$. Let $F : [\omega]^k \cup \{\emptyset\} \to \mathbb{V}_q$ be continuous with respect to the topologies $\mathcal{T}_p$ and $\mathbb{V}_q^\#$, and suppose that $F(\emptyset) = 0$. Then for some infinite $A \subseteq \omega$, F takes $[A]^k$ to 0.*

Corollary 4.2 *If p and q are distinct primes, then there is no 1-1 continuous function from $\mathbb{V}_p^\#$ into $\mathbb{V}_q^\#$.*

We now turn to the proof of Theorem 4.1. Of course, since F defines a k-ary sequence in $\mathbb{V}_q$, our intent is to apply Theorem 3.4 here, getting F restricted to some $[A]^k$ derived from some independent system $\mathcal{W}$. We first prove a preliminary lemma, which will allow us to handle the case where $\mathcal{W}$ contains any sequence of arity greater than one. This lemma implies in particular that in the statement of Ramsey's Theorem, one cannot expect to cover ω by finitely many homogeneous sets. Probably, much stronger "anti-Ramsey" lemmas can be proved, but this one is easy, and will suffice for our purposes, and is accomplished by the standard (Sierpiński) example.

Lemma 4.3 *Fix $r \geq 2$, and an infinite $A \subseteq \omega$. Then there is a $D \subseteq [\omega]^r$ with the following property: Whenever $n \geq r$, $\emptyset \neq S \subseteq [n]^r$, and $\mathcal{I}$ is a partition of ω into finitely many pieces, some $I_i \cap A$ contains elements $\alpha_0 < \cdots < \alpha_{n-1}$ such that exactly one $s \in S$ satisfies: $\{\alpha_k : k \in s\} \in D$.*

Proof. Of course, we may assume $A = \omega$, which simplifies the notation. Let $\lhd$ totally order ω isomorphically to the rationals. Let D be the set of all $s \in [\omega]^r$ such that $\lhd$ and $<$ agree on all pairs from D. Given S, let s be the lexically last element of S (where we compute lexical order by identifying each $s \in S$ with an increasing sequence of r numbers). Given $\mathcal{I}$, choose i so that I_i is dense in some $\lhd$-interval. Let j be the smallest element of s. Choose $\alpha_\ell \in I_i$, for $\ell \in s$, so that $\lhd$ agrees with $<$ on these α_ℓ. Then choose α_k, for $k \notin s$, so that $\lhd$ agrees with $>$ on these α_k; furthermore, if $k > j$, place α_k below (in $\lhd$) all the α_ℓ for $\ell \in s$, while if $k < j$, place α_k above all the α_ℓ for $\ell \in s$. ∎

To illustrate the lemma and its proof, suppose $r = 2$, $n = 8$, and S has the pair $(3, 6)$ as its lexically last element. Then $D \subseteq [\omega]^2$ is the set of pairs on which $\lhd$ and $<$ agree. Given any partition $\mathcal{I}$, we can always choose I_i to contain some $\alpha_0 < \cdots < \alpha_7$ such that $(\alpha_3, \alpha_6) \in D$, and $(\alpha_i, \alpha_j) \notin D$ whenever $i < j$ and either $i < 3$ or $i = 3 < j < 6$. To do this, we choose them in the order $\alpha_7 \lhd \alpha_5 \lhd \alpha_4 \lhd \alpha_3 \lhd \alpha_6 \lhd \alpha_2 \lhd \alpha_1 \lhd \alpha_0$.

Proof of Theorem 4.1. By Theorem 3.4, let $\mathcal{W}$ be an independent system and A an infinite set such that $F{\restriction}A$ is a derived sequence from $\mathcal{W}{\restriction}A$. If $F{\restriction}A$ is the identically 0 sequence, we are done. So, assume it is not identically 0 , and we shall derive a contradiction. Say $\mathcal{W}$ is the system, $\langle W^i : i < \ell\rangle$. Then we can write F as $\sum_{i<\ell} Y^i$, where each $Y^i{\restriction}A$ is a derived sequence from $W^i{\restriction}A$ alone.

Now, *fix* any i such that $Y^i{\restriction}A$ is not identically 0. Let π be a linear map from $\mathbb{V}_q$ onto $span(Y^i)$ such that π is the identity on $span(Y^i)$ and π takes $span(Y^j)$ for $j \neq i$ to 0. Let $G = \pi \circ F$. Then G is continuous. From now on, write Y for Y^i and W for W^i, so $G(s) = Y(s)$ for $s \in [A]^k$. Let r be the arity of W. Then $0 \le r \le k$. Our contradiction will come in three cases, depending on r.

If $r = 0$, then $W = \langle w_\emptyset\rangle$, and $G(s) = cw_\emptyset \neq 0$ for all $s \in [A]^k$, which contradicts continuity, since $\emptyset$ is a limit point of $[A]^k$ in $\mathcal{T}_p$.

If $r = 1$, then $W = \langle w_\alpha : \alpha < \omega\rangle$, and $G(s) = \sum_{j<k} c_j w_{\alpha_j} \neq 0$ for all $s = \{\alpha_0, \ldots, \alpha_{k-1}\} \in [A]^k$, where each $c_j \in \mathbb{Z}_q$. Let D be any subset of ω such that $A \cap D$ and $A \setminus D$ are both infinite. Let $\varphi \in \mathrm{Hom}(\mathbb{V}_q, \mathbb{Z}_q)$ be such that $\varphi(w_\alpha)$ is 1 when $\alpha \in D$ and 0 when $\alpha \notin D$. By continuity and Lemma 2.5, there must be a partition $\mathcal{I}$ such $\varphi(G(s)) = 0$ whenever $s \in V_p(\mathcal{I})$. Now, fix $m_0 < \cdots < m_{p-1} < k$ with $c_{m_0} + \cdots + c_{m_{p-1}} \neq 0$ (in $\mathbb{Z}_q$); this is possible since $q \neq p$ and $p < k$ and not all the c_j are 0. Let I_i and I_ℓ be two pieces of the partition $\mathcal{I}$ (possibly the same) such that $I_i \cap A \cap D$ and $I_\ell \cap A \setminus D$ are both infinite. Then, fix $\alpha_0 < \cdots < \alpha_{k-1}$ in A with $\alpha_{m_0}, \cdots, \alpha_{m_{p-1}} \in I_i \cap D$ and the other $\alpha_j \in I_\ell \setminus D$. Then $s = \{\alpha_0, \ldots, \alpha_{k-1}\} \in V_p(\mathcal{I})$, but $\varphi(G(s)) = c_{m_0} + \cdots + c_{m_{p-1}} \neq 0$, a contradiction.

If $r \geq 2$, then $W = \langle w_s : s \in [\omega]^r\rangle$. If $s \in [k]^r$ and $\alpha_0 < \cdots < \alpha_{k-1}$, let α_s abbreviate $\{\alpha_\ell : \ell \in s\}$. Then $G(\{\alpha_0, \ldots, \alpha_{k-1}\}) = \sum_{s \in S} c_s w_{\alpha_s}$, where $\emptyset \neq S \subseteq [k]^r$, and each $c_s \neq 0$. Fix D as in Lemma 4.3, and let $\varphi \in \mathrm{Hom}(\mathbb{V}_q, \mathbb{Z}_q)$ be such that $\varphi(w_s)$ is 1 when $s \in D$ and 0 when $s \notin D$. By continuity, there must be a partition $\mathcal{I}$ such $\varphi(G(\{\alpha_0, \ldots, \alpha_{k-1}\})) = 0$ whenever $\{\alpha_0, \ldots, \alpha_{k-1}\} \in V_p(\mathcal{I})$. Now, fix $\alpha_0 < \cdots < \alpha_{k-1}$ in A, such that all of them are in the same I_i, and such that exactly one $s \in S$ satisfies $\alpha_s \in D$. Then $\{\alpha_0, \ldots, \alpha_{k-1}\} \in$

$V_p(\mathcal{I})$ but $\varphi(G(\{\alpha_0, \ldots, \alpha_{k-1}\})) = c_s \neq 0$, a contradiction. ∎

5 Additional Remarks

Say G, H are two countably infinite Abelian groups. When are $G^\#, H^\#$ homeomorphic? We are still very far from answering this question, although we have shown that the answer isn't "always".

Perhaps the answer is "almost never". Specifically, define $G \sim H$ iff there are subgroups, G', H', of G, H, respectively, such that G' and H' are isomorphic, G' has finite index in G, and H' has finite index in H. It is easy to see that $\sim$ is an equivalence relation, and that $G \sim H$ implies that $G^\#$ and $H^\#$ are homeomorphic. We do not know if the converse holds.

This paper does not even settle what happens in the case of groups of finite exponent (satisfying $\exists n \in \omega \forall x \in G(nx = 0)$). For example, let $\mathbb{V}_n$ be the direct sum of $\aleph_0$ copies of $\mathbb{Z}_n$. We do not know whether $\mathbb{V}_4^\#$ is homeomorphic to $(\mathbb{V}_2 \times \mathbb{V}_4)^\#$. However, each of $\mathbb{V}_4$ and $\mathbb{V}_2 \times \mathbb{V}_4$ is embeddable in the other, so that the methods of Section 4, which establish non-homeomorphism by establishing non-embeddability, do not seem to apply here.

As the referee has pointed out, our methods can be pushed slightly further. For example, suppose G has finite exponent, p is prime, and G has no subgroup homeomorphic to $\mathbb{V}_p$. Then there is no 1-1 continuous function from $\mathbb{V}_p^\#$ into G. To see this, construct a chain of sub-groups,

$$\{0\} = G_0 \subset G_1 \subset \cdots \subset G_r = G \quad ,$$

such that each G_i/G_{i-1} is either finite or isomorphic to some $\mathbb{V}_q$, where $q = q_i$ is a prime different from p. Suppose $F : [\omega]^k \cup \{\emptyset\} \to G$ is continuous with respect to the topologies $\mathcal{T}_p$ and $G^\#$, $F(\emptyset) = 0$, $p \mid k$, and $p < k$. Inductively construct infinite sets

$$\omega = A_r \supset A_{r-1} \supset \cdots \supset A_0 \quad ,$$

such that F takes each $[A_i]^k$ to G_i. At each stage, apply Theorem 4.1 to G_i/G_{i-1} (or, just use Ramsey's Theorem if this group is finite).

References

[1] D. Booth, Ultrafilters on a countable set, *Annals of Mathematical Logic* 2 (1970) 1-24.

[2] W. W. Comfort, Problems on topological groups and other homogeneous spaces, in *Open Problems in Topology*, J. van Mill and G. M. Reed, eds., North-Holland Publishing Company, Amsterdam, 1990, pp. 313–347.

[3] E. K. van Douwen, Letters to W. W. Comfort, June 30, 1986 and May 9, 1987.

[4] E. K. van Douwen, The maximal totally bounded group topology on G and the biggest minimal G-space for Abelian groups G, *Topology and its Applications* 34 (1990) 69-91.

[5] J. E. Hart and K. Kunen, Weak measure extension axioms, *to appear.*

[6] K. P. Hart and J. van Mill, Discrete sets and the maximal totally bounded group topology, *J. Pure Appl. Alg.* 70 (1991) 73-80.

[7] K. Kunen, Some points in $\beta\mathbb{N}$, *Mathematical Proceedings of the Cambridge Philosophical Society* 80 (1976) 385-398.

[8] S. Watson, private communication, November, 1996.