On the Regularity of the Topology of Separate Continuity *

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Abstract

Given topological spaces X, Y, there is a unique topology $\mathcal{T}_+$ on $X \times Y$ such that, for all topological spaces Z, a function $f : X \times Y \to Z$ is continuous with respect to $\mathcal{T}_+$ iff f is separately continuous. We consider situations under which $\mathcal{T}_+$ is regular or normal. This is related to Eberlein compacta in the case that X, Y are compact, and to σ-sets in the case that X, Y are separable metric.

1 Introduction

If X, Y, Z are topological spaces, a function $f : X \times Y \to Z$ is *separately continuous* iff the maps $y \mapsto f(a, y)$ and $x \mapsto f(x, b)$ are continuous for each $a \in X$ and $b \in Y$. Clearly, this is weaker than continuity with respect to the usual product topology on $X \times Y$. However, separate continuity is equivalent to continuity with respect to the topology $\mathcal{T}_+$, defined by:

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Definition 1.1 *If $E \subseteq X \times Y$, then $E_a = \{y : (a, y) \in E\}$ (for $a \in X$) and $E^b = \{x : (x, b) \in E\}$ (for $b \in Y$). $U \subseteq X \times Y$ is* +open *iff U_a is open in Y for all $a \in X$ and U^b is open in X for all $b \in Y$. The* +open topology*, $\mathcal{T}_+$, is the collection of all +open sets. $X \otimes Y$ denotes $X \times Y$ with the topology $\mathcal{T}_+$.*

The following properties are immediate from the definition:

Proposition 1.2 *If X, Y, Z are any spaces, then:*

1. *$f : X \times Y \to Z$ is separately continuous iff f is continuous with respect to $\mathcal{T}_+$.*
2. *$E \subseteq X \otimes Y$ is closed iff E_a is closed in Y for all $a \in X$ and E^b is closed in X for all $b \in Y$.*
3. *$\mathcal{T}_+$ is T_2 iff X is T_2 and Y is T_2.*
4. *If X, Y are both T_2 and $F \subseteq X \times Y$ is a 1-1 function (that is, $|F_a| \leq 1$ and $|F^b| \leq 1$ for all $a \in X$ and $b \in Y$), then F is closed and discrete in $\mathcal{T}_+$.*
5. *If $A \subseteq X$ and $B \subseteq Y$, then the closure of $A \times B$ in $\mathcal{T}_+$ is $\overline{A} \times \overline{B}$.*
6. *If A is dense in X and B is dense in Y, then $A \times B$ is dense in $X \otimes Y$.*
7. *If X and Y are separable, then $X \otimes Y$ is separable.*

This topology has been mentioned only a few times in the literature. The terminology "$X \otimes Y$" was used by Knight, Moran, and Pym [15, 16], who called it a "tensor product", following Isbell [12], p. 53. They proved a number of basic facts about $X \otimes Y$; in particular, that it will often fail to be T_3. Recently, Velleman [29] used $\mathbb{R} \otimes \mathbb{R}$ to aid the exposition of several topics in elementary calculus; he coined the term "+open" because $U \subseteq \mathbb{R} \times \mathbb{R}$ is +open iff for each $(x, y) \in U$, one can find a small + sign centered at (x, y) and contained in U. He also pointed out that the +open topology is the *only* topology on $X \times Y$ for which Proposition 1.2.1 holds for all spaces Z. Another topology on the product, called $X \widetilde{\otimes} Y$ by Knight, Moran, and Pym [16], is the weak topology determined by the separately continuous *real-valued* functions on $X \times Y$; equivalently, this is the weakest topology for which Proposition 1.2.1 holds for all *completely regular* spaces Z. The recent summary by Henriksen [11] contains further information on $X \widetilde{\otimes} Y$ and the functorial relations between $X \widetilde{\otimes} Y$ and $X \otimes Y$. Since $X \widetilde{\otimes} Y$ is clearly completely regular, it is not directly relevant to our current paper.

The extensive literature on separately continuous functions on $X \times Y$ (see [23, 24] for references), however, does yield (implicitly) some facts about $X \otimes Y$. Non-trivial results on separate continuity go back to Baire [2], and the fact that $\mathbb{R}\otimes\mathbb{R}$ is not T_3 follows easily from the proof in Sierpiński [28] (see our Remark 2.2). Non-regularity of $\mathbb{R} \otimes \mathbb{R}$ also follows from a cardinal functions argument: a separable regular space can have weight no more than $\mathfrak{c}$, whereas $\mathbb{R} \otimes \mathbb{R}$ is separable (by Proposition 1.2.7) and of weight $2^{\mathfrak{c}}$ (by Lemma 2.1). The fact that $w(\mathbb{R}\otimes\mathbb{R}) \geq \mathfrak{c}^+$ was already pointed out by Popvassilev [25, 26], who has some more detailed results about such topologies on $\mathbb{R}^n$.

Other than Lemma 2.1, which is a tool for proving non-regularity, we do not go into detail on cardinal functions on $X \otimes Y$; this is considered more carefully in Hart [10]. Rather, we concentrate here on conditions which ensure the regularity or normality of $X \otimes Y$, and relate this question to more well-known topological notions, such as Eberlein compacta, Sierpiński sets, and σ-sets; terminology not in Engelking [5] will be defined where first used.

Our sharpest results involve the products of two separable metric spaces (Section 5) and two compact Hausdorff spaces (Section 4). Some general results, which apply to both situations, are proved in Section 2.

In the compact case: $X \otimes Y$ cannot be T_3 unless either X or Y is finite or X, Y are both scattered. If they are both scattered, then $X \otimes Y$ is sometimes, but not always, T_3. $X \otimes X$ is T_3 iff X is a strong Eberlein compactum, in which case $X \otimes X$ is also paracompact. Section 3 has some preliminary remarks on strong Eberlein compacta.

In the separable metric case: $X \otimes Y$ is not T_3 if $|X| = |Y| = \mathfrak{c}$. If X is countable and non-discrete, then $X \otimes Y$ is T_3 iff Y is a σ-set (every Borel subset is a relative F_σ), in which case $X \otimes Y$ is also paracompact. Since $X \otimes Y$ is trivially paracompact if one of X, Y is discrete, all possibilities are settled under CH. Under $\neg CH$, there are some independence results. If all σ-sets are countable (which is consistent by Theorem 22 of Miller [18]), then $X \otimes Y$ is T_3 iff both are countable or at least one of them is discrete. Under $MA + \neg CH$, we have some partial results; in particular (Corollary 5.8), $X \otimes Y$ is T_3, and in fact T_4, if $|X| < \mathfrak{p}$ and $|Y| < \mathfrak{p}$ (see Fremlin [7] for a discussion of MA and the cardinals $\mathfrak{p}, \mathfrak{m}, \mathfrak{c}$, etc.). Note that $X \otimes Y$ cannot be paracompact (or even collectionwise Hausdorff) when X, Y are uncountable separable metric spaces, since it is separable and has an uncountable closed discrete set (by (7) and (4) of Proposition 1.2).

2 Basics

A separable space of weight greater than $\mathfrak{c}$ cannot be regular, so the following lemma can sometimes be used to show that $X \otimes Y$ is non-regular. It applies immediately to $X = Y = \mathbb{R}$ (using Proposition 1.2.7), and with somewhat more work to all Čech-complete non-scattered spaces (see Theorem 2.7).

Lemma 2.1 *Suppose X, Y are T_2. Suppose that $w(X) \le \mathfrak{c}$ and each non-empty open subset of X has size at least $\mathfrak{c}$. Suppose that there are disjoint countable $D_\alpha \subset Y$ for $\alpha < \mathfrak{c}$ such that each D_α is dense in Y. Then $\chi((p,q), X \otimes Y) \ge 2^{\mathfrak{c}}$ for all $(p,q) \in X \times Y$.*

Proof. We shall in fact find $F_\delta \subseteq X \otimes Y$ for $\delta < 2^{\mathfrak{c}}$ such that each F_δ is closed and discrete, but each countable union $\bigcup_{n\in\omega} F_{\delta_n}$ is dense (for distinct δ_n). This is sufficient, because if (p,q) had character less than $2^{\mathfrak{c}}$, we could find a neighborhood W of (p,q) and distinct δ_n $(n \in \omega)$ such that each $W \cap (F_{\delta_n} \backslash \{(p,q)\}) = \emptyset$; since $\bigcup_n F_{\delta_n}$ is dense, (p,q) would be isolated, which is impossible, given the assumptions on X and Y.

By the assumption on X, we can find disjoint $B_\alpha \subset X$ for $\alpha < \mathfrak{c}$ such that each $|B_\alpha| = \mathfrak{c}$, and for all non-empty open $U \subseteq X$, there is an α with $B_\alpha \subseteq U$. Since the B_α are disjoint and have size $\mathfrak{c}$, we may fix $g_\delta : X \to \omega$ for $\delta < 2^{\mathfrak{c}}$ such that for each α, the sequence $\langle g_\delta \restriction B_\alpha : \delta < 2^{\mathfrak{c}} \rangle$ is σ-independent (see Engelking – Karłowicz [6]); that is, given distinct $\delta_n < 2^{\mathfrak{c}}$ and any $k_n \in \omega$, there is an $x \in B_\alpha$ such that $g_{\delta_n}(x) = k_n$ for all $n \in \omega$.

Let $B = \bigcup_{\alpha<c} B_\alpha$. Re-index the D_α as $\langle D_x : x \in B \rangle$, and then index each D_x as $\{d_x^n : n < \omega\}$. Define $F_\delta : B \to Y$ by $F_\delta(x) = d_x^{g_\delta(x)}$. Since the D_x are disjoint, F_δ is 1-1, so F_δ (i.e., its graph) is closed and discrete in $X \otimes Y$ by Proposition 1.2.4.

Now, fix distinct δ_n for $n < \omega$, and let $H = \bigcup_{n\in\omega} F_{\delta_n}$. To show that H is dense, we fix any non-empty open $N \subseteq X \otimes Y$, and show that $N \cap \overline{H} \neq \emptyset$. Fix $y \in Y$ such that $N^y \neq \emptyset$. We find an x so that $(x,y) \in N \cap \overline{H}$ as follows: First, fix $\alpha < \mathfrak{c}$ such that $B_\alpha \subseteq N^y$. Then, choose $x \in B_\alpha$ such that $g_{\delta_n}(x) = n$ for all $n \in \omega$. So, $F_{\delta_n}(x) = d_x^n$ for each n. Thus, $\{x\} \times D_x \subseteq H$, so $\{x\} \times Y \subseteq \overline{H}$, so $(x,y) \in N \cap \overline{H}$. ☺

Remark 2.2 Besides being useful for establishing non-regularity, this lemma is of interest because it computes the weight and character of $\mathbb{R} \otimes \mathbb{R}$. A more constructive proof of non-regularity is obtained by the method of Sierpiński [28], who does not explicitly mention $\mathbb{R} \otimes \mathbb{R}$: Fix $D \subseteq \mathbb{R} \times \mathbb{R}$ with D dense in the usual Tychonov topology. Sierpiński showed that if $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is separately continuous and $f \equiv 0$ on D, then $f \equiv 0$ everywhere. Since we may take D to be the graph of a 1-1 function, which is closed and discrete in $\mathbb{R} \otimes \mathbb{R}$ (by Proposition 1.2.4), this implies immediately that $\mathbb{R} \otimes \mathbb{R}$ is not completely regular. In fact, Sierpiński's proof establishes that whenever $U \supseteq D$ is +open, U must be +dense in $\mathbb{R} \otimes \mathbb{R}$; hence $\mathbb{R} \otimes \mathbb{R}$ is not even regular. For generalizations of Sierpiński's result and references to the literature, see [23, 24]. The proof of Lemma 2.3 below, which refutes regularity by a category argument, is close to the original Sierpiński argument. A similar proof is used also in (3.2) of [15] to derive the non-regularity of some $X \otimes Y$.

A *Baire space* is one in which every countable intersection of dense open sets is dense. A sequence of sets, $\langle F_i : i \in I \rangle$, is *point-finite* iff $\{i : y \in F_i\}$ is finite for each y.

Lemma 2.3 *Suppose that Y is Baire and contains a point-finite sequence of closed sets, $\langle F_n : n \in \omega \rangle$, such that $\bigcup_{n>m} F_n$ is dense in Y for each $m \in \omega$. Then $(\omega + 1) \otimes Y$ is not T_3.*

Proof. $\langle F_n : n \in \omega \rangle$ defines a set $F \subseteq \omega \times Y \subseteq (\omega + 1) \times Y$, and F is closed in the +open topology because each F^y is finite and hence closed. Fix a +open U with $F \subseteq U \subseteq (\omega + 1) \times Y$, and let H be the +closure of U. We shall show that $\{\omega\} \times Y \subseteq H$. This is sufficient, because if $(\omega + 1) \otimes Y$ were T_3, we could separate any point in $\{\omega\} \times Y$ from F by open sets.

Now, $B = \bigcap_{m<\omega} \bigcup_{n>m} U_n$ is dense in Y because Y is Baire. If $y \in B$, then $\omega \cap U^y$ is infinite, so that $(\omega, y) \in H$. Thus, $\{\omega\} \times B \subseteq H$, and hence $\{\omega\} \times Y \subseteq H$. ☺

Frequently, in proving $X \otimes Y$ is not regular, lemmas such as 2.1 or 2.3 do not apply directly to X, Y, but rather to some subspaces $X_1 \subseteq X$ and $Y_1 \subseteq Y$. One then applies the fact that T_3 is hereditary, plus:

Lemma 2.4 *If X_1 is closed in X and Y_1 is closed in Y, then the +open topology on $X_1 \times Y_1$ is the same as the relative topology it inherits from the +open topology on $X \times Y$.*

Proof. Every +closed subset of $X_1 \times Y_1$ is +closed in $X \times Y$. ☺

This lemma is also proved in [15], where it is pointed out that, unlike with the standard Tychonov topology, this lemma might fail if we drop the assumption that X_1 and Y_1 are closed. With the aid of this lemma, we can prove Theorem 2.7, which generalizes the non-regularity of $\mathbb{R}\otimes\mathbb{R}$ by replacing $\mathbb{R}$ by any non-scattered space which is Čech-complete – in particular, which is complete metric or compact Hausdorff. Basic facts about Čech-complete spaces are in §3.9 of [5]; in addition, we need the following two remarks:

Lemma 2.5 *Suppose that X is Čech-complete and not scattered. Then there is a compact $H \subseteq X$ and a continuous irreducible map from H onto $[0,1]$.*

Proof. By assumption, X is a G_δ in βX. By the standard tree argument, we may obtain a closed subset K of βX such that $K \subseteq X$ and K maps onto 2^ω, and hence onto $[0,1]$. Then, since K is compact, we may find a closed $H \subseteq K$ such that this map onto $[0,1]$ is irreducible on H. ☺

Lemma 2.6 *If X is Čech-complete and scattered but not discrete, then there is a closed $H \subseteq X$ homeomorphic to $\omega + 1$.*

Proof. Let I be the set of isolated points, and fix $p \in X \backslash I$ which is isolated in $X \backslash I$. Since X is regular, there is a neighborhood U of p such that $Z = \overline{U} \subseteq \{p\} \cup I$. If $J = Z \cap I$, then $Z = \{p\} \cup J$, and J is open and discrete in Z, while p is a limit point of J. Since Z is closed in X, it is also Čech-complete, and hence a G_δ in βZ. Now, $\beta Z = F \cup J$, where J is open and discrete in βZ and F is closed. Since Z is a G_δ, we may find open $U_n \subseteq \beta Z$, with $p \in U_n$, each $\overline{U}_{n+1} \subseteq U_n$, and $\bigcap_{n\in\omega} U_n = \bigcap_{n\in\omega} \overline{U}_n \subseteq \{p\} \cup J$. Then, if we choose distinct $x_n \in U_n \cap J$, the only possible limit point of the x_n in βZ is p, so that $\{x_n : n \in \omega\} \cup \{p\}$ is homeomorphic to $\omega + 1$. ☺

Theorem 2.7 *Suppose X, Y are both Čech-complete and non-discrete, and $X \otimes Y$ is T_3. Then X and Y are both scattered.*

Proof. First, we prove that $X \otimes Y$ is not T_3 in the case that neither X nor Y are scattered. Applying Lemmas 2.5 and 2.4, we may assume also that X and Y are compact, and that there are irreducible maps $f : X \to [0,1]$ and $g : Y \to [0,1]$. But now Lemma 2.1 applies. X is compact separable with no isolated points, so $w(X) \leq \mathfrak{c}$ and each non-empty open subset of X has size at least $\mathfrak{c}$. To obtain the $D_\alpha \subset Y$ for $\alpha < \mathfrak{c}$, let the E_α be disjoint countable dense subsets of $[0,1]$, and let D_α be countable with $g(D_\alpha) = E_\alpha$. So, $X \otimes Y$ is separable (by Proposition 1.2.7) and of weight larger than $\mathfrak{c}$, so it cannot be regular.

Now, assume that X is scattered and Y is not. Again, passing to closed subspaces (now applying Lemma 2.6) we may assume that $X = \omega + 1$ and that Y is compact separable and has no isolated points. Then, if $\{y_n : n \in \omega\}$ is dense in Y, we may apply Lemma 2.3 with $F_n = \{y_n\}$.

Later, we consider in more detail the regularity of $X \otimes Y$ when they are both compact scattered (Section 4), and when they are both separable metric (Section 5). Now, we take up a notion which is relevant to both cases:

Definition 2.8 *A space X is κ-*trite *iff whenever $\langle F_\alpha : \alpha < \kappa \rangle$ is a point-finite sequence of closed sets, there is a point-finite sequence of open sets, $\langle U_\alpha : \alpha < \kappa \rangle$, with each $F_\alpha \subseteq U_\alpha$. X is* trite *iff X is κ-trite for all κ.*

Clearly, the notion of κ-trite gets stronger as κ increases, since some of the F_α may be empty.

Definition 2.9 *$\mho A$ denotes the 1-point compactification of A, where the set A is given the discrete topology.*

Note that if κ is an infinite cardinal, then $\mho\kappa \cong \kappa + 1$ iff $\kappa = \omega$; as usual, if not stated otherwise, ordinals are presumed to have their ordinal topology.

In many cases, if $X \otimes Y$ is T_3, it is also T_4, and in fact paracompact. This holds, for example, if $X = Y$ and they are both compact (see Theorem 4.6); in fact, then $X \otimes X$ is actually *ultra-paracompact* (that is, every open cover has a disjoint clopen refinement). A simpler example of this phenomenon follows:

Lemma 2.10 *Assume that κ is an infinite cardinal, and consider the following properties:*

- *a. Y is κ-trite*
- *b. $\mho\kappa \otimes Y$ is T_4.*
- *c. $\mho\kappa \otimes Y$ is T_3.*
- *d. $\mho\kappa \otimes Y$ is paracompact.*
- *e. $\mho\kappa \otimes Y$ is ultra-paracompact.*

If Y is T_4, then $(a) \Leftrightarrow (b)$. If Y is paracompact, then $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d)$. If Y is ultra-paracompact, then $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d) \Leftrightarrow (e)$.

Proof. For $(b) \Rightarrow (a)$, if $\langle F_\alpha : \alpha < \kappa \rangle$ is a point-finite sequence of closed sets, it defines a closed set $F \subseteq \kappa \times Y \subset \mho\kappa \otimes Y$. Separating F from $\{\infty\} \times Y$ by open sets will produce the desired point-finite $\langle U_\alpha : \alpha < \kappa \rangle$.

For $(a) \Rightarrow (b)$, suppose that F, H are closed and disjoint in $\mho\kappa \otimes Y$. First, consider the special case that $(\{\infty\} \times Y) \cap F = \emptyset$. Then $\langle F_\alpha : \alpha < \kappa \rangle$ is point-finite. Since Y is κ-trite and T_4, we can choose open $U_\alpha \subseteq Y$ such that $\langle \overline{U}_\alpha : \alpha < \kappa \rangle$ is point-finite, each $F_\alpha \subseteq U_\alpha$, and each $H_\alpha \cap \overline{U}_\alpha = \emptyset$. Then $\langle U_\alpha : \alpha < \kappa \rangle$ defines an open U such that $F \subseteq U$ and $H \cap \overline{U} = \emptyset$.

As is typical in proofs of normality, we reduce the general case to the special case as follows. Since Y is T_4, we can find $V, W \subseteq Y$ with $\overline{V} \cap \overline{W} = \emptyset$, $F_\infty \subseteq V$, and $H_\infty \subseteq W$. On each of the two subspaces, $\mho\kappa \otimes (Y \backslash V)$ and $\mho\kappa \otimes (Y \backslash W)$, we can apply the special case to separate F, H, and then we can amalgamate the separations to separate F, H in $\mho\kappa \times Y$.

Now, assume that Y is paracompact. To prove $(a) \Leftrightarrow (b) \Leftrightarrow (c) \Leftrightarrow (d)$, it is sufficient to prove $(c) \Rightarrow (d)$, so assume (c), and let $\mathcal{U}$ be an +open cover of $\mho\kappa \otimes Y$. Note first that it is sufficient to find locally finite +open $\mathcal{V}$, $\mathcal{W}$ refining $\mathcal{U}$ such that $\{\infty\} \times Y \subseteq \bigcup \mathcal{W}$, with $\mathcal{W} = \{W_\xi : \xi < \lambda\}$, $\mathcal{V} = \{V_\xi : \xi < \lambda\}$, and each $\overline{W_\xi} \subseteq V_\xi$ (here, closures are taken with respect to the +open topology). Given $\mathcal{V}$, $\mathcal{W}$, we may, for each $\alpha \in \kappa$, let $\mathcal{E}_\alpha$ be a locally finite open refinement of $\mathcal{U}$ such that $(\{\alpha\} \times Y) \backslash \bigcup_\xi V_\xi \subseteq \bigcup \mathcal{E}_\alpha \subseteq (\{\alpha\} \times Y) \backslash \bigcup_\xi \overline{W_\xi}$. Then $\mathcal{V} \cup \bigcup_\alpha \mathcal{E}_\alpha$ is a locally finite open refinement of $\mathcal{U}$ which covers $\mho\kappa \otimes Y$.

To get $\mathcal{V}$, $\mathcal{W}$, first apply (c) to get open families $\{W'_\xi : \xi < \lambda\}$ and $\{V'_\xi : \xi < \lambda\}$ refining $\mathcal{U}$, with $\overline{W'_\xi} \subseteq V'_\xi$ and $\{\infty\} \times Y \subseteq \bigcup_\xi W'_\xi$. Then apply paracompactness of Y to choose open Q_ξ, P_ξ, with $\overline{Q_\xi} \subseteq P_\xi \subseteq (W'_\xi)_\infty$, such that $\{P_\xi : \xi < \lambda\}$ is locally finite and $\bigcup_\xi Q_\xi = Y$. Then let $W_\xi = W'_\xi \cap (\mho\kappa \times Q_\xi)$ and $V_\xi = V'_\xi \cap (\mho\kappa \times P_\xi)$.

Finally, if Y is ultra-paracompact, we need to prove $(c) \Rightarrow (e)$. The argument is similar, but now we must get also that each $V_\xi = W_\xi$ and is +clopen, and then take the $\mathcal{E}_\alpha$ to be +clopen families. Choose the W'_ξ and V'_ξ as before, but now choose disjoint clopen Q_ξ with $Q_\xi \subseteq (W'_\xi)_\infty$, such that $\bigcup_\xi Q_\xi = Y$. For each ξ, let $(W_\xi)_\infty = Q_\xi$, and let each $(W_\xi)_\alpha$ be some clopen set with $\overline{(W'_\xi)_\alpha} \cap Q_\xi \subseteq (W_\xi)_\alpha \subseteq (V'_\xi)_\alpha \cap Q_\xi$. Note that W_ξ is +clopen because for each $y \in Y$, $(W_\xi)^y$ is either empty (when $y \notin Q_\xi$) or cofinite (when $y \in Q_\xi$), so that each $(W_\xi)^y$ is clopen in $\mho\kappa$. ☺

Corollary 2.11 *If Y is ω-trite and Čech complete, then Y is scattered.*

Proof. If Y is not scattered, then, by Lemma 2.5, let H be a compact subspace of Y with no isolated points. Then $(\omega + 1) \otimes H$ is not T_3 (by Theorem 2.7). Since H is T_4, Lemma 2.10 applies to show that H, and hence Y, fails to be ω-trite.

We do not know if the converse to this corollary holds for Čech complete spaces, but it does hold for compact Hausdorff spaces (Corollary 4.2). Moreover, a compact Hausdorff space is trite (that is, κ-trite for all κ) iff it is a strong Eberlein compactum (Proposition 3.3). An uncountable separable metric space Y can never be ω_1-trite (just take the F_α to be distinct points); Y is ω-trite iff Y is a σ-set (Lemma 5.4).

When X and Y are totally disconnected, it is easy to see that $X \otimes Y$ is as well. But we do not know of any simple analog for $X \otimes Y$ of the (trivial) fact that the Tychonov product of 0-dimensional spaces is 0-dimensional. We do get a special case of this when X is countable and Y is Lindelöf. Then $X \otimes Y$ is also Lindelöf, so that the topological properties listed in $(b), (c), (d)$ of Lemma 2.10 are trivially equivalent for $X \otimes Y$, and when Y is 0-dimensional, we also get that $X \otimes Y$ is 0-dimensional (and hence ultraparacompact). When Y is just paracompact, we still get equivalences of the properties of $(b), (c), (d)$.

Theorem 2.12 *Suppose that X is countable and $X \otimes Y$ is T_3.*

a. If Y is paracompact, then $X \otimes Y$ is paracompact.

b. If Y is Lindelöf and 0-dimensional, then $X \otimes Y$ is 0-dimensional, and hence ultra-paracompact.

Proof. To prove (a), it suffices to show every open cover of $X \otimes Y$ has a σ-locally finite open refinement. In fact, since X is countable, it suffices to get, for each $x \in X$, a locally finite refinement covering $\{x\} \times Y$. So, fix an open cover $\mathcal{U}$ of $X \otimes Y$, and fix $x \in X$. Then $\{U_x : U \in \mathcal{U}\}$ covers Y, and so has a locally finite refinement $\mathcal{V}$. For each $V \in \mathcal{V}$, choose $U_V \in \mathcal{U}$ such that $V \subseteq (U_V)_x$. Now $\{X \times V : V \in \mathcal{V}\}$, and thus $\mathcal{W} = \{(X \times V) \cap U_V : V \in \mathcal{V}\}$, are also locally finite and cover $\{x\} \times Y$, and $\mathcal{W}$ refines $\mathcal{U}$.

To prove (b), it suffices to show that whenever H, K are disjoint closed subsets of $X \otimes Y$, they can be separated by disjoint clopen sets. Then the clopen sets will constitute a base for $X \otimes Y$.

Assume X is infinite (otherwise (b) is trivial), and list X as $\{x_n : n \in \omega\}$. For each $n \in \omega$ and any $A \subseteq X \times Y$, let A_n abbreviate A_{x_n}. In what follows, we use $A^{(n)}$ to denote a subset of $X \otimes Y$ rather than a fiber $A^n \subseteq X$.

We shall inductively choose open sets $U^{(n)}, V^{(n)} \subseteq X \otimes Y$ to construct disjoint clopen sets $U = \bigcup_{n\in\omega} U^{(n)}$ and $V = \bigcup_{n\in\omega} V^{(n)}$ with $H \subseteq U$ and $K \subseteq V$. To ensure that U, V will be clopen, each $U^{(n)}, V^{(n)}$ will be chosen so that

i. $\overline{U^{(n)}} \cap \overline{V^{(n)}} = \emptyset$,

ii. $\overline{U^{(n)}} \subseteq U^{(n+1)}$ and $\overline{V^{(n)}} \subseteq V^{(n+1)}$, and

iii. $\{x_n\} \times Y \subseteq U^{(n+1)} \cup V^{(n+1)}$.

First, to ensure that U and V separate H and K, choose $U^{(0)}$,$V^{(0)}$ satisfying (i) with $H \subseteq U^{(0)}$ and $K \subseteq V^{(0)}$ ($X \otimes Y$ is T_3 and Lindelöf, and hence T_4). Now suppose $U^{(k)}, V^{(k)}$ have been chosen, for $k \leq n$, so that $(i), (ii), (iii)$ hold. Since Y is Lindelöf and 0-dimensional, there is a clopen subset $E_n \subseteq Y$ such that $\big(\overline{U^{(n)}}\big)_n \subseteq E_n \subseteq Y \setminus \big(\overline{V^{(n)}}\big)_n$. This gives us closed subsets $H^{(n+1)} = \overline{U^{(n)}} \cup (\{x_n\} \times E_n)$ and $K^{(n+1)} = \overline{V^{(n)}} \cup (\{x_n\} \times (Y \setminus E_n))$, so we can choose open $U^{(n+1)}, V^{(n+1)} \subseteq X \otimes Y$ satisfying (i), with $H^{(n+1)} \subseteq U^{(n+1)}$ and $K^{(n+1)} \subseteq V^{(n+1)}$.

By (i) and (ii), U and V are disjoint, and by (iii), $U \cup V = X \otimes Y$, so U, V are clopen. ☺

The following generalization of the +open topology will be useful for simplifying some of the proofs, and may be of interest in its own right. This notion was also used by R. Brown [4] in defining a number of topologies on product spaces.

Definition 2.13 *Let $(Z; \mathcal{T})$ be a topological space and $\mathcal{E}$ any family of subsets of Z. Then $\mathcal{T}_\mathcal{E}$ is the family of all $U \subseteq Z$ such that $U \cap E$ is relatively open in E for all $E \in \mathcal{E}$.*

$\mathcal{T}_\mathcal{E}$ is clearly a topology on Z, and the following is easy from the definition:

Proposition 2.14 *For any $(Z; \mathcal{T})$:*

1. *If $\mathcal{E} \subseteq \mathcal{E}'$, then $\mathcal{T}_{\mathcal{E}'} \subseteq \mathcal{T}_\mathcal{E}$.*
2. *$\mathcal{T}_\mathcal{E} \supseteq \mathcal{T} = \mathcal{T}_{\{Z\}} = \mathcal{T}_{\mathcal{P}(Z)}$.*
3. *Suppose that every $E \in \mathcal{E}$ is closed in $\bigcup \mathcal{E}$. Let $\mathcal{E}'$ be the set of all finite unions of elements of $\mathcal{E}$. Then $\mathcal{T}_{\mathcal{E}'} = \mathcal{T}_\mathcal{E}$.*

4. *Let $Z' = \bigcup \mathcal{E}$, and let $\mathcal{T}'$ be the usual subspace topology Z' inherits from $\mathcal{T}$. Now we have topologies $\mathcal{T}_{\mathcal{E}}$ on Z and $\mathcal{T}'_{\mathcal{E}}$ on Z'. Then Z' is clopen in $\mathcal{T}_{\mathcal{E}}$, all points of $Z \backslash Z'$ are isolated in $\mathcal{T}_{\mathcal{E}}$, and $\mathcal{T}'_{\mathcal{E}}$ is the subspace topology which Z' inherits from $\mathcal{T}_{\mathcal{E}}$.*

For example, if $Z = X \times Y$, $\mathcal{T}$ is the usual Tychonov topology, and $\mathcal{E}$ is the family of all $\{a\} \times Y$ (for $a \in X$) and all $X \times \{b\}$ (for $b \in Y$), then $\mathcal{T}_{\mathcal{E}}$ is the +open topology. Here, assuming X, Y are T_1, every $E \in \mathcal{E}$ is closed in $\bigcup \mathcal{E} = Z$.

As another example, let $Z = \mathbb{R} \times \mathbb{R}$. Now, rather than the +open topology, which is not invariant under linear transformations, it might be more natural to let $\mathcal{E}$ be the family of all lines (or, for that matter, all algebraic curves). It is easy to modify the proof in Remark 2.2 to establish that $\mathcal{T}_{\mathcal{E}}$ is not regular (now choose D to be algebraically independent and dense). However, by Theorem 5.7, $\mathcal{T}_{\mathcal{E}}$ will be regular (and normal) if we replace $\mathbb{R}$ by some subfield of $\mathbb{R}$ of size less than $\mathfrak{p}$. A similar example, due to Zeeman [31], is related to special relativity. Let $Z = \mathbb{R}^4$ = space-time. Let $\mathcal{E}$ be the family of all time-like lines, together with all space-like 3-planes. Then the group of homeomorphisms of $(Z; \mathcal{T}_{\mathcal{E}})$ is precisely the group generated by the Lorentz transformations plus the translations and dilations.

We now make some remarks on the regularity or normality of $\mathcal{T}_{\mathcal{E}}$ in general. First, note that the assumption that every $E \in \mathcal{E}$ is closed in $\bigcup \mathcal{E}$ is important, even in the case that everything is countable. For example, let $(Z; \mathcal{T})$ be $\mathbb{Q}$, with the usual topology, let $S = \{2^{-n} : n \in \omega\}$, and let $\mathcal{E} = \{E_1, E_2\}$, where $E_1 = \mathbb{Q} \backslash S$ and $E_2 = \mathbb{Q} \backslash \{0\}$. Then $(Z; \mathcal{T}_E)$ is one of the standard examples of a countable non-regular T_2 space; S is closed, but 0 is in the closure of every neighborhood of S.

Next, note that if every $E \in \mathcal{E}$ is closed in $\bigcup \mathcal{E}$, *and* $\mathcal{E}$ is countable, then separation axioms for $\mathcal{T}$ can be used directly to prove separation axioms for $\mathcal{T}_{\mathcal{E}}$. For example,

Lemma 2.15 *Assume that $(Z; \mathcal{T})$ is T_4 and strongly 0-dimensional. Let $\mathcal{E}$ be a countable subset of $\mathcal{P}(Z)$ such that every $E \in \mathcal{E}$ closed in $\bigcup \mathcal{E}$. Then $(Z; \mathcal{T}_{\mathcal{E}})$ is T_4 and strongly 0-dimensional.*

Proof. By Proposition 2.14, we may assume that $\mathcal{E}$ is closed under finite unions and that $\bigcup \mathcal{E} = Z$. Then, fix $E_n \in \mathcal{E}$ for $n \in \omega$ such that $E_n \nearrow Z$ and $\forall E \in \mathcal{E} \exists n [E \subseteq E_n]$. Now, let H, K be disjoint and $\mathcal{T}_{\mathcal{E}}$-closed. Inductively construct $H = H_0 \subseteq H_1 \subseteq \cdots$ and $K = K_0 \subseteq K_1 \subseteq \cdots$ so that each H_n, K_n are disjoint and $\mathcal{T}_\mathcal{E}$-closed, and $H_{n+1} \cup K_{n+1} = E_n \cup H \cup K$. Then H and K are separated by the $\mathcal{T}_\mathcal{E}$-clopen sets $\bigcup_n H_n$ and $\bigcup_n K_n$.

This implies the following, which is also Theorem 7.1 of [16]:

Corollary 2.16 *If X and Y are countable and T_3, then $X \otimes Y$ is ultra-paracompact.*

The lemma may fail when $|\mathcal{E}| = \aleph_1$. For example, let $(Z; \mathcal{T})$ be the Tychonov product $(\omega_1 + 1) \times (\omega_1 + 1)$, and let $\mathcal{E}$ be the usual family for which $\mathcal{T}_\mathcal{E}$ is $(\omega_1 + 1) \otimes (\omega_1 + 1)$. Then, by Theorem 4.6, $\mathcal{T}_\mathcal{E}$ is not regular.

3 Strong Eberlein Compacta

Theorem 4.6 and Proposition 3.3 imply that for X compact Hausdorff, $X \otimes X$ is T_3 iff X is trite. Now, for compact spaces, "trite" is equivalent to a number of other properties. Most well-known is "strong Eberlein compact", but other equivalents are related to topological games and topological orders. In this section, we just discuss those equivalents which do not involve the +open topology.

Definition 3.1 *X is a* strong Eberlein compactum *iff for some κ and some closed $F \subseteq 2^\kappa$: X is homeomorphic to F and $\{\alpha : f(\alpha) = 1\}$ is finite for all $f \in F$.*

See, e.g., [1, 3, 8, 9, 22], for more about these and Eberlein compacta in general. It is well-known that strong Eberlein compacta are scattered, and we have just shown (Corollary 2.11) that trite compacta are also scattered.

To deal with compact scattered spaces, we use some basic facts and terminology. First, as is easy to see, scattered spaces are hereditarily disconnected, and hence (see [5]) compact scattered spaces are 0-dimensional. For any space X, one can derive a sequence of subsets of X: $X^{(0)} = X$, and $X^{(\alpha+1)}$ is the set of limit points of $X^{(\alpha)}$. For limit γ, $X^{(\gamma)} = \bigcap_{\alpha<\gamma} X^{(\alpha)}$. Then X is scattered iff some $X^{(\alpha)}$ is empty, and one defines the Cantor-Bendixon rank as follows:

Definition 3.2 *If X is compact Hausdorff, scattered, and nonempty, then* $\mathrm{rank}(X)$ *is the least α such that $X^{(\alpha+1)} = \emptyset$. If $x \in X$, then* $\mathrm{rank}(x, X)$ *is the least α such that $x \notin X^{(\alpha+1)}$.*

Equivalently (by compactness) the rank of X is the α such that $X^{(\alpha)}$ is finite and non-empty, and the rank of x in X is the α such that x is an isolated point of $X^{(\alpha)}$. Note that since $X^{(\alpha+1)}$ is closed and $X^{(\alpha)} \backslash X^{(\alpha+1)}$ is relatively discrete, each x of rank α has a neighborhood U such that $\mathrm{rank}(y, X) < \alpha$ for all $y \in U \backslash \{x\}$.

The following lists some easy equivalents to compact trite:

Proposition 3.3 *If X is compact Hausdorff, then the following are equivalent:*

1. *X is trite.*
2. *There is a point-finite sequence of clopen sets of the form $\langle V_p : p \in X \rangle$, where $p \in V_p$ for each $p \in X$.*
3. *X is a strong Eberlein compactum.*

Proof. (1) $\Rightarrow$ (2): The existence of open V_p is immediate from the definition, and we can shrink them to clopen sets because X is 0-dimensional.

(2) $\Rightarrow$ (1): Let $\langle F_\alpha : \alpha < \kappa \rangle$ be a point-finite sequence of closed sets. Let $U_\alpha = \bigcup \{V_p : p \in F_\alpha\}$. Clearly U_α is open and $F_\alpha \subseteq U_\alpha$. To see that $\langle U_\alpha : \alpha < \kappa \rangle$ is point-finite, fix x; then $E = \{p : x \in V_p\}$ is finite, and $\{\alpha : x \in U_\alpha\} = \bigcup_{p \in E} \{\alpha : p \in F_\alpha\}$.

(3) $\Rightarrow$ (2): Assume $X \subseteq 2^\kappa$ and $\{\alpha : p(\alpha) = 1\}$ is finite for all $p \in X$. Then let $V_p = \{q \in X : \forall \alpha [p(\alpha) \leq q(\alpha)]\}$.

(2) $\Rightarrow$ (3): Let the V_p be as in (2), and let $\varphi : X \to 2^X$ be such that $\varphi(x)(p) = 1$ iff $x \in V_p$. Then φ is continuous (giving 2^X the usual product topology), and $\mathrm{ran}(\varphi)$ is a strong Eberlein compactum (since $\langle V_p : p \in X \rangle$ is point-finite). φ might fail to be 1-1. To prevent this, note that X is scattered (by (2) $\Rightarrow$ (1) and Corollary 2.11), so we may assume also that $\forall x \in V_p \backslash \{p\} [\mathrm{rank}(x, X) < \mathrm{rank}(p, X)]$. This ensures that we never have distinct p, q with $p \in V_q$ and $q \in V_p$, so that the V_p separate points. Now φ is 1-1, and hence a homeomorphism.

A deeper result, due to Gruenhage [9], yields an equivalent in terms of a game he introduced in [8]:

Definition 3.4 *If X is any topological space and $p \in X$, the* Convergent Sequence Game*, $CSG(p, X)$ is played as follows: Alice and Bob take turns for ω plays, with Alice going first. At her n^{th} turn, Alice must play an open* *neighborhood* U_n *of* p*, and then Bob, at his* n^{th} *turn, must choose a point* $x_n \in U_n$*. Alice wins this play of the game iff the sequence* $\langle x_n : n \in \omega \rangle$ *converges to* p*.*

Theorem 3.5 (Gruenhage [9]) *If* X *is compact scattered, then* X *is a strong Eberlein compactum iff Alice has a winning strategy in* $CSG(p, X)$ *for all* $p \in X$*.*

Yet another equivalent to strong Eberlein compact is in terms of topological orders; see, e.g., Nachbin [21], for more information on these.

Definition 3.6 *A* compact order *is a pair* $(X; \leq)$*, where* X *is a compact Hausdorff space and* $\leq$ *is a partial order on* X *which is closed in* $X \times X$ *(with the usual Tychonov topology). If* $S \subseteq X$*, then* $S{\downarrow} = \{x \in X : \exists y \in S[x \leq y]\}$ *and* $S{\uparrow} = \{x \in X : \exists y \in S[x \geq y]\}$*. If* $x \in X$*, then* $x{\downarrow} = \{x\}{\downarrow}$ *and* $x{\uparrow} = \{x\}{\uparrow}$*.*

Lemma 3.7 *Suppose that* $(X; \leq)$ *is a compact order and* F *is a closed subset of* X*. Then* $F{\downarrow}$ *and* $F{\uparrow}$ *are closed.*

Note that the product of compact orders is a compact order. In particular, there is the natural compact product order on 2^κ, where $f \leq g$ iff $f(\alpha) \leq g(\alpha)$ for all α. It follows that one may characterize strong Eberlein compacta by:

Lemma 3.8 X *is a strong Eberlein compactum iff there is a partial order* $\leq$ *on* X *such that* $(X; \leq)$ *is a compact order satisfying:*

a. $(X; \leq)$ *has no infinite chains.*
b. $\langle q{\uparrow} : q \in X \rangle$ *is point-finite; equivalently, each* $q{\downarrow}$ *is finite.*
c. Each $q{\uparrow}$ *is clopen.*

Proof. If X is closed in 2^κ and $\{\alpha : q(\alpha) = 1\}$ is finite for all $q \in X$, then the natural product order satisfies (a)(b)(c). Conversely, given such an order, the $V_p = p{\uparrow}$ satisfy (2) of Proposition 3.3. ☺

In Section 4, it will be convenient to argue directly from (a)(b)(c). We now make some remarks, not needed for the rest of this paper, on which of these conditions can be dropped.

Condition (a) is redundant, since if C is any chain, we may choose $q \in \bigcap_{x \in C} x{\uparrow}$ and apply (b) to show that C is finite. However, (c) cannot be dropped, since if X is an arbitrary compact Hausdorff space and $\leq$ is the trivial order ($x \leq y$ iff $x = y$), then conditions (a)(b) are satisfied. Also, a 1-point compactification of a Mrówka Ψ space shows that (b) cannot be dropped. That is, let A_α, for $\alpha < \omega_1$, be almost disjoint subsets of ω. Let $X = \omega \cup \{p_\alpha : \alpha < \omega_1\} \cup \{\infty\}$, where elements of ω are isolated, basic neighborhoods of p_α are $\{p_\alpha\}$ together with a tail of A_α, and X is the 1-point compactification of $\omega \cup \{p_\alpha : \alpha < \omega_1\}$. Order X by placing elements of ω on top and incomparable to each other, placing ∞ on the bottom, and letting $p_\alpha < n$ iff $n \in A_\alpha$. Then (a)(c) hold, but this space cannot be a strong Eberlein compactum, since there is a countable set (i.e., ω) with uncountable closure. However, if the order is induced by a semilattice operation, then (a) alone is sufficient; see Junnila [14] for a proof and for further results on Eberlein compacta and compact orders and semilattices.

4 Compact Spaces

We consider here the regularity of $X \otimes Y$, where X, Y are both infinite compact Hausdorff spaces. By Theorem 2.7, this is only an issue when they are both scattered. We note first that in this case, $X \otimes Y$ will be T_3 whenever X is countable. By Theorem 2.12, that is equivalent to proving $X \otimes Y$ is ultra-paracompact. It turns out that the argument does not require compactness of X:

Lemma 4.1 *Suppose that X is countable and T_3, and Y is compact scattered and T_2. Then $X \otimes Y$ is ultra-paracompact.*

Proof. We fix X, and prove that $X \otimes Y$ is T_4 and strongly 0-dimensional by induction on rank(Y). We may assume that rank(Y) $= \alpha$ and that the result holds for compact scattered spaces of rank less than α. Since $Y^{(\alpha)}$ is finite, we may separate the points of $Y^{(\alpha)}$ by clopen sets, so we might as well assume that $Y^{(\alpha)} = \{p\}$. Fix closed disjoint $H, K \subseteq X \times Y$. We shall produce a +clopen U with $H \subseteq U$ and $K \cap U = \emptyset$. Separating H^p and K^p by clopen subsets of X, we may assume that H^p is empty. Also, to simplify notation, assume that X (as a set) is just ω.

Then the H_n, for $n \in \omega$, are closed subsets of Y with $p \notin H_n$. Choose clopen $C_n \subseteq Y$ with $Y = C_0 \supseteq C_1 \supseteq \cdots$ so that $p \in C_n$ and $C_{n+1} \cap H_n = \emptyset$ for each n. Then $C_n \backslash C_{n+1}$ is compact scattered and of rank less than α, so $\omega \otimes (C_n \backslash C_{n+1})$ is T_4 and strongly 0-dimensional. Thus, we may choose a +clopen $U_n \subseteq \omega \times (C_n \backslash C_{n+1})$ such that $H \cap (\omega \times (C_n \backslash C_{n+1})) \subseteq U_n$ and $U_n \cap (K \cup (n \times Y)) = \emptyset$. Then, let $U = \bigcup_{n \in \omega} U_n$. ☺

Corollary 4.2 *If Y is compact Hausdorff, then Y is ω-trite iff Y is scattered.*

Proof. The $\Rightarrow$ direction is Corollary 2.11. The $\Leftarrow$ direction follows by Lemmas 4.1 and 2.10. ☺

It now remains to investigate $X \otimes Y$ with both X, Y compact scattered and uncountable. Observe that Lemma 2.10 already gives some negative results here; for example, $\mho\omega_1 \otimes (\omega_1 + 1)$ is not T_3, since it is immediate from the pressing-down lemma that $\omega_1 + 1$ is not ω_1-trite. However, $\mho\omega_1 \otimes \mho\omega_1$ is T_3; more generally, $X \otimes Y$ regular, and in fact paracompact, when X, Y are both strong Eberlein compacta (Theorem 4.5). First, we prove the following general lemma for deriving paracompactness from a partial order:

Lemma 4.3 *Let Z be any Hausdorff space, and assume that $\leq$ is a partial order on Z satisfying:*

a. $(Z; \leq)$ *has no infinite chains.*

b. Each $q\!\downarrow$ *is finite.*

c. $K\!\uparrow$ *is clopen whenever* K *is closed.*

Then Z is ultra-paracompact.

Proof. For any U, let $U^* = Z \backslash ((Z \backslash U)\!\uparrow)$. By (c), U^* is clopen whenever U is open. Also, $U^* = \{x \in U : x\!\downarrow\, \subseteq U\}$ for every U, so that $U^* = U^{**} = U^*\!\downarrow$. If $F \subseteq Z$, let $U^*_F = (U \cup F)^* \backslash F = \{x \in U \backslash F : x\!\downarrow \backslash F \subseteq U\}$. Note that $U^*_F \subseteq U \backslash F$, and is clopen whenever F is clopen and U is open. $U^* = U^*_\emptyset$.

Now, let $\{U_\alpha : \alpha < \kappa\}$ be an open cover of Z. Define sets V^n_α and E^n for $n < \omega$ and $\alpha < \kappa$ by:

$$E^n = \bigcup \{V^m_\alpha : \alpha < \kappa \ \&\ m < n\} \quad .$$
$$V^n_\alpha = (U_\alpha \backslash \textstyle\bigcup_{\delta < \alpha} V^n_\delta)^*_{E^n} \quad .$$

Observe that this definition, and properties (1 – 4) in the following list, only depend on the order, not the topology:

1. $\emptyset \subseteq E^0 \subseteq E^1 \subseteq E^2 \subseteq \cdots$.
2. $V^n_\alpha \subseteq E^{n+1} \backslash E^n$.

3. $V^m_\alpha \cap V^n_\beta \neq \emptyset \quad \Rightarrow \quad m = n \;\&\; \alpha = \beta$.
4. $\alpha \neq \beta \quad \Rightarrow \quad V^n_\alpha\!\downarrow \cap V^n_\beta\!\downarrow \subseteq E^n$.
5. V^n_α is clopen.
6. E^n is clopen.
7. $\bigcup_{n<\omega} E^n = Z$.

Assuming this, the V^n_α form a disjoint clopen refinement of the U_α. To prove (1 – 7): (1 – 4) are immediate from the definition. Next, we prove (5),(6) by induction on n. Now (6) for $n = 0$ is obvious. For any n, (6) for n implies that each $p \in Z$ has a neighborhood U meeting at most one V^n_α (if $p \in E_n$, let $U = E_n$, and if $p \notin E_n$, let $U = p\!\uparrow$). Thus, assuming (6) for n, we prove by induction on α that each V^n_α is clopen, which is (5) for n, and then that $E^{n+1} = E^n \cup \bigcup_{\alpha<\kappa} V^n_\alpha$ is clopen, which is (6) for $n+1$.

Finally, for (7), if $\bigcup_{n<\omega} E^n \neq Z$, then by (a), there would be a minimal element, $q \in Z \backslash \bigcup_{n<\omega} E^n$, and then by (b), we can fix n with $q\!\downarrow \backslash E^n = \{q\}$. But then, if $q \in U_\alpha$, we would have $q \in V^n_\alpha$, a contradiction. ☺

This applies to topologies of the form $\mathcal{T}_\mathcal{E}$ (see Definition 2.13):

Corollary 4.4 *Let $(Z; \mathcal{T})$ be a strong Eberlein compactum, with partial order $\leq$ satisfying conditions (a)(b)(c) of Lemma 3.8. Let $\mathcal{E}$ be any family of closed subsets of Z such that for each $E \in \mathcal{E}$, $E\!\downarrow$ can be covered by finitely many elements of $\mathcal{E}$. Then $(Z; \mathcal{T}_\mathcal{E})$ is ultra-paracompact.*

Proof. We verify that $\leq$ also satisfies conditions (a)(b)(c) of Lemma 4.3. This is immediate if we can show that if $K \subseteq Z$ is $\mathcal{T}_\mathcal{E}$-closed, then $K\!\uparrow$ is also $\mathcal{T}_\mathcal{E}$-closed. Thus, we must show that $K\!\uparrow \cap E$ is $\mathcal{T}$-closed for each $E \in \mathcal{E}$. Say $E\!\downarrow \subseteq E_1 \cup \cdots \cup E_n$, where each $E_i \in E$. Then $K\!\uparrow \cap E = ((K \cap E_1)\!\uparrow \cup \cdots \cup (K \cap E_n))\!\uparrow) \cap E$, which is $\mathcal{T}$-closed by condition (c). ☺

In particular, letting $Z = X \times Y$ and letting $\mathcal{E}$ be usual family which generates the +open topology from the Tychonov topology:

Theorem 4.5 *If X and Y are both strong Eberlein compacta, then $X \otimes Y$ is ultra-paracompact.*

The converse of this theorem is false, since $(\omega+1) \otimes Y$ is ultra-paracompact whenever Y is compact scattered. The converse does hold when $X = Y$:

Theorem 4.6 *If X is compact Hausdorff, then the following are equivalent:*

1. *$X \otimes X$ is T_3.*
2. *$X \otimes X$ is ultra-paracompact.*
3. *X is a strong Eberlein compactum.*

Proof. (3) $\Rightarrow$ (2) follows by Theorem 4.5, and (2) $\Rightarrow$ (1) is trivial. For (1) $\Rightarrow$ (3), we apply Theorem 3.5. So, we fix $p \in X$ and produce a winning strategy for Alice in $CSG(p, X)$.

Let $\Delta = \{(x, x) : x \in X \backslash \{p\}\}$, and fix a +open $W \subseteq X \times X$ with $(p, p) \in W$ and $\Delta \cap \overline{W} = \emptyset$. We describe Alice's strategy σ: For her opening move, she chooses $U_0 = W^p$, so that $U_0 \times \{p\} \subseteq W$. At succeeding moves, she will always make sure that $U_0 \supseteq U_1 \supseteq \cdots$, so that Bob's x_n will always satisfy $(x_n, p) \in W$, and hence $p \in W_{x_n}$. Then Alice will choose U_{n+1} so that $p \in U_{n+1} \subseteq \overline{U}_{n+1} \subseteq U_n \cap W_{x_n}$.

To see that Alice has won, suppose that $q \neq p$ were a limit point of $\langle x_n : n \in \omega \rangle$. Then $q \in \bigcap_{n \in \omega} \overline{U}_n \subseteq \bigcap_{n \in \omega} W_{x_n}$. Thus, each $(x_n, q) \in W$, so that $(q, q) \in \overline{W}$, contradicting $\Delta \cap \overline{W} = \emptyset$. ☺

5 Separable Metric Spaces

If we try to copy the proof of Theorem 2.7 in this case, we get:

Theorem 5.1 *Suppose that X and Y are both non-discrete separable metric spaces and $X \otimes Y$ is T_3. Then:*

1. *Either $|X| < \mathfrak{c}$ or $|Y| < \mathfrak{c}$.*
2. *X and Y are both ω-trite.*

Proof. For (1), assume that $|X| = |Y| = \mathfrak{c}$. Passing to closed subspaces, we may assume also that in X and Y, every non-empty open subset has size $\mathfrak{c}$. Then, as in the proof of Theorem 2.7, Lemma 2.1 applies to show that $X \otimes Y$ is not T_3.

For (2), to prove Y is ω-trite, we may, passing to closed subspaces, assume also that $X = \omega + 1$, and then apply Lemma 2.10. ☺

Now, for separable metric spaces, "ω-trite" is equivalent to "σ-set" (see Lemma 5.4), which has already been studied in the literature.

Definition 5.2 *A* σ-set *is a separable metric space in which every F_σ is also a G_δ.*

This is the same as saying that every Borel set is both an F_σ and a G_δ. The most well-known example is a Sierpiński set; these exist under CH. Every Sierpiński set is a σ-set by Szpilrajn [27]; see also Theorem 4.1 of Miller [19], which discusses these and related notions, and also has examples (under CH) of σ-sets which are not Sierpiński sets. Also, by Miller (Theorem 22 of [18]), it is consistent with ZFC that all σ-sets are countable. By the following lemma, every σ-set is homeomorphic to a set of reals:

Lemma 5.3 *Every σ-set is 0-dimensional.*

Proof. Since the continuous real-valued functions separate points from closed sets, it is enough to note that no continuous function can map a σ-set onto $[0, 1]$. This is a result of I. Recław; see Theorem 17 of Miller [20] for a proof. ☺

Lemma 5.4 *For X a separable metric space, X is a σ-set iff X is ω-trite.*

Proof. If X is a σ-set and $\langle F_n : n < \omega\rangle$ is a point-finite sequence of closed sets, fix open V^i_m for $m, i \in \omega$ such that $\bigcup_{n\ge m} F_n = \bigcap_{i<\omega} V^i_m$. Let $U_n = \bigcap\{V^i_m : m, i \le n\}$. Then $F_n \subseteq U_n$, and $\langle U_n : n < \omega\rangle$ is point-finite.

Conversely, assume X is ω-trite, and let E be an F_σ. Note first that E is of the form $\bigcup_{n<\omega} F_n$, where each F_n is closed and $\langle F_n : n < \omega\rangle$ is point-finite. To see this, say $E = \bigcup_{n<\omega} C_n$, where each C_n is closed and $\emptyset \ne C_0 \subseteq C_1 \subseteq \cdots$. Let $K^i_n = \{x \in C_{n+1}\backslash C_n : 1/(i + 1) \le d(x, C_n) \le 1/i\}$, for $n < \omega, 0 < i < \omega$. Then $E = C_0 \cup \bigcup_{n,i} K^i_n$, and no point is in more than two different K^i_n, so, re-indexing, we get the F_n. Now, if $\langle U_n : n < \omega\rangle$ is a point-finite sequence of open sets with each $F_n \subseteq U_n$, then $E = \bigcap_{m<\omega}[\bigcup_{n<m} F_n \cup \bigcup_{n\ge m} U_n]$. Finally, note that for each $m < \omega$, $\bigcup_{n<m} F_n$ is a closed subset of a metric space, and hence is a G_δ, so that E itself is a G_δ. ☺

This lets us decide the regularity of the product of a countable metric space with a separable metric space:

Theorem 5.5 *Suppose that X is a countable non-discrete metric space and Y is a separable metric space. Then the following are equivalent:*

1. *$X \otimes Y$ is ultra-paracompact.*
2. *$X \otimes Y$ is T_3.*
3. *Y is a σ-set.*

Proof. (1) $\Rightarrow$ (2) is trivial, and (2) $\Rightarrow$ (3) follows from Theorem 5.1 and Lemma 5.4.

For (3) $\Rightarrow$ (1): Since Y is 0-dimensional (by Lemma 5.3), it is sufficient to prove that $X \otimes Y$ is T_3 (by Theorem 2.12), but it is just as easy to prove normality. So, fix closed disjoint $H, K \subseteq X \otimes Y$. We shall partition $X \otimes Y$ into clopen U, V with $H \subseteq U$ and $K \subseteq V$. To do this, inductively construct H_n and K_n with

α. $H = H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$ and $K = K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots$.

β. H_n and K_n closed and disjoint.

γ. $\forall x \exists n [(H_n)_x \cup (K_n)_x = Y]$.

δ. $\forall x \exists n \forall y [x \in ((H_n)^y)^\circ$ or $x \in ((K_n)^y)^\circ]$.

Assuming this construction can be done, set $U = \bigcup_n H_n$ and $V = \bigcup_n K_n$; items (β),(γ) and (δ) imply that each U_x and each U^y is clopen.

Since X is countable, to do the construction, it suffices to *fix* $x \in X$, assume we have H_{n-1} and K_{n-1}, and then define appropriate H_n and K_n. By (β), we can partition the strongly 0-dimensional Y into clopen A, B with $(H_{n-1})_x \subseteq A$ and $(K_{n-1})_x \subseteq B$. For each $\epsilon > 0$, since X is countable, the set $\{y \in Y : d(x, (K_{n-1})^y) < \epsilon\} = \{y \in Y : \exists x' \in (K_{n-1})^y \; [\; d(x, x') < \epsilon \;] \; \} = \bigcup \{K_{x'} : x' \in X \wedge d(x, x') < \epsilon\}$ is an F_σ. So $E_\epsilon = \{y \in A : d(x, (K_{n-1})^y) \geq \epsilon\}$ is a G_δ, and hence an F_σ (since Y is a σ-set). Since the E_ϵ cover the closed set A, we may find closed $A_j \subseteq A$ for $j < \omega$ such that $A = \bigcup_j A_j$ and each $\inf\{d(x, (K_{n-1})^y) : y \in A_j\} > 0$, and we may then find clopen R_j containing x such that each $(R_j \times A_j) \cap K_{n-1} = \emptyset$ and $diam(R_j) < 1/j$. Likewise, choose closed $B_j \subseteq B$ for $j < \omega$ and clopen S_j containing x such that $B = \bigcup_j B_j$ and each $(S_j \times B_j) \cap H_{n-1} = \emptyset$ and $diam(S_j) < 1/j$. Let $H_n = H_{n-1} \cup \bigcup_j (R_j \times A_j)$ and let $K_n = K_{n-1} \cup \bigcup_j (S_j \times B_j)$. Regarding item ($\gamma$), we have $(H_n)_x \cup (K_n)_x = A \cup B = Y$. Regarding item ($\delta$), we have $x \in ((H_n)^y)^\circ$ for $y \in A$ and $x \in ((K_n)^y)^\circ$ for $y \in B$. ☺

In particular, since countable metric spaces are σ-sets, $X \otimes Y$ is T_3 when both X, Y are countable, but this is true for non-metric X, Y as well (see

Corollary 2.16). By the results so far, we have a simple criterion for the regularity of $X \otimes Y$ in some models of set theory:

Corollary 5.6 *Let X, Y both be separable metric and not discrete.*

1. *If CH, then $X \otimes Y$ is T_3 iff one is countable and the other is a σ-set.*
2. *If all σ-sets are countable, then $X \otimes Y$ is T_3 iff both are countable.*

The hypothesis of (2) is consistent by Theorem 22 of Miller [18]. However, every set of reals of size less than $\mathfrak{p}$ is a σ-set, so one may consider what happens in models of ZFC in which $\mathfrak{p} > \aleph_1$. Now, the product of two σ-sets need not be T_3 in the +open topology; for example, this fails if they both have size $\mathfrak{c}$ by Theorem 5.1. However, $X \otimes Y$ is T_3 when $|X|, |Y| < \mathfrak{p}$. This follows easily from the following lemma, which generalizes the result to a $\mathcal{T}_\mathcal{E}$ (see Definition 2.13). The proof uses the method of Juhász and Weiss [13]; see also Theorem 7.1 of Weiss [30].

Theorem 5.7 *Suppose that $(Z; \mathcal{T})$ is a separable metric space, $\mathcal{E} \subseteq \mathcal{P}(Z)$, $|\mathcal{E}| < \mathfrak{p}$, and $|\bigcup \mathcal{E}| < \mathfrak{p}$, and every $E \in \mathcal{E}$ is closed in $\bigcup \mathcal{E}$. Then $(Z; \mathcal{T}_\mathcal{E})$ is normal and strongly 0-dimensional.*

Proof. We may assume that $\bigcup \mathcal{E} = Z$, by Proposition 2.14.4, so that each $E \in \mathcal{E}$ is closed in Z. Now, fix disjoint $\mathcal{T}_\mathcal{E}$-closed sets, H, K. We show how to separate them with $\mathcal{T}_\mathcal{E}$-clopen sets.

Let $\mathcal{B}$ be a countable family of $\mathcal{T}$-clopen sets such that $\mathcal{B}$ is closed under finite boolean combinations and $\mathcal{B}$ forms a base for $\mathcal{T}$. If $B \in \mathcal{B}$ and $Q, R \subseteq Z$, we say that B *separates* Q, R iff $Q \subseteq B$ and $R \cap B = \emptyset$. Let $\mathbb{P}$ be the set of all pairs (Q, R) such that Q, R are $\mathcal{T}$-closed subsets of Z, some element of $\mathcal{B}$ separates Q, R, and $(H \cup Q) \cap (K \cup R) = \emptyset$. Define $(Q', R') \leq (Q, R)$ iff $(Q', R') \supseteq (Q, R)$, so that $(\emptyset, \emptyset) \in \mathbb{P}$ is the largest element. $\mathbb{P}$ is σ-centered because $\{(Q, R) \in \mathbb{P} : B \text{ separates } Q, R\}$ is centered for each $B \in \mathcal{B}$.

For $S \subseteq E$, let $int_E(S)$ be the interior of S computed in the relative topology $\mathcal{T}$ induces on E. Whenever $x \in E \in \mathcal{E}$, let $D_{x,E} = \{(Q, R) \in \mathbb{P} : x \in int_E(Q \cap E) \cup int_E(R \cap E)\}$. To prove $D_{x,E}$ is dense, fix any $(Q, R) \in \mathbb{P}$. Since $(H \cup Q) \cap (K \cup R) = \emptyset$, assume that $x \notin (K \cup R)$ (the case $x \notin (H \cup Q)$ is similar). Since $(K \cap E) \cup R$ is $\mathcal{T}$-closed, fix $B \in \mathcal{B}$ with $x \in B$ and $B \cap ((K \cap E) \cup R) = \emptyset$. Let $Q' = Q \cup (B \cap E)$. Then $(Q', R) \leq (Q, R)$ and $x \in int_E(Q' \cap E)$, so we need to show that $(Q', R) \in \mathbb{P}$. Now $(H \cup Q') \cap$

$(K \cup R) = B \cap E \cap (K \cup R)$, which is empty by choice of B. Also, if $B_1 \in \mathcal{B}$ separates Q, R, then $B_1 \cup B$ separates Q', R.

Let $G \subseteq \mathbb{P}$ be a filter meeting all $D_{x,E}$. Let $U = \bigcup\{Q : \exists R[(Q,R) \in G]\}$ and $V = \bigcup\{R : \exists Q[(Q,R) \in G]\}$. Then U, V partition Z into disjoint $\mathcal{T}_{\mathcal{E}}$-open (and hence $\mathcal{T}_{\mathcal{E}}$-clopen) sets, and $H \subseteq U$ and $K \subseteq V$.

Corollary 5.8 *Suppose that X, Y are both separable metric spaces of size less than $\mathfrak{p}$. Then $X \otimes Y$ is normal and strongly 0-dimensional.*

Note that for separable metric X, Y, we have settled all cases for the regularity of $X \otimes Y$ under CH (Corollary 5.6), but not under MA. Left open is the situation where $\aleph_0 < |X| < \mathfrak{c} = |Y|$ and Y is a σ-set. As a partial result (Theorem 5.16), we prove regularity, and normality, in the case where Y is a generalized Sierpiński set.

We first note that for $|X| < \mathfrak{p}$, Silver's theorem says that X is a Q-set – that is, *every* subset is a relative G_δ and F_σ. We need the following generalization of Silver's result:

Lemma 5.9 *Suppose that X is a separable metric space, with $|X| < \mathfrak{p}$, and $f : X \to 2^\omega$. Then there are closed $F_n \subseteq X$ for $n \in \omega$ with $F_n \nearrow X$ and each $f \restriction F_n$ continuous.*

Proof. Let $\mathcal{A}$ be a countable clopen base for X, closed under finite boolean combinations. Call $U \subseteq X \times 2^\omega$ a *pre-function* iff $U = \bigcup_{j<\ell} A_j \times B_j$, where $\ell < \omega$, each B_j is a non-empty clopen subset of 2^ω, each $A_j \in \mathcal{A}$, and $\{A_j : j < \ell\}$ is a partition of X. Let $\mathbb{P}$ be the set of all $p = \langle U_n^p, \sigma_n^p : n \in \omega \rangle$ such that:

a. Each U_n^p is a pre-function.

b. $U_n^p = X \times 2^\omega$ for all but finitely many n.

c. $\sigma_0^p \subseteq \sigma_1^p \subseteq \sigma_2^p \cdots \subseteq X$ and $\bigcup_n \sigma_n^p$ is finite.

d. Each $f \restriction \sigma_n^p \subseteq U_n^p$.

Define $p \leq q$ iff $U_n^p \subseteq U_n^q$ and $\sigma_n^p \supseteq \sigma_n^q$ for each n.

Let G be a filter. Define $g_n = \bigcap\{U_n^p : p \in G\}$ and $X_n = \bigcup\{\sigma_n^p : p \in G\}$. Note that $X_0 \subseteq X_1 \subseteq X_2 \cdots$.

If G meets the dense set $\{p : \forall x(diam(U_n^p) \leq 2^{-m})\}$ for each m, n, then each g_n will be (the graph of) a continuous function, and $g_n \restriction X_n = f \restriction X_n$, so that $f \restriction X_n$ will be continuous. If G also meets the dense set $\{p : \exists n(x \in \sigma_n^p)\}$ for each $x \in X$, then $X_n \nearrow X$.

Now, let $F_n = \overline{X_n}$. The fact that $f \upharpoonright F_n$ is continuous follows from the fact that $X_n \nearrow X$. ☺

Corollary 5.10 (Silver) *Suppose that X is a separable metric space, $|X| < \mathfrak{p}$, and $S \subseteq X$. Then S is an F_σ in X.*

Proof. Apply Lemma 5.9 to the characteristic function of S. ☺

We may now modify the standard hyperspace construction and use this to study +closed sets:

Lemma 5.11 *Suppose that X, Y are separable metric spaces, with $|X| < \mathfrak{p}$, and suppose that $H \subseteq X \times Y$, with each H_x closed in Y. Then there are closed $F_n \subseteq X$ for $n \in \omega$ with each $H \cap (F_n \times Y)$ closed in $F_n \times Y$ (in the usual Tychonov topology) and $F_n \nearrow X$.*

Proof. Let $\mathcal{B}$ be a countable base for Y, and define $f : X \to 2^{\mathcal{B}}$ so that $f(x)(B)$ is 1 if $H_x \cap B \neq \emptyset$ and 0 if $H_x \cap B = \emptyset$. Apply Lemma 5.9 to get closed $F_n \nearrow X$ such that each $f \upharpoonright F_n$ is continuous. To prove $H \cap (F_n \times Y)$ is closed, fix $(x, y) \in (F_n \times Y) \setminus H$. Now, fix $B \in \mathcal{B}$ with $y \in B$ and $B \cap H_x = \emptyset$, so that $f(x)(B) = 0$. By continuity, there is a neighborhood U of x in F_n such that $f(x')(B) = 0$ for all $x' \in U$, so that $U \times B$ is a neighborhood of (x, y) in $F_n \times Y$ and $(U \times B) \cap H = \emptyset$. ☺

We now prove versions of these results for generalized Sierpiński sets.

Definition 5.12 *A* generalized Sierpiński *set with* associated measure μ *is a separable metric space of size $\mathfrak{c}$ on which μ is a finite regular Borel measure and each μ-null set has size less than $\mathfrak{c}$.*

Generalized Sierpiński sets can easily be proved to exist from MA (although not just from $\mathfrak{p} = \mathfrak{c}$). Under MA, such sets are σ-sets by the following lemma. The key observation in the proof is that every set of size less than $\mathfrak{c}$ is covered by a null F_σ.

Lemma 5.13 *If MA holds, Y is a generalized Sierpiński set, and $E \subseteq Y$ is in the $< \mathfrak{c}$-algebra generated by the open sets, then E is an F_σ set.*

Proof. Let μ be the associated measure. We first consider the special case where E is a μ-null set (and hence $|E| < \mathfrak{c}$). Let $\mathcal{B}$ be a countable base for the topology, closed under finite unions. Let $\mathbb{P}$ be the set of triples $p = \langle n_p, e_p, s_p \rangle$ such that $n_p \in \omega$, $e_p \in [E]^{<\omega}$, $s_p \in \mathcal{B}^{n_p}$, and $\mu(Y \backslash s_p(j)) \leq 2^{-j}$ for each $j < n_p$. Define $p \leq q$ iff $n_p \geq n_q$, $e_p \supseteq e_q$, $s_p \supseteq s_q$, and $s_p(j) \cap e_q = \emptyset$ whenever $n_q \leq j < n_p$. A filter in $\mathbb{P}$ which meets the obvious dense sets yields $U_j \in \mathcal{B}$ for $j < \omega$ such that if $D = \bigcap_{j<\omega} \bigcup_{k \geq j} U_k$, then $D \cap E = \emptyset$ and $\mu(Y \backslash D) = 0$. Then $E \subseteq (Y \backslash D)$, and $Y \backslash D$ is an F_σ set. Also, $|Y \backslash D| < \mathfrak{c}$, since Y is generalized Sierpiński, so that $Y \backslash D$ is Q-set. Hence, E is F_σ in $Y \backslash D$, and hence in Y.

Now, to prove the lemma in the general case, note that E must be μ-measurable by MA, so let F be an F_σ set such that $F \subseteq E$ and $\mu(E \backslash F) = 0$. Then, applying the special case, we have E as the union of the two F_σ sets, F and $E \backslash F$.

We remark that unlike for ordinary Sierpiński sets, one cannot prove in ZFC that generalized Sierpiński sets are σ-sets. For example, assume that $V \models CH$ and $V[G]$ adds a set S of $\aleph_2$ random reals, which we view as contained in $[0,1]$. Then, by Solovay (see [17]), in $V[G]$, S is a Sierpiński set. Hence $S' = S \cup (V \cap [0,1])$ is a generalized Sierpiński set, because $\mathfrak{c} = \aleph_2$ and any null subset of $S' \cap (V \cap [0,1])$ still has size at most $\aleph_1$. But S' is not a σ-set, since $\mathbb{Q} \cap [0,1]$ is a relative F_σ which is not a G_δ.

Lemma 5.14 *Assume MA. Assume that X, Y are 0-dimensional separable metric spaces, with X compact and Y a generalized Sierpiński set, with associated measure μ. Let $f : Y \to X$ be μ-measurable. Then there are closed $F_n \subseteq Y$ for $n \in \omega$ with $f \restriction F_n$ continuous and $\bigcup_n F_n = Y$.*

Proof. Just by using the measure, we can get closed A_n such that $f \restriction A_n$ is continuous and $Y \setminus \bigcup_n A_n$ is a null set, and hence of size less than $\mathfrak{c}$, and hence an F_σ by Lemma 5.13. Say $Y \setminus \bigcup_n A_n = \bigcup_i B_i$, where each B_i is closed. Applying Lemma 5.9 to each B_i, we get closed $C_{i,j} \subseteq B_i$ with each $f \restriction C_{i,j}$ continuous and $\bigcup_j C_{i,j} = B_i$. List $\{A_n : n \in \omega\} \cup \{C_{i,j} : i, j \in \omega\}$ as $\{D_n : n \in \omega\}$, and let $G_n = \bigcup_{m<n} D_m$.

Lemma 5.15 *Assume MA. Suppose that X, Y are separable metric spaces, with $|X| < \mathfrak{c}$, and Y a generalized Sierpiński set. Suppose that $H \subseteq X \times Y$, with each H^y closed in X. Then there are closed $G_n \subseteq Y$ for $n \in \omega$ with* *each $H \cap (X \times G_n)$ closed in $X \times G_n$ (with the usual Tychonov topology) and $G_n \nearrow Y$.*

Proof. Exactly as in Lemma 5.11. We use $|X| < \mathfrak{c}$ to prove that the f in that proof is measurable, so that Lemma 5.14 applies. ☺

Theorem 5.16 *Assume MA. Suppose that X, Y are separable metric, with $|X| < \mathfrak{c}$, and Y a generalized Sierpiński set. Then $X \otimes Y$ is T_4 and strongly 0-dimensional.*

Proof. Suppose H, K are closed and disjoint. Let $\mathcal{T}$ be the usual Tychonov topology. Applying Lemmas 5.11 and 5.15, there are closed $F_n \subseteq X$ and closed $G_n \subseteq Y$ such that all of the $H \cap (F_n \times Y)$, $H \cap (X \times G_n)$ $K \cap (F_n \times Y)$, and $K \cap (X \times G_n)$ are $\mathcal{T}$-closed. Let $\mathcal{E}$ be the family of all $F_n \times Y$ and all $X \times G_n$. Note that $\mathcal{T} \subseteq \mathcal{T}_\mathcal{E} \subseteq \mathcal{T}_+$, and H, K are $\mathcal{T}_\mathcal{E}$-closed. Now, $\mathcal{T}_\mathcal{E}$ is normal and strongly 0-dimensional by Lemma 2.15, so that we may separate H, K by $\mathcal{T}_\mathcal{E}$-clopen sets, and these sets remain clopen in the finer topology $\mathcal{T}_+$. ☺

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