First Countable Continua and Proper Forcing*

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August 1, 2006

Abstract

Assuming the Continuum Hypothesis, there is a compact first countable connected space of weight $\aleph_1$ with no totally disconnected perfect subsets. Each such space, however, may be destroyed by some proper forcing order which does not add reals.

1 Introduction

All topologies discussed in this paper are assumed to be Hausdorff. As in [12],

Definition 1.1 *A space X is* weird *iff X is compact and not scattered, and no perfect subset of X is totally disconnected.*

A subset P of X is *perfect* iff P is closed and has no isolated points. As usual, $\mathfrak{c}$ denotes the (von Neumann) cardinal $2^{\aleph_0}$. Big weird spaces (of size $2^{\mathfrak{c}}$) were produced from CH in Fedorchuk, Ivanov, and van Mill [10]. Small weird spaces (of size $\aleph_1$) were constructed from $\diamondsuit$ in [12], which proved:

Theorem 1.2 *Assuming $\diamondsuit$, there is a connected weird space which is hereditarily separable and hereditarily Lindelöf.*

*2000 Mathematics Subject Classification: Primary 54D05, 03E35. Key Words and Phrases: connected space, Continuum Hypothesis, proper forcing, irreducible map.

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§Both authors partially supported by NSF Grant DMS-0456653.

The weird spaces of [12], [10], and the earlier Fedorchuk [9] are all separable spaces of weight $\aleph_1$. Our $\diamondsuit$ example is also first countable, because it is compact and hereditarily Lindelöf. In contrast, the CH weird spaces of [10, 9] have no convergent ω-sequences. We do not know whether CH can replace $\diamondsuit$ in Theorem 1.2, but weakening hereditarily Lindelöf to first countable we do get:

Theorem 1.3 *Assuming CH, there is a separable first countable connected weird space of weight* $\aleph_1$.

This theorem cannot be proved by a *classical* CH construction. Classical CH arguments build the item of interest directly from an enumeration in type ω_1 of some natural set of size $\mathfrak{c}$ (e.g., $\mathbb{R}$, $\mathbb{R}^{<\omega_1}$, etc.). The result, then, is preserved by any forcing which does not add reals. These arguments include any CH proof found in Sierpiński's text [15], as well as most CH proofs in the current literature, including the constructions of the big weird spaces of [9, 10]. In contrast, every space satisfying Theorem 1.3 is destroyed by some proper forcing order which does not add reals.

Our proof of Theorem 1.3 uses classical CH arguments to make X weird, but then, to make X first countable, we adapt the method of Gregory [11] and Devlin and Shelah [2]. The methods of [11] and [2] are, as Hellsten, Hyttinen, and Shelah [13] pointed out, essentially the same. We review the method in Section 2, and use it to prove Theorem 1.3 in Section 4. Although [11] and [2] derive results from $2^{\aleph_0} < 2^{\aleph_1}$, for Theorem 1.3, we need CH; Section 5 explains why.

In Section 3, we show that each space satisfying Theorem 1.3 can be destroyed by a proper forcing which does not add reals; in $V[G]$, we add a point of uncountable character. More precisely, if X is a compactum in V, then in each generic extension $V[G]$, we still have the same set X with the natural topology obtained by using the open sets from V as a base. If X is first countable in V, then it must remain first countable in $V[G]$, but X need not be compact in $V[G]$. We get the point of uncountable character in the natural corresponding *compact* space $\widetilde{X}$ in $V[G]$. This compact space determined by X was described by Bandlow [1] (and later in [3, 4, 6]), and can be defined as follows:

Definition 1.4 *If X is a compactum in V and $V[G]$ is a forcing extension of V, then in $V[G]$ the corresponding compactum $\widetilde{X}$ is characterized by:*

1. *X is dense in $\widetilde{X}$.*
2. *Every $f \in C(X, [0,1]) \cap V$ extends to an $\widetilde{f} \in C(\widetilde{X}, [0,1])$ in $V[G]$.*
3. *The functions $\widetilde{f}$ (for $f \in V$) separate the points of $\widetilde{X}$.*

In forcing, $\overset{\diamond}{X}$ denotes the $\widetilde{X}$ of $V[G]$, while $\check{X}$ denotes the X of $V[G]$.

For example, if X is the $[0,1]$ of V, then $\widetilde{X}$ will be the unit interval of $V[G]$; note that in statement (2), asserted in $V[G]$, the "$[0,1]$" really refers to the unit interval of $V[G]$. If in V, we have $X \subseteq [0,1]^\kappa$, then $\widetilde{X}$ is simply the closure of X in the $[0,1]^\kappa$ of $V[G]$. If in V, X is the Stone space of a boolean algebra $\mathcal{B}$, then $\widetilde{X}$ will be the Stone space, computed in $V[G]$, of the same $\mathcal{B}$. In general, the weights of X and $\widetilde{X}$ will be the same (assuming that cardinals are not collapsed), but their characters need not be.

Following Eisworth and Roitman [8, 7], we call a partial order $\mathbb{P}$ *totally proper* iff $\mathbb{P}$ is proper and forcing with it does not add reals.

Theorem 1.5 *If X is compact, connected, and infinite, and X does not have a Cantor subset, then for some totally proper $\mathbb{P}$: $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\overset{\diamond}{X}$ is not first countable".*

The proof is in Section 3. Observe the importance of connectivity here. Suppose in V that X is the double arrow space, obtained from $[0,1]$ by doubling the points of $(0,1)$. Then in any $V[G]$, $\widetilde{X}$ is the compactum obtained from $[0,1]$ by doubling the points of $(0,1) \cap V$, and is hence first countable.

2 Predictors

In the following, λ^{ω_α} denotes the set of functions from ω_α into λ. Something like the next definition and theorem is implicit in both of [11, 2]:

Definition 2.1 *Let κ, λ be any cardinals and $\Psi : \kappa^{<\omega_1} \to \lambda$. If $f \in \kappa^{\omega_1}$, $g \in \lambda^{\omega_1}$, and $C \subseteq \omega_1$, then Ψ, f* predict *g* on *C iff $g(\xi) = \Psi(f \restriction \xi)$ for all $\xi \in C$. Ψ is a (κ, λ)*–predictor *iff for all $g \in \lambda^{\omega_1}$ there is an $f \in \kappa^{\omega_1}$ and a club C such that Ψ, f predict g on C.*

Theorem 2.2 *The following are equivalent whenever $2 \le \kappa \le \mathfrak{c}$ and $2 \le \lambda \le \mathfrak{c}$:*

1. *There is a (κ, λ)–predictor.*
2. *There is a $(\mathfrak{c}, \mathfrak{c})$–predictor.*
3. *$2^{\aleph_0} = 2^{\aleph_1}$.*

Proof. (3) $\to$ (1): Let $C = \omega_1 \setminus \omega$. List λ^{ω_1} as $\{g_\alpha : \alpha < \mathfrak{c}\}$, and choose $f_\alpha \in \kappa^{\omega_1}$ so that the $f_\alpha \restriction \omega$, for $\alpha < \mathfrak{c}$, are all distinct. Then we can define $\Psi : \kappa^{<\omega_1} \to \lambda$ so that $\Psi(f_\alpha \restriction \xi) = g_\alpha(\xi)$ for all $\xi \in C$.

(1) $\to$ (2): Fix a (κ, λ)–predictor $\Psi : \kappa^{<\omega_1} \to \lambda$. We shall define $\Phi : (\kappa^\omega)^{<\omega_1} \to (\lambda^\omega)$ so that it is a $(\kappa^\omega, \lambda^\omega)$–predictor in the sense of Definition 2.1. For $p \in (\kappa^\omega)^\xi$

and $n \in \omega$, define $p_{(n)} \in \kappa^\xi$ by: $p_{(n)}(\mu) = (p(\mu))(n) \in \kappa$. Then, for $p \in (\kappa^\omega)^{<\omega_1}$, define $\Phi(p) = \left\langle \Psi(p_{(n)}) : n \in \omega \right\rangle \in \lambda^\omega$.

(2) → (3): Fix a $(\mathfrak{c}, \mathfrak{c})$–predictor $\Psi : \mathfrak{c}^{<\omega_1} \to \mathfrak{c}$. Let $\Gamma : \mathfrak{c}^{<\omega_1} \times \mathfrak{c}^{<\omega_1} \to \mathfrak{c}$ be any 1-1 function. If $K \subseteq \omega_1$ is unbounded and $\xi < \omega_1$, let $\mathit{next}(\xi, K)$ be the least element of K which is greater than ξ.

For each $B \in \mathfrak{c}^{\omega_1}$, choose $G(n, B), F(n, B) \in \mathfrak{c}^{\omega_1}$ and clubs $C(n, B) \subseteq \omega_1$ for $n \in \omega$ as follows: Let $G(0, B) = B$. Given $G(n, B)$, let $C(n, B)$ be club of limit ordinals and let $F(n, B) \in \mathfrak{c}^{\omega_1}$ be such that $(G(n, B))(\xi) = \Psi((F(n, B)) \restriction \xi)$ for all $\xi \in C(n, B)$. Then define $G(n + 1, B)$ so that

$$(G(n+1, B))(\xi) = \Gamma\big(F(n, B) \restriction \mathit{next}(\xi, C(n, B)),\ G(n, B) \restriction \mathit{next}(\xi, C(n, B))\big)$$

for each ξ.

Now, fix $B, B' \in \mathfrak{c}^{\omega_1}$, and consider the statement:

$$\forall n \in \omega \left[G(n, B) \restriction \xi = G(n, B') \restriction \xi \right] \qquad (✰(\xi))$$

So, ✰(0) is true trivially, and ✰(ξ) implies ✰(ζ) whenever $\zeta < \xi$. We shall prove inductively that ✰(1) implies ✰(η) for all $\eta < \omega_1$. If we do this, then ✰(1) will imply $B = B'$, so we shall have $2^{\aleph_0} = 2^{\aleph_1}$, since there are $2^{\aleph_1}$ possible values for B but only $2^{\aleph_0}$ possible values for the sequence $\langle (G(n, B))(0) : n \in \omega \rangle$.

The induction is trivial at limits, so it is sufficient to fix η with $1 \le \eta < \omega_1$, assume ✰(η), and prove ✰($\eta + 1$) — that is, $(G(n, B))(\eta) = (G(n, B'))(\eta)$ for all n. Fix n. For $\xi < \eta$, we have $(G(n+1, B))(\xi) = (G(n+1, B'))(\xi)$, which implies:

a. $\mathit{next}(\xi, C(n, B)) = \mathit{next}(\xi, C(n, B'))$; call this γ_ξ.
b. $F(n, B) \restriction \gamma_\xi = F(n, B') \restriction \gamma_\xi$.
c. $G(n, B) \restriction \gamma_\xi = G(n, B') \restriction \gamma_\xi$.

Applying (a) for all $\xi < \eta$: $\eta \in C(n, B)$ iff $\eta \in C(n, B')$. If $\eta \notin C(n, B), C(n, B')$, then fix ξ with with $\xi < \eta < \gamma_\xi$; now (c) implies $(G(n, B))(\eta) = (G(n, B'))(\eta)$. If $\eta \in C(n, B), C(n, B')$, then η is a limit ordinal and (b) implies $F(n, B) \restriction \eta = F(n, B') \restriction \eta$; now $(G(n, B))(\eta) = (G(n, B'))(\eta) = \Psi((F(n, B)) \restriction \eta)$.

The *non*-existence of a $(2, 2)$–predictor is the weak version of $\diamondsuit$ discussed by Devlin and Shelah in [2], where they use it to prove that, assuming $2^{\aleph_0} < 2^{\aleph_1}$, every ladder system on ω_1 has a non-uniformizable coloring. By Shelah [14] (p. 196), each such coloring may be uniformized in some totally proper forcing extension.

A direct proof of (3) → (2), resembling the above proof of (3) → (1), would obtain C fixed at $\omega_1 \setminus \{0\}$, since one may choose the f_α so that the $f_\alpha(0)$, for $\alpha < \mathfrak{c}$, are all distinct. Gregory [11] used the failure of (2), with this specific C, to derive a result about trees under $2^{\aleph_0} < 2^{\aleph_1}$; see Theorem 3.14 below.

3 Some Totally Proper Orders

We consider *forcing posets*, $(\mathbb{P}; \leq, \mathbb{1})$, where $\leq$ is a transitive and reflexive relation on $\mathbb{P}$ and $\mathbb{1}$ is a largest element of $\mathbb{P}$. As usual, if $p, q \in \mathbb{P}$, then $p \not\perp q$ means that p, q are compatible (that is, have a common extension), and $p \perp q$ means that p, q are incompatible.

Definition 3.1 *Assume that X is compact, connected, and infinite. Let $\mathbb{K} = \mathbb{K}_X$ be the forcing poset consisting of all closed, connected, infinite subsets of X, with $p \leq q$ iff $p \subseteq q$ and $\mathbb{1}_{\mathbb{K}} = X$. In $\mathbb{K}$, define $p \perp\!\!\!\perp q$ iff $p \cap q = \emptyset$.*

Note that $p \perp q$ iff $p \cap q$ is totally disconnected. The stronger relation $p \perp\!\!\!\perp q$ will be useful in the proof that $\mathbb{K}$ is totally proper whenever X does not have a Cantor subset. First, we verify that $\mathbb{K}$ is separative; this follows easily from the following lemma, which is probably well-known; a proof is in [12]:

Lemma 3.2 *If P is compact, connected, and infinite, and $U \subseteq P$ is a nonempty open set, then there is a closed $R \subseteq U$ such that R is connected and infinite.*

In particular, in $\mathbb{K}$, if $p \not\leq q$, then we may apply this lemma with $U = p \setminus q$ to get $r \leq p$ with $r \perp q$, proving the following:

Corollary 3.3 *If X is compact, connected, and infinite, then $\mathbb{K}_X$ is separative and atomless.*

We collect some useful properties of the relation $\perp\!\!\!\perp$ on $\mathbb{K}$ in the following:

Definition 3.4 *A binary relation $\lightning$ on a forcing poset is a* strong incompatibility relation *iff*

1. *$p \lightning q$ implies $p \perp q$.*
2. *Whenever $p \perp q$, there are p_1, q_1 with $p_1 \leq p$, $q_1 \leq q$, and $p_1 \lightning q_1$.*
3. *$p \lightning q$ & $p_1 \leq p$ & $q_1 \leq q \;\rightarrow\; p_1 \lightning q_1$.*

This definition does not require $\lightning$ to be symmetric, but note that the relation $p \lightning q$ & $q \lightning p$ is symmetric and is also a strong incompatibility relation.

Lemma 3.5 *The relation $\perp\!\!\!\perp$ is a strong incompatibility relation on $\mathbb{K}_X$.*

Proof. Conditions (1) and (3) are obvious. For (2): Suppose that $p \perp q$. Let $F = p \cap q$, which is totally disconnected. Then by Lemma 3.2 there is an infinite connected $p_1 \subseteq p \backslash F$. Likewise, we get $q_1 \subseteq q \backslash F$. ☕

Definition 3.6 *If* $\mathbb{P}$ *is a forcing poset with a strong incompatibility relation* $\lightning$, *then a* strong Cantor tree *in* $\mathbb{P}$ (*with respect to* $\lightning$) *is a subset* $\{p_s : s \in 2^{<\omega}\} \subseteq \mathbb{P}$ *such that each* $p_{s^\frown\mu} < p_s$ *for* $\mu = 0, 1$, *and each* $p_{s^\frown 0} \lightning p_{s^\frown 1}$. *Then,* $\mathbb{P}$ *has the* weak Cantor tree property (WCTP) (*with respect to* $\lightning$) *iff whenever* $\{p_s : s \in 2^{<\omega}\} \subseteq \mathbb{P}$ *is a strong Cantor tree, there is at least one* $f \in 2^\omega$ *such that* $\mathbb{P}$ *contains some* $q = q_f$ *with* $q \le p_{f\restriction n}$ *for each* $n \in \omega$.

Note that if $\mathbb{P}$ has the WCTP, then the set of f for which q_f is defined must meet every perfect subset of the Cantor set 2^ω, since otherwise we could find a subtree of the given Cantor tree which contradicts the WCTP.

Lemma 3.7 *If* X *is compact, connected, and infinite, and* X *does not have a Cantor subset, then* $\mathbb{K}_X$ *has the WCTP.*

Definition 3.8 $\mathbb{P}$ *has the* Cantor tree property (CTP) *iff* $\mathbb{P}$ *has the WCTP with respect to the usual* $\perp$ *relation.*

$\mathbb{K}_X$ need not have the CTP (see Theorem 5.4). A countably closed $\mathbb{P}$ clearly has the CTP. In the case of trees, the CTP was also discussed in [13] (where it was called "$\aleph_0$ fan closed") and in [12]. The following modifies Lemma 3 of [13] and Lemma 5.5 of [12]:

Lemma 3.9 *If* $\mathbb{P}$ *has the WCTP, then* $\mathbb{P}$ *is totally proper.*

Proof. Define $q \le' p$ iff there is no r such that $r \le q$ and $r \perp p$. When $\mathbb{P}$ is separative, this is equivalent to $q \le p$.

Fix a suitably large regular cardinal θ, and let $M \prec H(\theta)$ be countable with $(\mathbb{P}; \le, \mathbb{1}, \lightning) \in M$, and fix $p \in \mathbb{P} \cap M$. It suffices (see [8]) to find a $q \le p$ such that whenever $A \subseteq \mathbb{P}$ is a maximal antichain and $A \in M$, there is an $r \in A \cap M$ with $q \le' r$. If $\mathbb{P}$ has an atom $q \le p$ such that $q \in M$, then we are done. Otherwise, then since $M \prec H(\theta)$, $\mathbb{P}$ must be atomless below p. Let $\{A_n : n \in \omega\}$ list all the maximal antichains which are in M. Build a strong Cantor tree $\{p_s : s \in 2^{<\omega}\} \subseteq \mathbb{P} \cap M$ such that, $p_{()} \le p$, and such that, when $n \in \omega$ and $s \in 2^n$, p_s extends some element of $A_n \cap M$. Then choose $f \in 2^\omega$ such that there is some $q \in \mathbb{P}$ with $q \le p_{f\restriction n}$ for each $n \in \omega$. ☕

Proof of Theorem 1.5. Let $\mathbb{P} = \mathbb{K}_X$. Working in $V[G]$, let $G' = \{\widetilde{p} : p \in G\}$; then $\bigcap G' = \{y\}$ for some $y \in \widetilde{X} \setminus X$. Since $\mathbb{P}$ does not add ω–sequences, $\bigcap E \supsetneqq \{y\}$ whenever E is a countable subset of G'. Thus, $\chi(y, \widetilde{X})$ is uncountable. ☕

These totally proper partial orders yield natural weakenings of PFA:

Definition 3.10 *If $\mathfrak{P}$ is a class of forcing posets, then* $\mathrm{MA}_{\mathfrak{P}}(\aleph_1)$ *is the statement that whenever $\mathbb{P} \in \mathfrak{P}$ and $\mathcal{D}$ is a family of $\leq \aleph_1$ dense subsets of $\mathbb{P}$, then there is a filter on $\mathbb{P}$ meeting each $D \in \mathcal{D}$.*

Trivially, PFA $\to \mathrm{MA}_{\mathrm{WCTP}}(\aleph_1) \to \mathrm{MA}_{\mathrm{CTP}}(\aleph_1)$, but in fact $\mathrm{MA}_{\mathrm{WCTP}}(\aleph_1) \leftrightarrow \mathrm{MA}_{\mathrm{CTP}}(\aleph_1)$ (see Lemma 3.13). Also, $\mathrm{MA}_{\mathrm{CTP}}(\aleph_1) \to 2^{\aleph_0} = 2^{\aleph_1}$ (see Corollary 3.15), so, the natural iteration of (totally proper) CTP orders with countable supports must introduce reals at limit stages. By the proof of Theorem 5.9 in [12], PFA does not follow from $\mathrm{MA}_{\mathrm{CTP}}(\aleph_1) + \mathrm{MA}(\aleph_1) + 2^{\aleph_0} = \aleph_2$, which in fact can be obtained by ccc forcing over L.

We now consider some CTP trees.

Definition 3.11 *Order $\lambda^{<\omega_1}$ by: $p \leq q$ iff $p \supseteq q$. Let $\mathbb{1} = \emptyset$, the empty sequence.*

So, $\lambda^{<\omega_1}$ is a tree, with the root $\mathbb{1}$ at the top. Viewed as a forcing order, it is equivalent to countable partial functions from ω_1 to λ. We often view $p \in \lambda^{<\omega_1}$ as a countable sequence and let $\mathrm{lh}(p) = \mathrm{dom}(p)$. Then $\mathrm{lh}(\mathbb{1}) = 0$.

Kurepa showed that SH is equivalent to the non-existence of Suslin trees. A similar proof shows that $\mathrm{MA}_{\mathrm{CTP}}(\aleph_1)$ is equivalent to the non-existence of Gregory trees:

Definition 3.12 *A* Gregory tree *is a forcing poset $\mathbb{P}$ which is a subtree of $\mathfrak{c}^{<\omega_1}$ and satisfies:*

1. *$\mathbb{P}$ has the CTP.*
2. *$\mathbb{P}$ is atomless.*
3. *$\mathbb{P}$ has no uncountable chains.*

It is easily seen that if any of conditions (1)(2)(3) are dropped, such trees may be constructed in ZFC. However:

Lemma 3.13 *The following are equivalent:*

1. $\mathrm{MA}_{\mathrm{CTP}}(\aleph_1)$.
2. $\mathrm{MA}_{\mathrm{WCTP}}(\aleph_1)$.
3. *There are no Gregory trees.*

Proof. (1) → (3): Let $\mathbb{P}$ be a Gregory tree. As with Suslin trees under MA($\aleph_1$), a filter G meeting the sets $D_\xi := \{p \in \mathbb{P} : \mathrm{lh}(p) \geq \xi\}$ yields an uncountable chain, and hence a contradiction, but to apply $\mathrm{MA_{CTP}}(\aleph_1)$, we must prove that each D_ξ is dense in $\mathbb{P}$. To do this, induct on ξ. The case $\xi = 0$ is trivial. For the successor stages, use the fact that $\mathbb{P}$ is atomless. For the limit stages, use the CTP.

(3) → (2): Fix $\mathbb{P}$ with the WCTP and dense sets $D_\xi \subseteq \mathbb{P}$ for $\xi < \omega_1$. We need to produce a filter $G \subseteq \mathbb{P}$ meeting each D_ξ. This is trivial if $\mathbb{P}$ has an atom, so assume that $\mathbb{P}$ is atomless.

Inductively define a subtree T of $2^{<\omega_1}$ together with a function $F : T \to \mathbb{P}$ as follows: $F(\mathbb{1}) = \mathbb{1}_\mathbb{P}$. If $t \in T$ and $\mathrm{lh}(t) = \xi$, then $t^\frown 0 \in T$ and $t^\frown 1 \in T$, and $F(t^\frown 0)$, $F(t^\frown 1)$ are extensions of $F(t)$ such that each $F(t^\frown i) \in D_\xi$ and $F(t^\frown 0) \not\Downarrow F(t^\frown 1)$; to accomplish this, given t and $F(t)$: first choose two $\perp$ extensions of $F(t)$, then extend these to be $\not\Downarrow$, and then extend these to be in D_ξ. If $\eta < \omega_1$ is a limit ordinal and $\mathrm{lh}(t) = \eta$, then $t \in T$ iff $\forall \xi < \eta\, [t \restriction \xi \in T]$ and $\exists q \in \mathbb{P}\, \forall \xi < \eta\, [q \leq F(t \restriction \xi)]$; then choose $F(t)$ to be some such q.

T is clearly atomless, and T has the CTP because $\mathbb{P}$ has the WCTP. If there are no Gregory trees, then T has an uncountable chain, so fix $g \in 2^{\omega_1}$ such that $g \restriction \xi \in T$ for all $\xi < \omega_1$, and let $G = \{y \in \mathbb{P} : \exists \xi < \omega_1\, [F(g \restriction \xi) \leq y]\}$. ☕

Theorem 3.14 (Gregory [11]) *If $2^{\aleph_0} < 2^{\aleph_1}$ then there is a Gregory tree.*

Corollary 3.15 $\mathrm{MA_{CTP}}(\aleph_1)$ *implies that* $2^{\aleph_0} = 2^{\aleph_1}$.

4 A Weird Space

We now prove Theorem 1.3. The basic construction is an inverse limit in ω_1 steps, and we follow approximately the terminology in [5, 12]. We build a compact space $X_{\omega_1} \subseteq [0,1]^{\omega_1}$ by constructing inductively $X_\alpha \subseteq [0,1]^{1+\alpha} \cong [0,1] \times [0,1]^\alpha$. Usually, one has $X_\alpha \subseteq [0,1]^\alpha$ in these constructions, but for finite α, the notation will be slightly simpler if we start at stage 0 with $X_0 = [0,1] = [0,1]^1$; of course, $1+\alpha = \alpha$ for infinite α.

Definition 4.1 $\pi^\beta_\alpha : [0,1]^{1+\beta} \twoheadrightarrow [0,1]^{1+\alpha}$ *is the natural projection.*

As usual, $\pi : X \twoheadrightarrow Y$ means that π is a *continuous* map from X *onto* Y. These constructions always have $\pi^\beta_\alpha(X_\beta) = X_\alpha$ whenever $0 \leq \alpha \leq \beta \leq \omega_1$. This determines X_γ for limit γ, so the meat of the construction involves describing how to build $X_{\alpha+1}$ given X_α.

A classical CH argument can ensure that X_{ω_1} is weird, but by Theorem 1.5, such an argument cannot make X_{ω_1} first countable. However, the same classical argument will let us construct a binary tree of spaces, resulting in a weird space $X_g \subseteq [0,1]^{\omega_1}$ for each $g \in 2^{\omega_1}$. We shall show that if no X_g were first countable, then there would be a $(\mathfrak{c}, 2)$–predictor $\Psi : [0,1]^{<\omega_1} \to 2$; so CH ensures that some X_g is first countable.

Our tree will give us an X_p for each $p \in 2^{\leq\omega_1}$. We now list requirements (R1)(R2)(R3)$\cdots$(R17) on the construction; a proof that all the requirements can be satisfied, and that they yield a weird space, concludes this section. We begin with the requirements involving the inverse limit:

R1. $X_{\mathbb{1}} = [0,1]$, where $\mathbb{1}$ is the empty sequence.
R2. X_p is an infinite closed connected subspace of $[0,1]^{1+\mathrm{lh}(p)}$.
R3. $\pi^\beta_\alpha \restriction X_p : X_p \twoheadrightarrow X_{p\restriction\alpha}$, and is irreducible, whenever $\beta = \mathrm{lh}(p) \geq \alpha$.

When $\gamma = \mathrm{lh}(p) \leq \omega_1$ is a limit, (R2)(R3) force:

$$X_p = \{x \in [0,1]^\gamma : \forall\alpha < \gamma\, [\pi^\gamma_\alpha(x) \in X_{p\restriction\alpha}]\} \quad . \qquad (❀)$$

To simplify notation for the restricted projection maps, we shall use:

Definition 4.2 *If $\beta = \mathrm{lh}(p) \geq \alpha$ and $r = p\restriction\alpha$, define $\pi^p_r = \pi^\beta_\alpha \restriction X_p : X_p \twoheadrightarrow X_r$.*

As in [12], each of $X_{p^\frown 0}$ and $X_{p^\frown 1}$ is obtained from X_p as the graph of a "$\sin(1/x)$" curve. We choose h_q, u_q, and v^n_q for $n < \omega$ and $q \in 2^{<\omega_1}$ of successor length, satisfying, for $i = 0, 1$:

R4. $u_{p^\frown i} \in X_p$ and $h_{p^\frown i} \in C(X_p \backslash \{u_{p^\frown i}\},\ [0,1])$ and $X_{p^\frown i} = \overline{h_{p^\frown i}}$.
R5. $v^n_{p^\frown i} \in X_p \backslash \{u_{p^\frown i}\}$, and $\langle v^n_{p^\frown i} : n \in \omega\rangle \to u_{p^\frown i}$, and all points of $[0,1]$ are limit points of $\langle h_{p^\frown i}(v^n_{p^\frown i}) : n \in \omega\rangle$.

As usual, we identify $h_{p^\frown i}$ with its graph. So, if $\alpha = \mathrm{lh}(p)$, then $X_{p^\frown i}$ is a subset of $[0,1]^{1+\alpha} \times [0,1]$, which we identify with $[0,1]^{1+\alpha+1}$. We shall say that the point $u_{p^\frown i}$ gets *expanded* in the passage from X_p to $X_{p^\frown i}$; the other points get *fixed*. (R3) follows from (R4) plus (❀). Also, if $\delta < \alpha$, then $\pi^p_{p\restriction\delta} : X_p \twoheadrightarrow X_{p\restriction\delta}$, and $(\pi^p_{p\restriction\delta})^{-1}\{x\}$ is a singleton unless x is in the countable set $\{\pi^{p\restriction\xi}_{p\restriction\delta}(u_{p\restriction(\xi+1)}) : \delta \leq \xi < \alpha\}$.

We now explain how points in $X_g \subset [0,1]^{\omega_1}$ can predict g, in the sense of Definition 2.1. We shall get A_q and B_q for $q \in 2^{<\omega_1}$ of successor length, satisfying:

R6. For $i = 0, 1$: $A_{p^\frown i}, B_{p^\frown i} \subseteq X_p$ and $A_{p^\frown i} = X_p \setminus B_{p^\frown i}$.
R7. For $i = 0, 1$ and $\xi < \mathrm{lh}(p) : A_{p^\frown i} \supseteq (\pi^p_{p\restriction\xi})^{-1}(A_{p\restriction(\xi+1)})$.
R8. $B_{p^\frown 0} \cap B_{p^\frown 1} = \emptyset$
R9. For $i = 0, 1$: $u_{p^\frown i} \in B_{p^\frown i}$.

Observe that some care must be exercised here in the inductive construction; otherwise, at some stage (R7) might imply that $A_{p^\frown i} = X_p$, so that $B_{p^\frown i} = \emptyset$, making (R9) impossible.

(R6)(R7)(R9) imply that points in $A_{p^\frown i}$ are forever fixed in the passage from X_p to any future X_q with $q \leq p^\frown i$; only points in $B_{p^\frown i}$ can get expanded. Points which are forever fixed must wind up having countable character, and (R8) lets us use a point of uncountable character in X_g to predict g:

Lemma 4.3 *Assume that we have* (R1 – R9)*, and assume that* $2^{\aleph_0} < 2^{\aleph_1}$*. Then* X_g *is first countable for some* $g \in 2^{\omega_1}$*.*

Proof. We shall define $\Psi : [0,1]^{<\omega_1} \to 2$, and prove that Ψ is a $(\mathfrak{c}, 2)$–predictor if every X_g contains a point of uncountable character.

Say $\mathrm{lh}(p) = \alpha < \omega_1$ and $\delta < \alpha$. If $x \in B_{p^\frown i} \subseteq X_p$ then, by (R6)(R7), $\pi^\alpha_\delta(x) \in B_{p\restriction(\delta+1)} \subseteq X_{p\restriction\delta}$. Applying (R8), if $x \in [0,1]^{1+\alpha}$ and $x \in B_{p^\frown i} \cap B_{r^\frown j}$, then $p = r$ and $i = j$; to prove this, consider the least $\delta < \alpha$ such that $p(\delta) \neq r(\delta)$.

Set $\Psi(x) = 0$ if $\mathrm{lh}(x) < \omega$. Now, say $x \in [0,1]^\alpha$, where $\omega \leq \alpha < \omega_1$ (so $1 + \alpha = \alpha$). If there exist $p \in 2^\alpha$ and $i \in 2$ such that $x \in B_{p^\frown i}$, then these p, i are unique, and set $\Psi(x) = i$. If there are no such p, i, then set $\Psi(x) = 0$.

Now, assume that for each g, we can find $z = z_g \in X_g$ with $\chi(z, X_g) = \aleph_1$. Let $C = \omega_1 \backslash \omega$. We shall show that Ψ, z predict g on C. For $\xi \in C$, let $p^\frown i = g\restriction(\xi+1)$. Then $z\restriction\xi = \pi^g_p(z) \in X_p$, and $z\restriction\xi$ must be in $B_{p^\frown i}$, since if it were in $A_{p^\frown i}$, then $(\pi^g_p)^{-1}(\pi^g_p(z)) = \{z\}$, so that $\chi(z, X_g) = \aleph_0$. Thus, $\Psi(z\restriction\xi) = i = g(\xi)$. ☕

Since every X_g clearly has weight $\aleph_1$, we are done if we can make every X_g weird. Since points in $A_{p^\frown i}$ are forever fixed, we must make sure that $A_{p^\frown i}$ has no Cantor subsets. Conditions (R6)(R8) say that $A_{p^\frown 0} \cup A_{p^\frown 1} = X_p$, so $A_{p^\frown 0}$ and $A_{p^\frown 1}$ must be Bernstein sets. Note that Condition (R7) may present a problem at limit stages. When $\mathrm{lh}(p) = \alpha$ we have $A_{p^\frown i} \supseteq \bigcup_{\xi<\alpha} (\pi^p_{p\restriction\xi})^{-1}(A_{p\restriction(\xi+1)})$. Points in $A_{p\restriction(\xi+1)}$ are forever fixed, so each $(\pi^p_{p\restriction\xi})^{-1}(A_{p\restriction(\xi+1)})$ will have no Cantor subsets. Without further requirements, though, $\bigcup_{\xi<\alpha} (\pi^p_{p\restriction\xi})^{-1}(A_{p\restriction(\xi+1)})$ may contain a Cantor subset. So, we make sure each such union is disjoint from some set in a tree of Bernstein sets:

Definition 4.4 *For any topological space Y and $p \in 2^{<\omega_1}$, a* Bernstein tree in Y rooted in p *is a family of subsets of Y, $\{D^q : q \le p\}$, satisfying:*

1. *For each q, neither D^q nor $Y \backslash D^q$ contains a Cantor subset.*
2. *Each $D^{q^\frown 0} \cap D^{q^\frown 1} = \emptyset$.*
3. *If $r \le q$ then $D^r \subseteq D^q$.*

Note that if Y itself does not contain a Cantor subset, then (1) is trivial, and we may take all $D^q = \emptyset$ to satisfy (2) and (3).

Now, in our construction, we also build D^q_p for $q \le p \in 2^{<\omega_1}$ satisfying:

R10. For each $p \in 2^{<\omega_1}$: $\{D^q_p : q \le p\}$ is a Bernstein tree in X_p rooted in p.

R11. If $q \le p \le r$ and $\pi = \pi^p_r : X_p \twoheadrightarrow X_r$ and $x \in X_p$ with $\pi^{-1}(\pi(x)) = \{x\}$, then $x \in D^q_p$ iff $\pi(x) \in D^q_r$.

R12. For each $p \in 2^{<\omega_1}$ and $i \in 2$: $B_{p^\frown i} = D^{p^\frown i}_p$ and $A_{p^\frown i} = X_p \setminus D^{p^\frown i}_p$.

Of course, (R12) simply defines $A_{p^\frown i}$ in terms of the D^q_p, and then (R10) guarantees that no $A_{p^\frown i}$ has a Cantor subset, but we need to verify that the conditions (R1 – R12) can indeed be satisfied. First, three easy lemmas about Bernstein trees. A standard inductive construction in $\mathfrak{c}$ steps shows:

Lemma 4.5 *If Y is a separable metric space, then there is a Bernstein tree in Y rooted in $\mathbb{1}$.*

Using the fact that every uncountable Borel subset of the Cantor set contains a perfect subset, we get:

Lemma 4.6 *Assume that Y is any topological space, Z is a Borel subset of Y, and $\{D^q : q \le p\}$ is a family of subsets of Y satisfying (2)(3) of Definition 4.4. Then $\{D^q : q \le p\}$ is a Bernstein tree in Y iff both $\{D^q \cap Z : q \le p\}$ is a Bernstein tree in Z and $\{D^q \backslash Z : q \le p\}$ is a Bernstein tree in $Y \backslash Z$.*

Combining these two lemmas:

Lemma 4.7 *If Y is a separable metric space, Z is a Borel subset of Y, and $\{E^q : q \le p\}$ is a Bernstein tree in Z rooted in p, then there is a Bernstein tree $\{D^q : q \le p\}$ in Y rooted in p such that each $D^q \cap Z = E^q$.*

Returning to the construction:

Lemma 4.8 *There exist X_p for $p \in 2^{\le \omega_1}$ satisfying Conditions* (R1 – R12).

Proof. We start with $X_{\mathbb{1}} = [0,1]$, and we obtain the $D^q_{\mathbb{1}}$ by applying Lemma 4.5.

If $\alpha = \mathrm{lh}(p) > 0$ and we have done the construction for $p{\restriction}\xi$ for all $\xi < \mathrm{lh}(p)$, then X_p is determined either by (R4) when $\mathrm{lh}(p)$ is a successor or by (❁) when $\mathrm{lh}(p)$ is a limit. If $\alpha < \omega_1$, we construct the D^q_p to satisfy (R10)(R11) as follows: For $\xi < \alpha$, use π_ξ for $\pi^p_{p{\restriction}\xi}$. Let $Z_\xi = \{x \in X_p : \pi_\xi^{-1}(\pi_\xi(x)) = \{x\}\}$, and let $Z = \bigcup_{\xi<\alpha} Z_\xi$. Observe that Z and all the Z_ξ are Borel sets. Let $\{E^q_p : q \le p\}$ be the Bernstein tree in Z rooted in p defined by saying that for $x \in Z_\xi$: $x \in E^q_p$ iff $\pi_\xi(x) \in D^q_{p{\restriction}\xi}$. Note that, by (R11) applied inductively, this is independent of which ξ is used. To obtain the D^q_p from the E^q_p, apply Lemma 4.7. Note that, by (R11) applied inductively once again, these D^q_p work for X_p.

The $A_{p^\frown i}$ and $B_{p^\frown i}$ (for $i = 0, 1$) are now defined by (R12), and we must verify that this definition satisfies (R7): Assume that $\xi < \mathrm{lh}(p) = \alpha$ and $x \in X_p$ and $\pi_\xi(x) \in A_{p{\restriction}(\xi+1)}$. We must show that $x \in A_{p^\frown i}$; equivalently, by (R12), that $x \notin D^{p^\frown i}_p$. Now $\pi_\xi(x) \in A_{p{\restriction}(\xi+1)}$ implies that $\pi_\xi^{-1}(\pi_\xi(x)) = \{x\}$ (using (R4 – R9) inductively), so that $x \notin D^{p^\frown i}_p$ iff $\pi_\xi(x) \notin D^{p^\frown i}_{p{\restriction}\xi}$. By (R10) for $p{\restriction}\xi$ and Definition 4.4(3), $D^{p^\frown i}_{p{\restriction}\xi} \subseteq D^{p{\restriction}(\xi+1)}_{p{\restriction}\xi}$. So $A_{p{\restriction}(\xi+1)} = X_{p{\restriction}\xi} \setminus D^{p{\restriction}(\xi+1)}_{p{\restriction}\xi}$ gives us (R7).

Since the $B_{p^\frown i}$ are nonempty, there is no problem choosing the $u_{p^\frown i}$, $v^n_{p^\frown i}$, and $h_{p^\frown i}$ to satisfy (R4)(R5)(R9), and then the $X_{p^\frown i}$ are defined by (R4). ☕

Finally, we must make each X_g weird. Observe:

Lemma 4.9 *Conditions* (R1 – R5) *imply that if $F \subseteq X_p$ is closed and connected then $(\pi^q_p)^{-1}(F)$ is connected for all $q \le p$.*

Now, we shall make sure that whenever F is a perfect subset of X_g, there is some $\alpha < \omega_1$ such that $(\pi^g_{g{\restriction}(\alpha+1)})^{-1}(\{u_{g{\restriction}(\alpha+1)}\} \times [0,1]) \subseteq F$ (recall that our construction gave us $\{u_{g{\restriction}(\alpha+1)}\} \times [0,1] \subset X_{g{\restriction}(\alpha+1)} \subset X_{g{\restriction}\alpha} \times [0,1]$) . By Lemma 4.9, this implies that F is not totally disconnected. The argument in [12] obtained this α by using $\diamondsuit$ to capture F. Here, we replace this use of $\diamondsuit$ by a classical CH argument. First, as in [12], construct $\mathcal{F}_p$ for $p \in 2^{<\omega_1}$ so that:

R13. $\mathcal{F}_p$ is a countable family of uncountable closed subsets of X_p.

R14. If $F \in \mathcal{F}_p$ and $q \le p$ then $(\pi^q_p)^{-1}(F) \in \mathcal{F}_q$.

R15. For each $F \in \mathcal{F}_p$, either $u_{p^\frown i} \notin F$, or $u_{p^\frown i} \in F$ and $v^n_{p^\frown i} \in F$ for all but finitely many n.

R16. $\{u_{p^\frown i}\} \times [0,1] \in \mathcal{F}_{p^\frown i}$.

We may satisfy (R13)(R14)(R16) simply by defining

$$\mathcal{F}_p = \left\{ (\pi^p_{p\restriction\xi})^{-1}\{u_{p\restriction(\xi+1)}\} : \xi < \mathrm{lh}(p) \right\} \quad .$$

Requirements (R4)(R14)(R15) imply:

Lemma 4.10 *$\pi^q_p : (\pi^q_p)^{-1}(F) \twoheadrightarrow F$ is irreducible whenever $F \in \mathcal{F}_p$ and $q \le p$.*

Then, we use CH rather than $\diamondsuit$ to get:

R17. Whenever $p \in 2^{<\omega_1}$ and F is an uncountable closed subset of X_p, there is a β with $\mathrm{lh}(p) < \beta < \omega_1$ such that for all $q < p$ with $\mathrm{lh}(q) = \beta$ and for each $x \in \{u_{q^\frown 0}, u_{q^\frown 1}\} \cup \{v^n_{q^\frown i} : n \in \omega \ \&\ i \in 2\}$, the projections $\pi = \pi^q_p$ satisfy $\pi(x) \in F$ and $|\pi^{-1}(\pi(x))| = 1$.

Proof of Theorem 1.3. Assuming that we can obtain (R1 – R17), note that each X_g is separable, because each $\pi^g_{\mathbb{1}} : X_g \twoheadrightarrow X_{\mathbb{1}}$ is irreducible. Then, to finish, by Lemma 4.3, it suffices to show that each X_g is weird. Fix a perfect $H \subseteq X_g$; we shall show that it is not totally disconnected. First, fix $\alpha < \omega_1$ such that, if we set $p = g\restriction\alpha$ and $F = \pi^g_p(H)$, then F is perfect (the set of all such α form a club). Then, fix $\beta > \alpha$ as in (R17), let $q = g\restriction\beta$, and let $i = g(\beta)$, so that $q^\frown i = g \restriction (\beta+1)$. Let $K = (\pi^q_p)^{-1}(F)$. Then $\pi^g_q(H) \subseteq K$, and this inclusion may well be proper. However, $u_{q^\frown i} \in \pi^g_q(H)$ and $v^n_{q^\frown i} \in \pi^g_q(H)$ for each $n \in \omega$ because $\pi(u_{q^\frown i}) \in F$ and $\pi(v^n_{q^\frown i}) \in F$ and $|\pi^{-1}(\pi(u_{q^\frown i}))| = |\pi^{-1}(\pi(v^n_{q^\frown i}))| = 1$. It follows (using (R5)) that $E := \{u_{q^\frown i}\} \times [0,1] \subseteq \pi^g_{q^\frown i}(H)$. Since $E \in \mathcal{F}_{q^\frown i}$ by (R16) and H maps onto E, Lemma 4.10 implies that $(\pi^g_{q^\frown i})^{-1}(E) \subseteq H$. Since $(\pi^g_{q^\frown i})^{-1}(E)$ is connected by Lemma 4.9, H cannot be totally disconnected.

Next, to obtain conditions (R1 – R17), we must augment the proof of Lemma 4.8: Fix in advance a map ψ from $\omega_1 \backslash \{0\}$ onto $\omega_1 \times \omega_1$, such that $\alpha < \beta$ whenever $\psi(\beta) = (\alpha, \xi)$. Now, given X_p, use CH and let $\{F^p_\xi : \xi < \omega_1\}$ be a listing of all uncountable closed subsets of X_p. Whenever $0 < \beta < \omega_1$ and $\psi(\beta) = (\alpha, \xi)$ and $\mathrm{lh}(q) = \beta$, we may set $p = q\restriction\alpha$ and $F = F^p_\xi \subseteq X_p$. It is sufficient to show how to accomplish (R17) with these specific α, β, p, q, F.

Choose a perfect $K \subset F$ which is disjoint from $\{\pi^{(q\restriction\zeta)}_p(u_{q\restriction(\zeta+1)}) : \alpha \le \zeta < \beta\}$. Then π^q_p is 1-1 on $(\pi^q_p)^{-1}(K)$, so choosing all $u_{q^\frown i}$ and $v^n_{q^\frown i}$ in $(\pi^q_p)^{-1}(K)$ will ensure (R17). Now *fix* $i \in 2$, and write u and v^n for $u_{q^\frown i}$ and $v^n_{q^\frown i}$. To ensure (R15) and (R9), we modify the argument of [12]. Let $\{Q^n : n \in \omega\}$ list $\mathcal{F}_q$. Let d be a metric on $(\pi^q_p)^{-1}(K)$. For each $s \in 2^{<\omega}$, choose a perfect $L_s \subseteq (\pi^q_p)^{-1}(K)$. Make these into a tree, in the sense that each $L_{s^\frown 0} \cap L_{s^\frown 1} = \emptyset$, each $\mathrm{diam}(L_s) \le 2^{-\mathrm{lh}(s)}$, and

$L_{s^\frown 0} \subseteq L_s$ and $L_{s^\frown 1} \subseteq L_s$. Also make sure that whenever $\mathrm{lh}(s) = n+1$ we have either $L_s \subseteq Q^n$ or $L_s \cap Q^n = \emptyset$. Let $v[s^\frown \ell]$ be any point in $L_{s^\frown \ell} \backslash L_s$. For $f \in 2^\omega$, let $\{u[f]\} = \bigcap_n L_{f \restriction n}$. For any $f \in 2^\omega$, if we set $u = u[f]$ and $v^n = v[f \restriction (n+1)]$, then (R15) will hold. Now, (R9) requires $u \in B_{q^\frown i}$. Since $B_{q^\frown i}$ is a Bernstein set and $\{u[f] : f \in 2^\omega\}$ is a Cantor set, we may choose f so that $u[f] \in B_{q^\frown i}$. ☕

If $H \subseteq X_g$ is closed and for some initial segment $p = g \restriction \alpha$ the projection $\pi^g_p(H) \in \mathcal{F}_p$, then, by irreducibility, $H = (\pi^g_p)^{-1}(\pi^g_p(H))$, so that H is a G_δ. To make X_g hereditarily Lindelöf, it suffices to capture projections for each closed $H \subseteq X_g$ this way, but it is not clear whether this can be done without using $\diamondsuit$.

5 Remarks and Examples

One cannot replace "CH" by "$2^{\aleph_0} < 2^{\aleph_1}$" in the statement of Theorem 1.3, since by Proposition 5.3, it is consistent with any cardinal arithmetic that every non-scattered compactum of weight less than $\mathfrak{c}$ contains a copy of the Cantor set. As usual, define

Definition 5.1 $\mathrm{cov}(\mathcal{M})$ *is the least* κ *such that* $\mathbb{R}$ *is the union of* κ *meager sets.*

Note that $\mathrm{cov}(\mathcal{M})$ is the least κ such that $\mathrm{MA}(\kappa)$ for countable partial orders fails. Using this, we easily see:

Lemma 5.2 *If* $\kappa < \mathrm{cov}(\mathcal{M})$ *and* $E_\alpha \subset [0,1]$ *is meager for each* $\alpha < \kappa$*, then* $[0,1] \setminus \bigcup_{\alpha<\kappa} E_\alpha$ *contains a copy of the Cantor set.*

Proposition 5.3 *If* X *is compact and not scattered, and* $w(X) < \mathrm{cov}(\mathcal{M})$*, then* X *contains a copy of the Cantor set.*

Proof. Replacing X by a subspace, we may assume that we have an irreducible map $\pi : X \twoheadrightarrow [0,1]$. Let $\mathcal{B}$ be an open base for X with $|\mathcal{B}| < \mathrm{cov}(\mathcal{M})$ and $\emptyset \neq \mathcal{B}$.

Whenever $U, V \in \mathcal{B}$ with $\overline{U} \cap \overline{V} = \emptyset$, let $E_{U,V} = \pi(\overline{U}) \cap \pi(\overline{V})$. Then $E_{U,V} \subset [0,1]$ is nowhere dense because π is irreducible. Applying Lemma 5.2, let $K \subset [0,1]$ be a copy of the Cantor set disjoint from all the $E_{U,V}$. Note that $|\pi^{-1}\{y\}| = 1$ for all $y \in K$. Thus, $\pi^{-1}(K)$ is homeomorphic to K. ☕

Note that one can force "$\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$" by adding $\mathfrak{c}$ Cohen reals, which does not change cardinal arithmetic, but in the statement of Proposition 5.3, "$\mathrm{cov}(\mathcal{M})$" cannot be replaced by "$\mathfrak{c}$". If CH holds in V, then one may force $\mathfrak{c}$ to be arbitrarily large by adding random reals, and *any* random real extension $V[G]$ will have a compact non-scattered space of weight $\aleph_1$ which does not contain a Cantor subset. In fact, Dow and Fremlin [4] show that if X is a compact F-space in V, then in a random real extension $V[G]$, the corresponding compact space $\widetilde{X}$ has no convergent ω–sequences, and hence no Cantor subsets.

The weird space constructed in [12] also failed to satisfy the CSWP (the complex version of the Stone–Weierstrass Theorem). Using the method there, we can modify the proof of Theorem 1.3 to get:

Theorem 5.4 *Assuming CH, there is a separable first countable connected weird space X of weight $\aleph_1$ such that X fails the CSWP and $\mathbb{K}_X$ fails the CTP.*

Proof. First, in the proof of Theorem 1.3, replace $[0,1]$ by $\overline{D}$, the closed unit disc in the complex plane, so that we may view X as a subspace of the $\aleph_1$–dimensional polydisc. Then, as in [12], by carefully choosing the functions $h_{p^\frown i}$, one can ensure that the restriction to X of the natural analog of the disc algebra refutes the CSWP of X. To refute the CTP, construct in $\overline{D}$ a Cantor tree $\{p_s : s \in 2^{<\omega}\} \subseteq \mathbb{K}_{\overline{D}}$ such that each p_s is a wedge of the disc with center 0 and radius $2^{-\mathrm{lh}(s)}$; then each $\bigcap_{n\in\omega} p_{f\restriction n} = \{0\}$. Then, since we may assume the point 0 is not expanded in the construction of X, the inverse images of the p_s yield a counterexample to the CTP of $\mathbb{K}_X$. ☕

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