Bohr Topologies and Compact Function Spaces *

Joan E. Hart† and Kenneth Kunen‡§

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Abstract

We consider (discrete) structures, $\mathfrak{A}$, for a countable language. $\mathfrak{A}^\#$ denotes $\mathfrak{A}$ with its Bohr topology. Let Y be a compact Hausdorff space. Then Y is homeomorphic to a subspace of some $\mathfrak{A}^\#$ iff Y is Talagrand compact.

1 Introduction

1.1 Summary

The results of this paper use the theory of compact function spaces to characterize the possible compact subspaces of topological structures endowed with the Bohr topology. We begin by reviewing some background on Bohr topologies, and then explain how this relates to function spaces.

Throughout this paper, a *language*, $\mathcal{L}$, is a *countable* (possibly finite) set of function and constant symbols. Then, a *structure* $\mathfrak{A}$ for $\mathcal{L}$ is a non-empty set A, together with actual functions on A and elements of A, corresponding to the function and constant symbols of $\mathcal{L}$. The language $\mathcal{L}$ is needed when

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†University of Dayton, Dayton, OH 45469, U.S.A., jhart@udayton.edu

‡University of Wisconsin, Madison, WI 53706, U.S.A., kunen@math.wisc.edu

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we talk about logical formulas being true or false in $\mathfrak{A}$. Homomorphisms are always between structures for the same language.

$\mathfrak{X}$ is a *topological structure* iff $\mathfrak{X}$ is a structure together with a topology which makes all the functions of $\mathfrak{X}$ continuous; then $\mathfrak{X}$ is a *compact structure* iff this topology is compact Hausdorff. A *compactification* of a structure $\mathfrak{A}$ is a pair $(\mathfrak{X}, \varphi)$ such that $\mathfrak{X}$ is a compact structure and φ is a homomorphism from $\mathfrak{A}$ into $\mathfrak{X}$ such that $\mathrm{ran}(\varphi)$ is dense in X.

Following Holm [17], for every structure $\mathfrak{A}$, there is a largest compactification of $\mathfrak{A}$, which is now called the *Bohr-Holm compactification*, and is denoted by $(\mathsf{b}\mathfrak{A}, \Phi_{\mathfrak{A}})$. Then, $\mathfrak{A}^{\#}$ denotes the topological structure obtained by giving $\mathfrak{A}$ the coarsest topology which makes $\Phi_{\mathfrak{A}}$ continuous; equivalently, this is the coarsest topology which makes φ continuous for *all* compactifications $(\mathfrak{X}, \varphi)$ of $\mathfrak{A}$. See [14] for further details.

Every compact Hausdorff space Y is a subspace of $\mathsf{b}\mathfrak{A}$ for some $\mathfrak{A}$ (in fact, $\mathfrak{A}$ can be taken to be an abelian group). But not every such Y can be embedded in an $\mathfrak{A}^{\#}$. For Y compact Hausdorff, Y is a subspace of some $\mathfrak{A}^{\#}$ iff Y is Talagrand compact (see Theorem 3.19). The Talagrand compacta are described in Section 2. Every Eberlein compactum is Talagrand compact, and every Talagrand compactum is Corson compact. The Corson and Eberlein compacta can be defined as follows:

Definition 1.1 *For* $y \in \mathbb{R}^J$*:* $\mathrm{supt}(y) = \{j \in J : y(j) \neq 0\}$ *and* $\mathrm{supt}_\varepsilon(y) = \{j \in J : |y(j)| > \varepsilon\}$. $\Sigma(J)$ *is the set of all* $y \in \mathbb{R}^J$ *such that* $\mathrm{supt}(y)$ *is countable and* $\Sigma_*(J)$ *is the set of all* $y \in \mathbb{R}^J$ *such that* $\mathrm{supt}_\varepsilon(y)$ *is finite for all* $\varepsilon > 0$. $\Sigma(J)$ *and* $\Sigma_*(J)$ *have the usual product topology.*

Definition 1.2 *A space* Y *is* Corson compact *iff for some set* J, Y *is homeomorphic to a compact subspace of* $\Sigma(J)$. Y *is* Eberlein compact *iff for some set* J, Y *is homeomorphic to a compact subspace of* $\Sigma_*(J)$.

It is not true in general that a compact subspace of $\mathfrak{A}^{\#}$ must be an Eberlein compactum, but this is true if $\mathfrak{A}$ is *nice*:

Definition 1.3 *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are structures (for the same language), then* $Hom(\mathfrak{A}, \mathfrak{B})$ *is the set of homomorphisms from* $\mathfrak{A}$ *into* $\mathfrak{B}$. *If* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *are topological structures, then* $Hom_c(\mathfrak{X}, \mathfrak{Y})$ *is the set of continuous homomorphisms from* $\mathfrak{X}$ *into* $\mathfrak{Y}$.

Definition 1.4 *A compact structure* $\mathfrak{X}$ *is* nice *iff there is a single compact second countable structure* $\mathfrak{N}$ *such that* $Hom_c(\mathfrak{X}, \mathfrak{N})$ *separates the points of* X. *A structure* $\mathfrak{A}$ *(without a topology) is* nice *iff* $\mathsf{b}\mathfrak{A}$ *is nice.*

Every group is nice, since $\mathfrak{N}$ can be the product of the unitary groups, $\prod_{1\leq n<\omega} U(n)$, and every boolean algebra is nice, since $\mathfrak{N}$ can be the two-element algebra. A few other examples are given in [14], but not much is known in general about which structures are nice. For example, it is unknown whether every compact semilattice is nice, although the obvious $\mathfrak{N} = ([0,1]; \wedge)$ does not work by Lawson [19]. Actually, as applied to arbitrary structures, the notion of "nice" is a bit pathological, since a compact group with an added constant can fail to be nice (Example 3.25).

As we have indicated, requiring the $\mathfrak{A}$ of Theorem 3.19 to be nice gives us the Eberlein compacta. More precisely, Theorem 3.13 shows that a compact Hausdorff space Y is Eberlein compact iff Y is homeomorphic to some closed subspace of some $\mathfrak{A}^\#$ for some nice $\mathfrak{A}$. The theorem also shows that $\mathfrak{A}$ can be taken to be self-bohrifying:

Definition 1.5 *If $\mathfrak{X}$ is a topological structure, then $\mathfrak{X}_d$ denotes the structure $\mathfrak{X}$ stripped of its topology. A compact structure $\mathfrak{X}$ is* self-bohrifying *iff its Bohr-Holm compactification is just the identity map $\mathfrak{X}_d \hookrightarrow \mathfrak{X}$.*

Equivalently, $\mathfrak{X}$ is self-bohrifying iff $Hom_c(\mathfrak{X},\mathfrak{Y}) = Hom(\mathfrak{X},\mathfrak{Y})$ for all compact structures $\mathfrak{Y}$.

These notions are considered in their generality in [14], but they occur much earlier in the literature for the special case of groups. For connected compact Lie groups, van der Waerden [29] (see also [16], Theorem 5.64) showed that self-bohrifying implies semisimple, and the converse goes back to Anderson and Hunter [1]. Also, by Moran [22], some infinite products of finite groups are self-bohrifying (and, of course, disconnected). It is still not known exactly which products of finite groups are self-bohrifying.

By Argyros and Negrepontis [2], every Talagrand (in fact, Gul'ko) compactum with the countable chain condition is second countable. This result applies to every compact subspace of an $\mathfrak{A}^\#$, by Theorem 3.19. In particular, if $\mathfrak{X}$ is self-bohrifying and contains a group operation, then $\mathfrak{X}$ is second countable. Some important special cases of this were already known. If $\mathfrak{X} = (X; \cdot)$ is just a group, this fact is due to Moran [22], who proved the stronger result that for each n, $\mathfrak{X}$ has only finitely many inequivalent irreducible representations of degree n. When $\mathfrak{X}$ is a topological ring, the result is due to Ursul [28]. Comfort and Remus [7] contains further results on self-bohrifying rings.

Theorem 3.19 says that the Talagrand compacta are precisely those compact spaces which are contained in some $\mathfrak{A}^\#$, and that this $\mathfrak{A}$ may be taken to be self-bohrifying. We do not have a simple characterization of the class of spaces which *themselves* can be made into self-bohrifying structures; call these the *self-bohrable* compacta. Observe that this class does not contain all Eberlein compacta, or even all metric continua. For example, let Y be a Cook continuum [9, 24]; that is, Y is a metric continuum and the only continuous functions from Y into Y are the constant functions and the identity function. It follows that if $f : Y^n \to Y$ is continuous, then f is either constant or of the form $f(y_1, \ldots, y_n) = y_i$. All these functions extend continuously to $\beta(Y_d)$. Thus, if Y is made into a topological structure $\mathfrak{Y}$, then $\mathsf{b}\mathfrak{Y}_d$ will be $\beta(Y_d)$, not Y. However, it is true that many of the "common" metric continua, such as finite polyhedra and countable products thereof, are self-bohrable by our Corollary 3.22.

The following is a variation on Theorem 4.2 of Comfort [6]:

Proposition 1.6 *If $\mathfrak{X}$ is a compact structure, $|X| \geq 2$, and κ is infinite, then $\mathfrak{X}^\kappa$ is not self-bohrifying.*

Proof. Let $\mathcal{U}$ be any non-principal ultrafilter on κ, and for $x \in X^\kappa$, let $\varphi(x) = \mathcal{U}\text{-}\lim\langle x_\alpha : \alpha < \kappa\rangle$; equivalently, if $x : \kappa \to X$ and $\overline{x} : \beta\kappa \to X$ is the usual extension of x to the Čech compactification, then $\varphi(x) = \overline{x}(\mathcal{U})$.

Then φ is a homomorphism of $\mathfrak{X}^\kappa$ into $\mathfrak{X}$. To see that φ is not continuous, fix a, b in X with $a \neq b$, and let $\vec{a}, \vec{b} \in X^\kappa$ be the corresponding constant sequences. Observe that $\varphi(\vec{a}) = a$, and every neighborhood of $\vec{a}$ contains a sequence x with $\varphi(x) = b$ (take x equal to b on all but finitely many $\alpha \in \kappa$).

In the present paper, this proposition is of interest primarily for motivation. For example, $[0, 1]^\omega$ is self-bohrable by Corollary 3.22, but the proof of this cannot be by making $[0, 1]$ self-bohrable (Corollary 3.10), and then just taking an infinite power; one needs to add more functions. Of course, an *uncountable* power cannot be made self-bohrifying using *any* countable collection of functions, since it is not Corson compact.

1.2 Technical Remarks

Compact structures are Hausdorff by definition, but $\mathfrak{A}^\#$ will fail to be Hausdorff whenever distinct points of A get identified by the map $\Phi_\mathfrak{A} : \mathfrak{A} \to \mathsf{b}\mathfrak{A}$. However, this additional complication can be avoided in this paper, since we are considering only compact *Hausdorff* spaces Y contained in $\mathfrak{A}^\#$. For $a, b \in A$, define $a \sim b$ iff $\Phi_\mathfrak{A}(a) = \Phi_\mathfrak{A}(b)$ (equivalently, iff $\varphi(a) = \varphi(b)$ for all compactifications $(\mathfrak{X}, \varphi)$ of $\mathfrak{A}$). Then $\mathfrak{A}^\#$ is Hausdorff iff $\sim$ is the identity relation. One can always replace a structure $\mathfrak{A}$ by the quotient $\mathfrak{A}/\sim$, and then consider embeddings of Y into the Hausdorff $(\mathfrak{A}/\sim)^\#$ (see [14], Lemma 2.3.11). Thus, we lose no generality by considering only structures $\mathfrak{A}$ for which $\mathfrak{A}^\#$ is Hausdorff; so, we may always view $\mathfrak{A}$ as a substructure of $\mathsf{b}\mathfrak{A}$, with $\Phi_{\mathfrak{A}}$ the identity map and $\mathfrak{A}^\#$ the subspace topology. Dropping explicit mention of $\Phi_{\mathfrak{A}}$ simplifies the notation somewhat; for example, the fact (see [14], Lemma 2.3.9) that $\mathsf{b}\mathfrak{A}$ preserves some of the properties of $\mathfrak{A}$ may be stated as:

Lemma 1.7 *Suppose that $\mathfrak{X}$ is any compactification of $\mathfrak{A}$, with $\mathfrak{A} \hookrightarrow \mathfrak{X}$. Suppose that $\psi(v_1, \ldots, v_n)$ is a positive logical formula, $a_1, \ldots, a_n \in A$, and $\psi(a_1, \ldots, a_n)$ is true in $\mathfrak{A}$. Then $\psi(a_1, \ldots, a_n)$ is true in $\mathfrak{X}$ also.*

Paper [14] also discusses how much freedom one has with the language of structures in computing their Bohr compactifications; it includes remarks on excluding relations, as well as details on dropping inessential functions. For orders, for example, rather than use the relation symbol $\leq$, we use the binary function $\wedge$. For groups, $\mathsf{b}(G; \cdot)$ can be identified with $\mathsf{b}(G; \cdot, ^{-1})$. In the present paper, we only need the fact, which can easily be checked directly, that constants can be dropped. Thus, if $(A; \wedge)$ is a semilattice with a 0, we can identify $\mathsf{b}(A; \wedge)$ and $\mathsf{b}(A; \wedge, 0)$.

However, the notion of "nice" is very sensitive to the addition or deletion of constants; see Example 3.25.

1.3 Semilattices

Although in functional analysis, the linear space $\Sigma_*(J)$ is the natural object to consider, for our purposes it will be more convenient to intersect this with the compact semilattice $[0,1]^J$. The following variation on Definition 1.2 is standard (see Corollary 2.9 for a similar proof).

Lemma 1.8 *A space Y is Eberlein compact iff for some set J, Y is homeomorphic to a compact subspace of $\Sigma_*(J) \cap [0,1]^J$.*

Now, our first step in embedding an Eberlein compactum into a self-bohrifying structure will be to embed it into a compact semilattice. Of course, an arbitrary subspace of $[0,1]^J$ need not be a semilattice, but we can just close it downward:

Definition 1.9 *If Z is any partially ordered set and Y is a subset of Z, then $Y\!\downarrow = \{z \in Z : \exists y \in Y[z \leq y]\}$ and $Y\!\uparrow = \{z \in Z : \exists y \in Y[z \geq y]\}$.*

Observe that if Y is closed in the semilattice $[0,1]^J$, then $Y\!\downarrow$ is closed also; it is thus immediate from Lemma 1.8 that every Eberlein compactum is contained in an Eberlein compact semilattice; in fact, one of this special form (a $Y\!\downarrow$) is actually self-bohrable by Lemma 3.12. There is a similar description of the Talagrand compacta (see Section 2); we use this in Section 3 when we show how to embed Talagrand compacta into self-bohrifying structures.

1.4 Related Notions

Some properties weaker than self-bohrifying have also been considered in the literature. A compact structure $\mathfrak{X}$ is *self-compactifying* (see [14]) iff its topology is determined by its algebraic structure; that is, the given topology is the only one which makes the structure into a compact structure. This is strictly weaker than self-bohrifying; for example, Lawson [20] showed that every compact semilattice is self-compactifying, while by [14], the only self-bohrifying compact semilattices are the ones with no infinite chains. We shall use Lawson's methods in Section 3, however, when we make Talagrand compact semilattices self-bohrifying by adding additional functions to the structure.

A still weaker property, called the *van der Waerden property* by Lawson [20] and Anderson and Hunter [1], holds iff every automorphism of $\mathfrak{X}$ is continuous. This is strictly weaker than self-compactifying. For example, let $\mathfrak{Z}$ be the structure $(\mathbb{Z} \cup \{+\infty, -\infty\};\ S, 0, +\infty, -\infty)$. Here, the language has three constant symbols, $0, +\infty, -\infty$, interpreted in the obvious way, together with a unary "successor" function, interpreted so that $S(n) = n+1$, $S(+\infty) = +\infty$, and $S(-\infty) = -\infty$. Because of the constants in the language, the only automorphism is the identity, so the obvious compact topology trivially has the van der Waerden property. However, there is a second compact Hausdorff topology, obtained by making $+\infty$ a limit of the negative integers and $-\infty$ a limit of the positive integers.

In the case that $\mathfrak{X}$ is a finite dimensional compact connected group, [1] shows that it has the van der Waerden property iff it is is self-bohrifying.

2 Some Classes of Compacta

The literature contains several equivalent definitions of the Eberlein and Talagrand compacta. In this section, we rephrase these in a form more suitable to our purpose.

Definition 2.1 *$C(X, Z)$ is the* set *of continuous functions from X to Z. $C_p(X, Z)$ is $C(X, Z)$ given the topology of pointwise convergence (that is, regarded as a subspace of Z^X, with the usual product topology). $C(X) = C(X, \mathbb{R})$ and $C_p(X) = C_p(X, \mathbb{R})$.*

Proposition 2.2 *Y is Eberlein compact iff Y is compact and is homeomorphic to a subspace of $C_p(X)$ for some compact Hausdorff X.*

The equivalence of this with several other standard definitions of the Eberlein compacta is discussed in Arkhangel'skii [3]; in particular, the equivalence with our Definition 1.2 is the Amir-Lindenstrauss Theorem. Gul'ko [13] (or see [3].IV.4.12) gave a proof of this theorem which can be generalized (Mercourakis [21]) to yield similar characterizations of the Talagrand and Gul'ko compacta. We describe these after a few preliminary remarks.

Definition 2.3 *A space Z is* K-analytic *iff Z is T_3 and is the continuous image of a Čech-complete Lindelöf space.*

We remark that X is Čech-complete Lindelöf iff X is homeomorphic to a closed subspace of a product of a Polish space and a compact space. For more on K-analytic spaces, see Rogers and Jayne [25]. In particular, see [25]§2.8 for this particular description of the K-analytic spaces.

Proposition 2.4 *Y is Talagrand compact iff Y is compact and is homeomorphic to a subspace of $C_p(X)$ for some Čech-complete Lindelöf space X.*

Proof. Following Arkhangel'skii ([4]§6) and Okunev ([23]§4), Y is Talagrand compact iff $C_p(Y)$ is K-analytic. If Y is Talagrand compact with $Z = C_p(Y)$, then we may identify Y with a subspace of $C_p(Z)$. So, assume that $Y \subseteq C_p(Z)$ and we have a Čech-complete Lindelöf space X and a continuous map f from X onto Z. Define $g : Y \to C_p(X)$ so that $g(y) = y \circ f$. Then g is 1-1 (since f is onto), and g is continuous.

Conversely, suppose that Y is a compact subspace of $C_p(X)$, where X is Čech-complete Lindelöf, and hence K-analytic. Then $C_p(Y)$ is also K-analytic (see Arkhangel'skii [3], Theorem IV.2.13), so Y is Talagrand compact. ☺

To state the appropriate generalization of the Amir-Lindenstrauss Theorem, we extend the notion of Σ_* products (Definition 1.1):

Definition 2.5 *For any set J and topological space S, $\mathcal{I}(S, J)$ is the ideal of subsets $E \subseteq S \times J$ such that $E \cap (K \times J)$ is finite whenever $K \subseteq S$ is compact. Define $\pi : S \times J \to S$ by $\pi(s, j) = s$.*

Lemma 2.6 *The following are equivalent whenever S is T_2 and first countable and $E \subseteq S \times J$:*

1. *$E \in \mathcal{I}(S, J)$.*
2. *$\pi(E)$ is closed and discrete in S and $E \cap (\{s\} \times J)$ is finite for all $s \in S$.*
3. *For all $s \in S$, there is a neighborhood W of s such that $E \cap (W \times J)$ is finite.*

Definition 2.7 *For any set J and topological space S: $\Sigma_*(S, J)$ is the set of all $y \in \mathbb{R}^{S\times J}$ such that $\mathrm{supt}_\varepsilon(y) \in \mathcal{I}(S, J)$ whenever $\varepsilon > 0$. $\Sigma_*(S, J)$ has the usual product topology. $\Sigma_*^I(S, J) = \Sigma_*(S, J) \cap [0, 1]^{S\times J}$.*

The following is immediate from Theorem 3.2 of Mercourakis [21]:

Theorem 2.8 *A compact Hausdorff Y is Talagrand compact iff for some set J, Y is homeomorphic to a subspace of $\Sigma_*(\omega^\omega, J)$.*

A similar characterization of the Gul'ko compacta (Theorem 3.3 of [21]) replaces ω^ω by an arbitrary separable metric space. Thus, every Gul'ko compactum is Corson compact, a fact due earlier to Gul'ko [13]. In Section 3 we plan to use the following easy corollary of Theorem 2.8:

Corollary 2.9 *Suppose that Y is Talagrand compact. Then Y is homeomorphic to a subspace of $\Sigma_*^I(\omega^\omega, J)$ for some J.*

Proof. Compressing $\mathbb{R}$, we may assume that $Y \subseteq \Sigma_*(\omega^\omega, J_0) \cap (-1, 1)^{\omega^\omega \times J_0}$. Let $J = J_0 \times \{0, 1\}$. Then, define a 1-1 map $f : Y \to \Sigma_*^I(\omega^\omega, J)$ so that $f(y)(s, (j, 0)) = \max(0, y(s, j))$ and $f(y)(s, (j, 1)) = \max(0, -y(s, j))$. ☺

The point of getting Y in $\Sigma_*^I(\omega^\omega, J)$ is that if we view $[0,1]^{\omega^\omega \times J}$ as a semilattice in the usual way, then $Y{\downarrow} \subseteq \Sigma_*^I(\omega^\omega, J)$, so that we get a Talagrand compactum which is also a compact semilattice.

Finally, for the nested families of compacta we have mentioned:

$$\text{Eberlein} \subsetneqq \text{Talagrand} \subsetneqq \text{Gul'ko} \subsetneqq \text{Corson}$$

the least possible weights are known for examples witnessing that the inclusions are proper:

Remark 2.10 *If $\mathcal{K}$ is a non-empty class of compacta, let $W(\mathcal{K})$ be the least weight of a member of $\mathcal{K}$. Then:*
$W(\text{Corson } \setminus \text{Gul'ko }) = \aleph_1$.
$W(\text{Gul'ko } \setminus \text{Talagrand }) = \aleph_1$.
$W(\text{Talagrand } \setminus \text{Eberlein }) = \mathfrak{b}$.

Proof. By an example of G. A. Sokolov (see [3]§IV.6), there is a Corson compactum of weight $\aleph_1$ which is not Gul'ko compact. There is also a Gul'ko compactum of weight $\aleph_1$ which is not Talagrand compact. To see this, use the example of Talagrand [27], but instead of using all well-founded trees, just choose one tree of each countable rank.

To get a Talagrand compactum of weight $\mathfrak{b}$ which is not Eberlein compact, use Talagrand's example ([26], or [3]§IV.6), but fix an unbounded family $B \subseteq \omega^\omega$, and restrict the adequate family to contain only subsets of B. To see that there is no example of smaller weight, note that if Y is compact in $\Sigma_*(\omega^\omega, J)$ and $w(Y) < \mathfrak{b}$, then there are compact $K_i \subseteq \omega^\omega$ for $i < \omega$ such that the projection of Y on $\Sigma_*(\bigcup_i K_i, J)$ is 1-1. ☺

Here, $\mathfrak{b}$ is the least size of an unbounded family in ω^ω; see van Douwen [10] or Fremlin [11] for more on such cardinals.

3 Subspaces of Self-Bohrifying Structures

In this section, we shall produce self-bohrifying structures which are compact semilattices. Observe that a compact semilattice $(Z; \wedge)$ is also a compact order (i.e., $\leq$ is closed in $Z \times Z$). Some basic results on compact orders and semilattices are contained in [12]. In any compact order, the following relation between the order and the topology is useful; it is easily proved using Nachbin's theorem ([12], VI.1.9) that a compact order is locally convex.

Lemma 3.1 *Suppose that* $(Z; \leq)$ *is a compact order,* $(D; \sqsubseteq)$ *is any directed set, and* $\langle z_d : d \in D \rangle$ *is a net in* Z. *If* $\forall d_1 d_2 \, [d_1 \sqsubseteq d_2 \Rightarrow z_{d_1} \leq z_{d_2}]$, *then* $\lim_{d \in D} z_d$ *exists in* Z *and equals* $\bigvee_{d \in D} z_d$. *If* $\forall d_1 d_2 \, [d_1 \sqsubseteq d_2 \Rightarrow z_{d_1} \geq z_{d_2}]$, *then* $\lim_{d \in D} z_d$ *exists in* Z *and equals* $\bigwedge_{d \in D} z_d$.

In particular, $\bigvee_{\alpha<\theta} z_\alpha$ exists whenever $\langle z_\alpha : \alpha < \theta \rangle$ is increasing (i.e., $\alpha \leq \beta \rightarrow z_\alpha \leq z_\beta$), and $\bigwedge_{\alpha<\theta} z_\alpha$ exists whenever $\langle z_\alpha : \alpha < \theta \rangle$ is decreasing. Note also that a compact semilattice is a complete semilattice; that is, $\bigwedge_{\alpha<\theta} z_\alpha$ exists for all sequences. The following lemma, which is a variation on Theorem 13 of Lawson [20], will let us prove that a semilattice is self-bohrifying by just looking at increasing and decreasing sequences.

Lemma 3.2 *Let* $(Z; \wedge)$ *be a compact semilattice and* L *a sub-semilattice of* Z. *Then* L *is closed in* Z *iff for all regular* θ *and all* θ*-sequences* $\mathsf{x} = \langle x_\alpha : \alpha < \theta \rangle$ *from* L, *the following two conditions are satisfied:*

1. *If* x *is increasing, then* $\bigvee_{\alpha<\theta} x_\alpha \in L$.
2. *If* x *is decreasing, then* $\bigwedge_{\alpha<\theta} x_\alpha \in L$.

Proof. One direction is trivial from Lemma 3.1. For the other direction, assume (1,2). By a standard argument, closure under decreasing $\bigwedge$s implies that if $E \subseteq L$ then $\bigwedge E \in L$ (induct on $|E|$); hence L is a complete sub-semilattice of Z. Likewise, if $E \subseteq L$ and is directed upward ($\forall x, y \in E \, \exists z \in E \, [x \leq z \; \& \; y \leq z]$), then $\bigvee E$ exists in Z and $\bigvee E \in L$. So, L is closed under arbitrary $\bigwedge$s and directed $\bigvee$s. Hence, L is closed by [12], VI.2.9. ☺

Corollary 3.3 *Let* $\mathfrak{Y} = (Y; \wedge)$ *be a compact semilattice. Suppose that* $\mathfrak{Y}_d \subseteq \mathfrak{Z}$, *where* $\mathfrak{Z}$ *is a compactification of* $\mathfrak{Y}_d$. *Then* $\mathfrak{Y} = \mathfrak{Z}$ *iff for all regular* θ *and all* θ*-sequences* $\mathsf{a} = \langle a_\alpha : \alpha < \theta \rangle$ *from* Y *the following two conditions are satisfied:*

↑. *If* a *is increasing and* $a_\alpha \nearrow b$ *in* $\mathfrak{Y}$ *and* $a_\alpha \nearrow p$ *in* $\mathfrak{Z}$, *then* $p = b$.
↓. *If* a *is decreasing and* $a_\alpha \searrow b$ *in* $\mathfrak{Y}$ *and* $a_\alpha \searrow p$ *in* $\mathfrak{Z}$, *then* $p = b$.

Proof. Apply Lemma 3.2. $\mathfrak{Z}$ is a semilattice by Lemma 1.7. ☺

Note that item ↑ could fail if $b > p$, and item ↓ could fail if $b < p$; then $p \in Z \backslash Y$.

By Corollary 3.3, if $(Y; \wedge)$ is a compact semilattice with no infinite chains, then it is self-bohrifying (since ↑ and ↓ are vacuous); this was proved in [14] by a different argument. The converse also holds [14], because every chain in $(Y;\wedge)$ is closed and discrete in $Y^{\#}$. Adding extra functions to $(Y;\wedge)$ yields a much wider class of self-bohrifying structures. Also, adding a subtraction function allows us to simplify Corollary 3.3 to Lemma 3.8 by shifting all the monotonic convergent sequences of $\mathfrak{Y}$ to sequences decreasing to 0.

Definition 3.4 *Let $\mathfrak{Y} = (Y;\wedge)$ be a compact semilattice. Suppose that $\mathfrak{Y}_d \subseteq \mathfrak{Z}$. A* bad sequence *in Y (with respect to $\mathfrak{Z}$) is a decreasing θ-sequence $\mathsf{a} = \langle a_\alpha : \alpha < \theta\rangle$ from Y, where θ is regular and $a_\alpha \searrow 0$ in $\mathfrak{Y}$ but $a_\alpha \searrow p > 0$ in $\mathfrak{Z}$. The element $a \in Y$ is* bad *iff there is a bad sequence a with $a_0 = a$.*

Note that $p \in Z\backslash Y$. If a is bad and $a \leq b$ in $\mathfrak{Y}$, then b is bad also. This "bad" notion is useful in semilattices to which we have added a subtraction:

Definition 3.5 *A* subtraction semilattice *is a structure $(A;\wedge,\dot{-},0)$ such that $(A;\wedge,0)$ is a semilattice and $\dot{-}$ is a binary function on A satisfying:*

1. $x \dot{-} 0 = x$
2. $(x \wedge z) \dot{-} y = (x \dot{-} y) \wedge (z \dot{-} y)$
3. $(y \dot{-} (x \wedge z)) \wedge (y \dot{-} z) = y \dot{-} z$
4. $y \dot{-} (y \dot{-} x) = y \wedge x$

The unit interval gives us a natural example of these structures:

Definition 3.6 *$\mathfrak{I}$ denotes the subtraction semilattice $([0,1];\wedge,\dot{-},0)$, where $x \dot{-} y = \max(x-y,0)$.*

It is easily verified that $\mathfrak{I}$ is a topological subtraction semilattice; hence, so is any substructure of any $\mathfrak{I}^J$ which contains 0 and is closed under $\wedge$ and $\dot{-}$. In particular, any $\mathfrak{Y} \subseteq \mathfrak{I}^J$ with $Y = Y{\downarrow}$ is a topological subtraction semilattice. When discussing these structures, the $\wedge$ and $\dot{-}$ will always denote coordinate-wise $\wedge$ and $\dot{-}$. In any semilattice, $x \vee y$ denotes $\mathrm{lub}(x,y)$ *when* this lub exists.

Lemma 3.7 *Suppose $(A;\wedge,\dot{-},0)$ is a subtraction semilattice. Then:*

A. $x \leq z \Rightarrow (x \dot{-} y) \leq (z \dot{-} y)$
B. $x \leq z \Rightarrow (y \dot{-} x) \geq (y \dot{-} z)$
C. $y \geq (y \dot{-} z)$
D. $y \dot{-} y = 0$

E. $[x \le y \ \& \ y \mathbin{\dot-} x = 0] \Rightarrow x = y$

F. $y \mathbin{\dot-} (y \wedge x) = y \mathbin{\dot-} x$

G. $x \ge y \Leftrightarrow y \mathbin{\dot-} x = 0$

H. *Each* $a{\downarrow}$ *is a distributive lattice.*

Proof. Apply Definition 3.5: (A) and (B) follow from (2) and (3), (C) follows from (B) and (1), and $y \mathbin{\dot-} y = y \mathbin{\dot-} (y \mathbin{\dot-} 0) = 0$ from (1) and (4). (E) follows from (4) and (1). For (F), apply (4) and then (C) to get

$$y \mathbin{\dot-} (y \wedge x) = y \mathbin{\dot-} (y \mathbin{\dot-} (y \mathbin{\dot-} x)) = y \wedge (y \mathbin{\dot-} x) = y \mathbin{\dot-} x \quad .$$

For (G): use (F), (D) for $\Rightarrow$; and (E) (replacing x by $y \wedge x$), (F) for $\Leftarrow$.

For (H), fix a, and define $\sim\! x = a \mathbin{\dot-} x$. By (4), $\sim\sim\! x = x$, so that (using (B)) $\sim$ is an order-reversing bijection of $a{\downarrow}$ onto $a{\downarrow}$. Hence $(a{\downarrow}; \wedge, \vee)$ is a lattice, with $x \vee z = \sim\!(\sim\! x \wedge \sim\! z)$.

To prove distributivity, it is sufficient (see Birkhoff [5], p. 39) to fix $c_1, c_2, d \in a{\downarrow}$, assume that $c_1 \wedge d = c_2 \wedge d = p$ and $c_1 \vee d = c_2 \vee d = q$, and prove that $c_1 = c_2$. We may also assume that $q = a$ (if not, work in the lattice $q{\downarrow}$). Then $0 = \sim\! a = \sim\!(c_2 \vee d) = (a \mathbin{\dot-} c_2) \wedge (a \mathbin{\dot-} d)$, and hence $(c_1 \mathbin{\dot-} c_2) \wedge (c_1 \mathbin{\dot-} d) = 0$ (applying (A)). But by (F), $c_1 \mathbin{\dot-} d = c_1 \mathbin{\dot-} (c_1 \wedge d) = c_1 \mathbin{\dot-} p \ge c_1 \mathbin{\dot-} c_2$ (applying (B)). Hence, $c_1 \mathbin{\dot-} c_2 = 0$, so $c_1 \le c_2$ by (G). Likewise, $c_2 \le c_1$, so $c_1 = c_2$. ☺

In this paper, we shall not pursue further the study of the variety of subtraction semilattices. All of our examples are of the form $Y = Y{\downarrow} \subseteq [0,1]^J$, for which (A–H) are obvious. The point of deriving these facts from the equations in Definition 3.5 is only to guarantee (via Lemma 1.7), that (A–H) also hold in every compactification of Y.

We remark that in (H), the proof of distributivity is patterned on the proof (see [5], p. 294) that lattice-ordered groups are distributive. Also, in any lattice-ordered abelian group, $0{\uparrow}$ is a subtraction semilattice, with $x \mathbin{\dot-} y = (x - y) \vee 0$.

Lemma 3.8 *Let* $\mathfrak{Y} = (Y; \wedge, \dot-, 0)$ *be a compact subtraction semilattice. Suppose that* $\mathfrak{Y}_d \subseteq \mathfrak{Z}$*, where* $\mathfrak{Z}$ *is a compactification of* $\mathfrak{Y}_d$*. Then* $\mathfrak{Y} = \mathfrak{Z}$ *iff there are no bad elements in* $\mathfrak{Y}$ *(with respect to* $\mathfrak{Z}$*).*

Proof. Assume that there are no bad elements.

To verify condition $\downarrow$ of Corollary 3.3, suppose $a_\alpha \searrow b$ in $\mathfrak{Y}$ and $a_\alpha \searrow p$ in $\mathfrak{Z}$; so $b \le p$. Let $c_\alpha = a_\alpha \mathbin{\dot-} b \in Y$. Then $c_\alpha \ge c_\beta$ for $\alpha \ge \beta$, by Lemma

3.7(A). In $\mathfrak{Z}$, let $q = \lim_\alpha c_\alpha = \bigwedge_\alpha c_\alpha$ (see Lemma 3.1). Then $q = p \dot{-} b$ by by continuity of $\dot{-}$. So, $c_\alpha \searrow (p \dot{-} b)$ in $\mathfrak{Z}$, and, likewise, $c_\alpha \searrow (b \dot{-} b) = 0$, in $\mathfrak{Y}$. Since c_0 is not bad, $p \dot{-} b = 0$, so $b = p$ by Lemma 3.7(E).

To verify condition $\uparrow$ of Corollary 3.3, suppose $a_\alpha \nearrow b$ in $\mathfrak{Y}$ and $a_\alpha \nearrow p$ in $\mathfrak{Z}$; so $b \geq p$. Let $c_\alpha = b \dot{-} a_\alpha \in Y$. Again, $c_\alpha \searrow (b \dot{-} b) = 0$ in $\mathfrak{Y}$, and hence $c_\alpha \searrow 0$ in $\mathfrak{Z}$. But, in $\mathfrak{Z}$, $c_\alpha \searrow (b \dot{-} p)$, so that $b \dot{-} p = 0$, which implies $b = p$, since $b \geq p$.

Lemma 3.9 *Suppose that* $\mathfrak{Y} = (Y; \wedge, \dot{-})$, *where* $Y = Y\!\downarrow$ *is closed in* $[0,1]^J$. *If* $a \in Y$ *is bad with respect to some compactification of* $\mathfrak{Y}_d$, *then* $\mathrm{supt}(a)$ *is infinite.*

Proof. Following the notation of Definition 3.4, let $\vec{a}$ be a bad θ-sequence with $a_0 = a$. Assume $F = \mathrm{supt}(a)$ is finite (so θ must be ω). By passing to a subsequence, we may assume that for all α, $\mathrm{supt}(a_\alpha) = F$ and $a_\alpha(j) > a_{\alpha+1}(j)$ for all $j \in F$. Then for each α, there is a β such that $a_\beta \leq a_\alpha \dot{-} a_{\alpha+1}$; hence $p \leq a_\alpha \dot{-} a_{\alpha+1}$. Taking the limit in Z, we have $p \leq p \dot{-} p = 0$, a contradiction.

In particular, if all elements of Y have finite support, then Y is self-bohrifying.

Corollary 3.10 $\mathfrak{I}$ *is self-bohrifying.*

Likewise, $\mathfrak{I}^n$ is self-bohrifying for each finite n. $\mathfrak{I}^\omega$ is not self-bohrifying (Proposition 1.6), but we shall now (Lemma 3.12) make the Hilbert cube self-bohrifying by adding another function to $(Y; \wedge, \dot{-})$. The same method handles Eberlein compacta in general.

Definition 3.11 *If* $y \in [0,1]^J$ *and* $L \subseteq J$, *let* $(y \restriction L)(j)$ *be* 0 *for* $j \notin L$ *and* $y(j)$ *for* $j \in L$.

Lemma 3.12 *Assume that* $Y = Y\!\downarrow$ *is closed in* $[0,1]^J$. *Then there is a continuous* $f : Y^2 \to Y$ *such that if* $\mathfrak{Y} = (Y; \wedge, \dot{-}, f)$, *then*

1. $\mathfrak{Y}$ *is nice.*
2. *If* $a \in Y$ *is bad with respect to some compactification of* $\mathfrak{Y}_d$, *then for some* $\varepsilon > 0$, $\mathrm{supt}_\varepsilon(a)$ *is infinite and* $a \restriction \mathrm{supt}_\varepsilon(a)$ *is bad.*

Proof. Assume $|Y| > 1$ (otherwise the result is trivial), so we may assume that J contains the element 0 and that $y(0) > 0$ for some $y \in Y$. Rescaling coordinate 0, we may assume that $\sup\{y(0) : y \in Y\} = \max\{y(0) : y \in Y\} = 1$, and then let $c \in Y$ be the element such that $c(0) = 1$ and $c(j) = 0$ whenever $j \neq 0$. Define $f(y,z)(j) = y(j) \dot{-} z(0)$.

For (1): Let $\mathfrak{N} = ([0,1]^2; \wedge, \dot{-}, f_{\mathfrak{N}})$, where $2 = \{0,1\}$ and $f_{\mathfrak{N}}$ is defined as was f. For $j \neq 0$, define $\varphi_j(y) = (y(0), y(j))$. Then the φ_j are homomorphisms and separate points of Y.

For (2): Note that $f(y, \varepsilon c)(j) = y(j) \dot{-} \varepsilon$, so $\mathrm{supt}_\varepsilon(y) = \mathrm{supt}(f(y, \varepsilon c))$. Now, assume that a is bad with respect to some compactification $\widetilde{\mathfrak{Y}} = (\widetilde{Y}; \wedge, \dot{-}, f)$ of $\mathfrak{Y}$. We shall show that $f(a, \varepsilon c)$ is bad for some $\varepsilon > 0$, so we are done by Lemma 3.9, since $(a \restriction \mathrm{supt}_\varepsilon(a)) \geq f(a, \varepsilon c)$. Let $\mathsf{a} = \langle a_\alpha : \alpha < \theta \rangle$ be a bad sequence from Y, with $a_0 = a$ and $a_\alpha \searrow 0$ in $\mathfrak{Y}$ but $a_\alpha \searrow p > 0$ in $\widetilde{\mathfrak{Y}}$.

For each (fixed) $\varepsilon \in [0,1]$, we have $f(a_\alpha, \varepsilon c) \searrow f(p, \varepsilon c)$ in $\widetilde{\mathfrak{Y}}$. To see this, note that the sequence $\langle f(a_\alpha, \varepsilon c) : \alpha < \theta \rangle$ is decreasing because $\forall x, y, z[f(x,z) \wedge f(y,z) = f(x \wedge y, z)]$ is true in $\mathfrak{Y}$ and hence in $\widetilde{\mathfrak{Y}}$. Then, by continuity, the limit of the sequence must be $f(p, \varepsilon c)$.

If $f(a, \varepsilon c)$ is not bad, then $f(p, \varepsilon c) = 0$. Now, c is not bad (by Lemma 3.9), so $\frac{1}{n}c \to 0$ in $\widetilde{\mathfrak{Y}}$ as $n \to \infty$. But $f(p, 0) = p \neq 0$, since $\forall x[f(x,0) = x]$ is true in $\mathfrak{Y}$ and hence in $\widetilde{\mathfrak{Y}}$. Hence, there must be an n such that $f(p, \frac{1}{n}c) \neq 0$, so that $f(a, \frac{1}{n}c)$ is bad. ☺

Theorem 3.13 *For Z compact Hausdorff, the following are equivalent:*

1. *Z is Eberlein compact.*
2. *Z is closed in some nice self-bohrifying compact structure for a finite language.*
3. *Z is closed in some nice self-bohrifying compact structure for a countable language.*
4. *For some nice structure $\mathfrak{A}$ for a countable language, Z is homeomorphic to a closed subset of $\mathfrak{A}^\#$.*

Proof. (1) → (2) is by Lemmas 1.8 and 3.12. For (4) → (1): The whole $\mathfrak{A}^\#$ is contained in C_p of some compactum (see §2.10 of [14]), so Z is Eberlein compact by Proposition 2.2. ☺

We now proceed to embed every Talagrand compactum into a self-bohrifying structure. This is accomplished by a continued study of bad elements.

Lemma 3.14 *Let $\mathfrak{Y} = (Y; \wedge, \dot{-})$ be a compact subtraction semilattice and let $\widetilde{\mathfrak{Y}} \supseteq \mathfrak{Y}_d$ be a compactification of $\mathfrak{Y}_d$. Fix $a \in Y$. Then a is bad (with respect to $\widetilde{\mathfrak{Y}}$) iff $a\!\downarrow$ (computed in $\widetilde{\mathfrak{Y}}$) is not a subset of Y.*

Proof. Let $\mathfrak{X}, \widetilde{\mathfrak{X}}$ be the substructures $a\!\downarrow$ computed in $\mathfrak{Y}, \widetilde{\mathfrak{Y}}$ respectively. For the non-trivial direction, assume that $\widetilde{X} \neq X$. Observe that $\widetilde{\mathfrak{X}}$ is a compactification of $\mathfrak{X}$, since if $q \in \widetilde{X}$, then q is a limit of a net, $\langle y_d : d \in D \rangle$ from Y, but then $q = q \wedge a$ is also a limit of the net $\langle y_d \wedge a : d \in D \rangle$ from X. Hence, by Lemma 3.8 applied to $\mathfrak{X}, \widetilde{\mathfrak{X}}$, there is bad sequence in $\mathfrak{X}$, which yields a bad sequence below a in $\mathfrak{Y}$. ☺

Lemma 3.15 *In the notation of Lemma 3.14, assume that a is bad (with respect to $\widetilde{\mathfrak{Y}}$) and that $a = a_1 \vee a_2$ in $\mathfrak{Y}$. Then at least one of a_1, a_2 is bad.*

Proof. Let $\mathfrak{X}, \widetilde{\mathfrak{X}}$ be exactly as in the proof of Lemma 3.14. Fix $q \in \widetilde{X} \backslash X$. Then, applying distributivity (see Lemma 3.7), $q = q \wedge a = q \wedge (a_1 \vee a_2) = (q \wedge a_1) \vee (q \wedge a_2)$. Since $q \notin X$, at least one of $(q \wedge a_1), (q \wedge a_2)$ fails to be in X, so by Lemma 3.14, at least one of a_1, a_2 is bad. ☺

The version of this lemma for infinite joins is false, but adding some functions gives a structure where it *is* true:

Lemma 3.16 *Suppose that $Y = Y\!\downarrow$ is closed in $[0,1]^J$, and J is partitioned into $\{J_n : n \in \omega\}$. Then there are continuous $f, g : Y^2 \to Y$ such that $g(y,z) \leq y$ for all y, z, and such that $\mathfrak{Y} = (Y; \wedge, \dot{-}, f, g)$ has the property that whenever $a \in Y$ is bad with respect to a compactification $\widetilde{\mathfrak{Y}}$ of $\mathfrak{Y}_d$, then there is an $n \in \omega$ such that $a \restriction J_n$ is bad with respect to $\widetilde{\mathfrak{Y}}$.*

Proof. We may fix $c \in Y$ with $\mathrm{supt}(c)$ infinite. If there is no such c, then the lemma is trivial by Lemma 3.9. We may now assume that $\omega \subseteq J$ and that $\mathrm{supt}(c) = \omega$. Also, since $Y = Y\!\downarrow$, we may assume that $c(n) \searrow 0$ as $n \to \infty$. Let f be any function which guarantees that whenever $y \in Y$ is bad with respect to some compactification of $\mathfrak{Y}_d$, then for some $\varepsilon > 0$, $\mathrm{supt}_\varepsilon(y)$ is infinite (see Lemma 3.12). In particular, we know that c is not bad, so that, by Lemma 3.14, $c\!\downarrow$ is the same in $\mathfrak{Y}$ and $\widetilde{\mathfrak{Y}}$. Thus, if $c_n = c \restriction \{0, 1, \ldots, n\}$, then in $\widetilde{\mathfrak{Y}}$ as well as in $\mathfrak{Y}$, $c_n \to c$ as $n \to \infty$.

Now, define $g(y,z)(j) = y(j) \cdot (z(n) \wedge c(n))/c(n)$ for $j \in J_n$. Then $g(y,c) = y$, and $g(y, c_n) = y \restriction (J_0 \cup J_1 \cup \cdots \cup J_n)$. By Lemma 3.15, it is sufficient to assume that a is bad and prove that some $g(a, c_n)$ is bad.

Let $a_\alpha \searrow 0$ in Y, with $a_0 = a$, and $a_\alpha \searrow p > 0$ in $\widetilde{Y}$. Then (as in the proof of Lemma 3.12), $g(a_\alpha, c_n) \searrow g(p, c_n)$ in $\widetilde{Y}$. If no $g(a, c_n)$ is bad, then each $g(p, c_n) = 0$. But then, letting $n \to \infty$, we have $p = g(p, c) = 0$, a contradiction. ☺

Lemma 3.17 *If Y is Talagrand compact, then there is a compact self-bohrifying structure $\mathfrak{Y}$ for a finite language such that X is a subspace of Y.*

Proof. By Corollary 2.9, we may assume that $X \subseteq Y \subseteq \Sigma^I_*(\omega^\omega, J) \subseteq [0,1]^{\omega^\omega \times J}$, where Y is compact and $Y = Y{\downarrow}$. For each $i \in \omega$, partition ω^ω into $\{S^i_n : n \in \omega\}$, where $S^i_n = \{s : s(i) = n\}$.

We first get $\mathfrak{Y} = (Y; \wedge, \dot{-}, f, g_0, g_1, g_2, \ldots)$ with a countable language. Applying Lemma 3.12, let f have the property that whenever $y \in Y$ is bad, there is an $\varepsilon > 0$ such that $\mathrm{supt}_\varepsilon(a)$ is infinite and $y \restriction \mathrm{supt}_\varepsilon(a)$ is also bad. Applying Lemma 3.16, let g_i guarantee that whenever y is bad, some $y \restriction S^i_n$ is bad (it is clear from the proof that the same f works with each g_i). Now, if $\mathfrak{Y}$ is not self-bohrifying, we can fix a bad $a \in Y$ (with respect to $\mathsf{b}\mathfrak{Y}$), and then fix $\varepsilon > 0$ so that $b = a \restriction \mathrm{supt}_\varepsilon(a)$ is bad also. Note that $\mathrm{supt}(b) = \mathrm{supt}_\varepsilon(b)$, so by Lemma 2.6, $\mathrm{supt}(b) \cap (\{s\} \times J)$ is finite for every $s \in \omega^\omega$, and $\pi(\mathrm{supt}(b))$ is closed and discrete in ω^ω. Now, inductively choose $b_i \in Y$ with $b_0 = b$ and each $b_{i+1} = b_i \restriction (S^i_{n_i} \times J)$, where n_i is chosen so that b_{i+1} is bad. We now have defined an $s \in \omega^\omega$, where $s(i) = n_i$ for each i. Since $\pi(\mathrm{supt}(b_{i+1}))$ is infinite (by Lemma 3.9), this s is a common accumulation point of all the $\pi(\mathrm{supt}(b_{i+1}))$, which is impossible since $\pi(\mathrm{supt}(b))$ is closed and discrete.

Finally, to get a finite language, we show how to code all the g_i by one function. By Lemma 3.16, we can have $g_i(y, z) \le y$ for all y, so that $g_i(y, z) = y \dot{-} (y \dot{-} g_i(y, z))$. Fix $d \in Y$ with $\mathrm{supt}(d) = \{u_n : n \in \omega\} \subseteq \omega^\omega \times J$, such that each $d(u_n) = 1/r_n$, where $r_0 < r_1 < \cdots$ are all positive integers and $\sum_n d(u_n) \le 1$. Let $h(y, z, x) = \sum_n (x(u_n) \wedge d(u_n))(y \dot{-} g_n(y, z))$. Then the structure $(Y; \wedge, \dot{-}, f, h)$ is self-bohrifying. To see this, we show that each g_i is defined by a finite composition of the functions h and $\dot{-}$. Let $\mathrm{supt}(d_i) = \{u_i\}$ and $d_i(u_i) = 1/r_i$. Then $h(y, z, d_i) = (y \dot{-} g_i(y, z))/r_i$, so that $g_i(y, z) = y\ (\dot{-})^{r_i}\ h(y, z, d_i)$, where $y\ (\dot{-})^0\ x = y$ and $y\ (\dot{-})^{\ell+1}\ x = (y\ (\dot{-})^\ell\ x) \dot{-} x$. ☺

Lemma 3.18 *If $\mathfrak{A}$ is any structure for a countable language, and Z is a compact subset of $\mathfrak{A}^\#$, then Z is Talagrand compact.*

Proof. Following the notation of §2.10 of [14], let $P = P_{\mathcal{L}} = \prod_{s \in \mathcal{L}} F_s$ be the space of all compact $\mathcal{L}$-structures whose domain is the Hilbert cube, Q.

Here $F_s = C(Q^n, Q)$ (with the uniform metric) whenever $s \in \mathcal{L}$ is a function symbol of arity $n \geq 1$, and $F_s = Q$ whenever s is a constant symbol. So, P is a Polish space. Then, $Homq(\mathfrak{A})$, the set of all $(p, \varphi) \in P \times Q^A$ such that φ is a homomorphism into the structure p, is closed in $P \times Q^A$, and hence is a Čech-complete Lindelöf space. Now, define $\Phi : A \to Q^{Homq(\mathfrak{A})}$ by $\Phi(a)(p, \varphi) = \varphi(a)$. Then the Bohr compactification of $\mathfrak{A}$ is $(\mathsf{b}\mathfrak{A}, \Phi)$, where $\mathsf{b}\mathfrak{A} = \mathrm{cl}(\mathrm{ran}(\Phi)) \subseteq Q^{Homq(\mathfrak{A})}$. Elements of $\mathsf{b}\mathfrak{A}$ will not in general be continuous functions from $Homq(\mathfrak{A})$ into Q, but $\mathrm{ran}(\Phi) \subseteq C_p(Homq(\mathfrak{A}), Q) \hookrightarrow C_p(H, [0,1])$, where $H = Homq(\mathfrak{A}) \times (\omega + 1)$ is also Čech-complete Lindelöf. Thus, $Z \subseteq \mathfrak{A}^\# \hookrightarrow C_p(H)$ (see [14], Lemma 2.10.4), so Z is Talagrand compact by Proposition 2.4. ☺

The last two lemmas yield:

Theorem 3.19 *For Z compact Hausdorff, the following are equivalent:*

1. *Z is Talagrand compact.*
2. *Z is closed in some self-bohrifying compact structure for a finite language.*
3. *Z is closed in some self-bohrifying compact structure for a countable language.*
4. *For some structure $\mathfrak{A}$ for a countable language, Z is homeomorphic to a closed subset of $\mathfrak{A}^\#$.*

Next, we make a few remarks on the following notion:

Definition 3.20 *A compact Hausdorff space Z is* self-bohrable *iff Z* itself *can be made into a self-bohrifying structure $\mathfrak{Z}$ with finitely many functions.*

As pointed out in the Introduction, Cook compacta show that not every metric compactum is self-bohrable. However, Eberlein compacta of the form $Y = Y{\downarrow}$ are self-bohrable by Lemma 3.12. This includes the Hilbert cube, represented as $\prod_n [0, 2^{-n}]$. Also, the Hilbert cube, along with many other metric compacta, such as compact manifolds, are self-bohrable, by Corollary 3.22 below.

Lemma 3.21 *Let $X \subseteq Y$ be compact Hausdorff, and suppose that X is a retract of Y and that there is continuous map from X onto Y. If X is self-bohrable, then so is Y.*

Proof. Let $\rho : Y \to X$ be a retraction. Let $\mathfrak{X} = (X; f_1, \ldots, f_n)$ be a self-bohrifying $\mathcal{L}$-structure. By using ρ, we can get a structure $\mathfrak{Y}$ with base set Y such that $\mathfrak{X}$ is a substructure of $\mathfrak{Y}$; that is, extend f_i to Y by defining $f_i(y_1, y_2, \ldots) = f_i(\rho(y_1), \rho(y_2), \ldots)$. Likewise, if f_0 maps X onto Y, we may extend f_0 to map Y to Y by letting $f_0(y) = f_0(\rho(y))$. Let $\mathfrak{Y}^+ = (Y; f_0, f_1, \ldots, f_n)$. Now, let $\mathfrak{Y}^+_d \subseteq \mathsf{b}\mathfrak{Y}^+_d = (\widetilde{Y}; \widetilde{f}_0, \widetilde{f}_1, \ldots, \widetilde{f}_n)$. We shall show that $\widetilde{Y} = Y$:

Let $\widetilde{X}$ be the closure of X in $\widetilde{Y}$. Then $(\widetilde{X}; \widetilde{f}_1, \ldots, \widetilde{f}_n)$ is a compactification of $\mathfrak{X}$, so that $\widetilde{X} = X$. Hence, $Y = f_0(X) = \widetilde{f}_0(\widetilde{X})$ is both closed and dense in $\widetilde{Y}$, so that $Y = \widetilde{Y}$. ☺

Corollary 3.22 *Suppose that Y is compact Hausdorff, $[0,1] \subseteq Y$, and there is a continuous map from $[0,1]$ onto Y. Then Y is self-bohrable.*

Lemma 3.21 extends easily to products:

Lemma 3.23 *Let $X_1, X_2 \subseteq Y$ be compact Hausdorff, with X_1, X_2 retracts of Y, and assume that there is continuous map from $X_1 \times X_2$ onto Y. If X_1, X_2 are self-bohrable, then so is Y.*

Corollary 3.24 *If X_1, X_2 are self-bohrable, then so is $X_1 \times X_2$.*

We conclude with an example to show that the notion of "nice" is not very nice. Simply adding a particular constant symbol to a nice group yields a self-bohrifying structure which fails to be nice.

Example 3.25 *There is a compact group X with an element $c \in X$ such that the structure $(X; \cdot, c)$ (where the language has a binary function and a constant) is not nice.*

Proof. Let $X = \mathbb{T}^{\omega_1}$, where $\mathbb{T}$ is the circle group. Fix $c \in X$ such that $\{c^n : n \in \omega\}$ is dense in X (see [18], §4). Let $\mathfrak{N} = (N; *, d)$ be any other compact structure for the same language. If $\varphi, \psi \in Hom_c(\mathfrak{X}, \mathfrak{N})$, then φ, ψ must agree on $\{c^n : n \in \omega\}$ (because they are homomorphisms); it follows by continuity that $\varphi = \psi$. Thus, $Hom_c(\mathfrak{X}, \mathfrak{N})$ is either a singleton or empty, so it cannot separate the points of X if N is second countable. ☺

4 Questions

Note that the self-bohrifying structures into which we have inserted Eberlein compacta (Theorem 3.13) and Talagrand compacta (Theorem 3.19) are somewhat artificial. We do not know if there is a "natural" class of structures which can be used in these theorems.

If Z is Eberlein compact, then there is a group G such that Z is homeomorphic to a subset of $G^{\#}$, but G cannot in general be taken to be self-bohrifying, since self-bohrifying groups are second countable. If Z is a metric compactum, then Z can be embedded in $\prod_{n=3}^{\infty} SO(n)$, which is self-bohrifying. See [15] for details.

We do not know a good criterion for a totally disconnected compact (i.e., profinite) group to be self-bohrifying, even in the following special case:

Question 4.1 (Comfort and Remus [8]) *Suppose that $\langle G_n : n \in \omega \rangle$ is a list of finite nonabelian simple groups, with no group listed infinitely often. Is $\prod_n G_n$ self-bohrifying?*

The proviso that no group be listed infinitely often is necessary by Proposition 1.6.

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