

# Transfinite Sequences of Continuous and Baire Class 1 Functions

Márton Elekes and Kenneth Kunen\*

March 11, 2002

## Abstract

The set of continuous or Baire class 1 functions defined on a metric space  $X$  is endowed with the natural pointwise partial order. We investigate how the possible lengths of well-ordered monotone sequences (with respect to this order) depend on the space  $X$ .

## Introduction

Any set  $\mathcal{F}$  of real valued functions defined on an arbitrary set  $X$  is partially ordered by the pointwise order; that is,  $f \leq g$  iff  $f(x) \leq g(x)$  for all  $x \in X$ . Then,  $f < g$  iff  $f \leq g$  and  $g \not\leq f$ ; equivalently,  $f(x) \leq g(x)$  for all  $x \in X$  and  $f(x) < g(x)$  for at least one  $x \in X$ . Our aim will be to investigate the possible lengths of the increasing or decreasing well-ordered sequences of functions in  $\mathcal{F}$  with respect to this order. A classical theorem (see Kuratowski [7], §24.III, Theorem 2') asserts that if  $\mathcal{F}$  is the set of Baire class 1 functions (that is, pointwise limits of continuous functions) defined on a Polish space  $X$  (that is, a complete separable metric space), then there exists a monotone sequence of length  $\xi$  in  $\mathcal{F}$  iff  $\xi < \omega_1$ . P. Komjáth [5] proved that the corresponding question concerning Baire class  $\alpha$  functions for  $2 \leq \alpha < \omega_1$  is independent of *ZFC*.

---

\*Partially supported by NSF Grant DMS-0097881.

2000 Mathematics Subject Classification: Primary 26A21; Secondary 03E17, 54C30.

Key words and Phrases: Baire class 1, separable metric space, transfinite sequence of functions.

In the present paper we investigate what happens if we replace the Polish space  $X$  by an arbitrary metric space.

Section 1 considers chains of continuous functions. We show that for any metric space  $X$ , there exists a chain in  $C(X, \mathbb{R})$  of order type  $\xi$  iff  $|\xi| \leq d(X)$ . Here,  $|A|$  denotes the cardinality of the set  $A$ , while  $d(X)$  denotes the density of the space  $X$ , that is

$$d(X) = \max(\min\{|D| : D \subseteq X \text{ \& } \overline{D} = X\}, \omega) \ .$$

In particular, for separable  $X$ , every well-ordered chain has countable length, just as for Polish spaces.

Section 2 considers chains of Baire class 1 functions on separable metric spaces. Here, the situation is entirely different from the case of Polish spaces, since on some separable metric spaces, there are well-ordered chains of every order type less than  $\omega_2$ . Furthermore, the existence of chains of type  $\omega_2$  and longer is independent of  $ZFC + \neg CH$ . Under  $MA$ , there are chains of all types less than  $\mathfrak{c}^+$ , whereas in the Cohen model, all chains have type less than  $\omega_2$ .

We note here that instead of examining well-ordered sequences, which is a classical problem, we could try to characterize all the possible order types of linearly ordered subsets of the partially ordered set  $\mathcal{F}$ . This problem was posed by M. Laczkovich, and is considered in detail in [3].

## 1 Sequences of Continuous Functions

**Lemma 1.1** *For any topological space  $X$ : If there is a well-ordered sequence of length  $\xi$  in  $C(X, \mathbb{R})$ , then  $\xi < d(X)^+$ .*

**Proof.** Let  $\{f_\alpha : \alpha < \xi\}$  be an increasing sequence in  $C(X, \mathbb{R})$ , and let  $D \subseteq X$  be a dense subset of  $X$  such that  $d(X) = \max(|D|, \omega)$ . By continuity, the  $f_\alpha \upharpoonright D$  are all distinct; so, for each  $\alpha < \xi$ , choose a  $d_\alpha \in D$  such that  $f_\alpha(d_\alpha) < f_{\alpha+1}(d_\alpha)$ . For each  $d \in D$  the set  $E_d = \{\alpha : d_\alpha = d\}$  is countable, because every well-ordered subset of  $\mathbb{R}$  is countable. Since  $\xi = \bigcup_{d \in D} E_d$ , we have  $|\xi| \leq \max(|D|, \omega) = d(X)$ .  $\square$

The converse implication is not true in general. For example, if  $X$  has the countable chain condition (ccc), then every well-ordered chain in  $C(X, \mathbb{R})$  is countable (because  $X \times \mathbb{R}$  is also ccc). However, the converse is true for metric spaces:

**Lemma 1.2** *If  $(X, \rho)$  is any non-empty metric space and  $\prec$  is any total order of the cardinal  $d(X)$ , then there is a chain in  $C(X, \mathbb{R})$  which is isomorphic to  $\prec$ .*

**Proof.** First, note that every countable total order is embeddable in  $\mathbb{R}$ , so if  $d(X) = \omega$ , then the result follows trivially using constant functions. In particular, we may assume that  $X$  is infinite, and then fix  $D \subseteq X$  which is dense and of size  $d(X)$ . For each  $n \in \omega$ , let  $D_n$  be a subset of  $D$  which is maximal with respect to the property  $\forall d, e \in D_n [d \neq e \rightarrow \varrho(d, e) \geq 2^{2-n}]$ . Then  $\bigcup_n D_n$  is also dense, so we may assume that  $\bigcup_n D_n = D$ . We may also assume that  $\prec$  is a total order of the set  $D$ . Now, we shall produce  $f_d \in C(X, \mathbb{R})$  for  $d \in D$  such that  $f_d < f_e$  whenever  $d \prec e$ .

For each  $n$ , if  $c \in D_n$ , define  $\varphi_c^n(x) = \max(0, 2^{-n} - \varrho(x, c))$ . For each  $d \in D$ , let  $\psi_d^n = \sum \{\varphi_c^n : c \in D_n \text{ \& } c \prec d\}$ . Since every  $x \in X$  has a neighborhood on which all but at most one of the  $\varphi_c^n$  vanish, we have  $\psi_d^n \in C(X, [0, 2^{-n}])$ , and  $\psi_d^n \leq \psi_e^n$  whenever  $d \prec e$ . Thus, if we let  $f_d = \sum_{n < \omega} \psi_d^n$ , we have  $f_d \in C(X, [0, 2])$ , and  $f_d \leq f_e$  whenever  $d \prec e$ . But also, if  $d \in D_n$  and  $d \prec e$ , then  $\psi_d^n(d) = 0 < 2^{-n} = \psi_e^n(d)$ , so actually  $f_d < f_e$  whenever  $d \prec e$ .  $\square$

Putting these lemmas together, we have:

**Theorem 1.3** *Let  $(X, \varrho)$  be a metric space. Then there exists a well-ordered sequence of length  $\xi$  in  $C(X, \mathbb{R})$  iff  $\xi < d(X)^+$ .*

**Corollary 1.4** *A metric space  $(X, \varrho)$  is separable iff every well-ordered sequence in  $C(X, \mathbb{R})$  is countable.*

## 2 Sequences of Baire Class 1 Functions

If we replace continuous functions by Baire class 1 functions, then Corollary 1.4 becomes false, since on some separable metric spaces, we can get well-ordered sequences of every type less than  $\omega_2$ . To prove this, we shall apply some basic facts about  $\subset^*$  on  $\mathcal{P}(\omega)$ . As usual, for  $x, y \subseteq \omega$ , we say that  $x \subseteq^* y$  iff  $x \setminus y$  is finite. Then  $x \subset^* y$  iff  $x \setminus y$  is finite and  $y \setminus x$  is infinite. This  $\subset^*$  partially orders  $\mathcal{P}(\omega)$ .

**Lemma 2.1** *If  $X \subset \mathcal{P}(\omega)$  is a chain in the order  $\subset^*$ , then on  $X$  (viewed as a subset of the Cantor set  $2^\omega \cong \mathcal{P}(\omega)$ ), there is a chain of Baire class 1 functions which is isomorphic to  $(X, \subset^*)$ .*

**Proof.** Note that for each  $x \in X$ ,

$$\{y \in X : y \subseteq^* x\} = \bigcup_{m \in \omega} \{y \in X : \forall n \geq m [y(n) \leq x(n)]\} ,$$

which is an  $F_\sigma$  set in  $X$ . Likewise, the sets  $\{y \in X : y \supseteq^* x\}$ ,  $\{y \in X : y \subset^* x\}$ , and  $\{y \in X : y \supset^* x\}$ , are all  $F_\sigma$  sets in  $X$ , and hence also  $G_\delta$  sets. It follows that if  $f_x : X \rightarrow \{0, 1\}$  is the characteristic function of  $\{y \in X : y \subset^* x\}$ , then  $f_x : X \rightarrow \mathbb{R}$  is a Baire class 1 function. Then,  $\{f_x : x \in X\}$  is the required chain.  $\square$

**Lemma 2.2** *For any infinite cardinal  $\kappa$ , suppose that  $(\mathcal{P}(\omega), \subset^*)$  contains a chain  $\{x_\alpha : \alpha < \kappa\}$  (i.e.,  $\alpha < \beta \rightarrow x_\alpha \subset^* x_\beta$ ). Then  $(\mathcal{P}(\omega), \subset^*)$  contains a chain  $X$  of size  $\kappa$  such that every ordinal  $\xi < \kappa^+$  is embeddable into  $X$ .*

**Proof.** Let  $S = \bigcup_{1 \leq n < \omega} \kappa^n$ . For  $s = (\alpha_1, \dots, \alpha_{n-1}, \alpha_n) \in S$ , let  $s^+ = (\alpha_1, \dots, \alpha_{n-1}, \alpha_n + 1)$ . Starting with the  $x_{(\alpha)} = x_\alpha$ , choose  $x_s \in \mathcal{P}(\omega)$  by induction on  $\text{length}(s)$  so that  $x_s = x_{s \smallfrown 0} \subset^* x_{s \smallfrown \alpha} \subset^* x_{s \smallfrown \beta} \subset^* x_{s^+}$  whenever  $s \in S$  and  $0 < \alpha < \beta < \kappa$ . Let  $X = \{x_s : s \in S\}$ . Then, whenever  $x, y \in X$  with  $x \subset^* y$ , the ordinal  $\kappa$  is embeddable in  $(x, y) = \{z \in X : x \subset^* z \subset^* y\}$ . From this, one easily proves by induction on  $\xi < \kappa^+$  (using  $\text{cf}(\xi) \leq \kappa$ ) that  $\xi$  is embeddable in each such interval  $(x, y)$ .  $\square$

Since  $\mathcal{P}(\omega)$  certainly contains a chain of type  $\omega_1$ , these two lemmas yield:

**Theorem 2.3** *There is a separable metric space  $X$  on which, for every  $\xi < \omega_2$ , there is a well-ordered chain of length  $\xi$  of Baire class 1 functions.*

Under  $CH$ , this is best possible, since there will be only  $2^\omega = \omega_1$  Baire class 1 functions on a separable metric space, so there could not be a chain of length  $\omega_2$ . Under  $\neg CH$ , the existence of longer chains of Baire class 1 functions depends on the model of set theory. It is consistent with  $\mathfrak{c} = 2^\omega$  being arbitrarily large that there is a chain in  $(\mathcal{P}(\omega), \subset^*)$  of type  $\mathfrak{c}$ ; for example, this is true under  $MA$  (see [2]). In this case, there will be a separable  $X$  with well-ordered chains of all lengths less than  $\mathfrak{c}^+$ . However, in the Cohen model, where  $\mathfrak{c}$  can also be made arbitrarily large, we never get chains of type  $\omega_2$ . We shall prove this by using the following lemma, which relates it to the rectangle problem:

**Lemma 2.4** *Suppose that there is a separable metric space  $Y$  with an  $\omega_2$ -chain of Borel subsets,  $\{B_\alpha : \alpha < \omega_2\}$  (so,  $\alpha < \beta \rightarrow B_\alpha \subsetneq B_\beta$ ). Then in  $\omega_2 \times \omega_2$ , the well-order relation  $<$  is in the  $\sigma$ -algebra generated by the set of all rectangles,  $\{S \times T : S, T \in \mathcal{P}(\omega_2)\}$ .*

**Proof.** Each  $B_\alpha$  has some countable Borel rank. Since there are only  $\omega_1$  ranks, we may, by passing to a subsequence, assume that the ranks are bounded. Say, each  $B_\alpha$  is a  $\Sigma_\mu^0$  set for some fixed  $\mu < \omega_1$ .

Let  $J = \omega^\omega$ , and let  $A \subseteq Y \times J$  be a universal  $\Sigma_\mu^0$  set; that is,  $A$  is  $\Sigma_\mu^0$  in  $Y \times J$  and every  $\Sigma_\mu^0$  subset of  $Y$  is of the form  $A^j = \{y : (y, j) \in A\}$  for some  $j \in J$  (see [7], §31). Now, for  $\alpha, \beta < \omega_2$ , fix  $y_\alpha \in B_{\alpha+1} \setminus B_\alpha$ , and fix  $j_\beta \in J$  such that  $A^{j_\beta} = B_\beta$ . Then  $\alpha < \beta$  iff  $(y_\alpha, j_\beta) \in A$ . Thus,  $\{(y_\alpha, j_\beta) : \alpha < \beta < \omega_2\}$  is a Borel subset of  $\{y_\alpha : \alpha < \omega_2\} \times \{j_\beta : \beta < \omega_2\}$ , and is hence in the  $\sigma$ -algebra generated by open rectangles, so  $<$ , as a subset of  $\omega_2 \times \omega_2$ , is in the  $\sigma$ -algebra generated by rectangles.  $\square$

**Theorem 2.5** *Assume that the well-order relation  $<$  on  $\omega_2$  is not in the  $\sigma$ -algebra generated by the set of all rectangles. Then no separable metric space can have a chain of length  $\omega_2$  of Baire class 1 functions.*

**Proof.** Suppose that  $\{f_\alpha : \alpha < \omega_2\}$  is a chain of Baire class one functions on the separable metric space  $X$ . Let  $B_\alpha = \{(x, r) \in X \times \mathbb{R} : r \leq f_\alpha(x)\}$ . Then the  $B_\alpha$  form an  $\omega_2$ -chain of Borel subsets of the separable metric space  $X \times \mathbb{R}$ , so we have a contradiction by Lemma 2.4.  $\square$

Finally, we point out that the hypothesis of this theorem is consistent, since it holds in the extension  $V[G]$  formed by adding  $\geq \omega_2$  Cohen reals to a ground model  $V$  which satisfies *CH*. This fact was first proved in [6]. It also follows from the more general principle  $HP_2(\omega_2)$  of Brendle, Fuchino, and Soukup [1]. They define this principle, prove that it holds in Cohen extensions (and in a number of other forcing extensions), and show the following:

**Lemma 2.6**  *$HP_2(\kappa)$  implies that if  $R$  is any relation on  $\mathcal{P}(\omega)$  which is first-order definable over  $H(\omega_1)$  from a fixed element of  $H(\omega_1)$ , then there is no  $X \subseteq \mathcal{P}(\omega)$  such that  $(X; R)$  is isomorphic to  $(\kappa; <)$ .*

These matters are also discussed in [4], which indicates how such statements are verified in Cohen extensions. Here,  $H(\omega_1)$  denotes the set of hereditarily countable sets.

**Lemma 2.7**  *$HP_2(\omega_2)$  implies that in  $\omega_2 \times \omega_2$ , the well-order relation  $<$  is not in the  $\sigma$ -algebra generated by the set of all rectangles,  $\{S \times T : S, T \in \mathcal{P}(\omega_2)\}$ .*

**Proof.** Suppose that  $<$  were in this  $\sigma$ -algebra. Then we would have fixed  $K_n \subseteq \omega_2$  for  $n < \omega$  such that  $<$  is in the  $\sigma$ -algebra generated by all the  $K_m \times K_n$ .

For each  $\alpha$ , let  $u_\alpha = \{n \in \omega : \alpha \in K_n\}$ . There is then a formula  $\varphi(x, y, z)$  and a fixed  $w \in H(\omega_1)$  such that for all  $\alpha, \beta < \omega_2$ ,  $\alpha < \beta$  iff  $H(\omega_1) \models \varphi(u_\alpha, u_\beta, w)$ ; here,  $w$  encodes the particular countable boolean combination used to get  $<$  from the  $K_n$ . Now, if  $X = \{u_\alpha : \alpha < \omega_2\}$ , then  $\varphi$  defines a relation  $R$  on  $H(\omega_1)$  such that  $(X; R)$  is isomorphic to  $(\omega_2; <)$ , contradicting Lemma 2.6.  $\square$

## References

- [1] J. Brendle, S. Fuchino, and L. Soukup, Coloring ordinals by reals, to appear.
- [2] E. K. van Douwen, The integers and topology, in *Handbook of Set-Theoretic Topology*, North-Holland, Amsterdam, 1984, pp. 111-167.
- [3] M. Elekes, Linearly ordered families of Baire 1 functions, *Real Analysis Exchange*, to appear.
- [4] I. Juhász and K. Kunen, The power set of  $\omega$ , *Fundamenta Mathematicae*, Vol **170** (2001), 257-265.
- [5] P. Komjáth, Ordered families of Baire-2-functions, *Real Analysis Exchange*, Vol **15** (1989-90), 442-444.
- [6] K. Kunen, *Inaccessibility Properties of Cardinals*, Doctoral Dissertation, Stanford, 1968.
- [7] K. Kuratowski, *Topology, Vol. 1*, Academic Press, 1966.

DEPARTMENT OF ANALYSIS, EÖTVÖS LORÁND UNIVERSITY, BUDAPEST,  
PÁZMÁNY PÉTER SÉTÁNY 1/C, 1117, HUNGARY

*Email address:* emarci@cs.elte.hu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON,  
WI 53706, USA

*Email address:* kunen@math.wisc.edu

*URL:* <http://www.math.wisc.edu/~kunen>