Forcing and Differentiable Functions*

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Abstract

We consider covering $\aleph_1 \times \aleph_1$ rectangles by countably many smooth curves, and differentiable isomorphisms between $\aleph_1$-dense sets of reals.

1 Introduction

In this paper, we consider two different issues, both related to the question of obtaining differentiable real–valued functions where classical results only produced functions or continuous functions.

Regarding the first issue, the text of Sierpiński [12] shows that CH is equivalent to his Proposition P_2, which is the statement that the plane "est une somme d'une infinité dénombrable de courbes". Here, a "curve" is just the graph of a function or an inverse function, so P_2 says only that $\mathbb{R}^2 = \bigcup_{i\in\omega}(f_i \cup f_i^{-1})$, where each f_i is (the graph of) a function from $\mathbb{R}$ to $\mathbb{R}$, with no assumption of continuity. The proof actually shows, in ZFC, that for every $E \in [\mathbb{R}]^{\aleph_1}$, there are $f_i : \mathbb{R} \to \mathbb{R}$ with $E^2 \subseteq \bigcup_{i\in\omega}(f_i \cup f_i^{-1})$, and that this is false for all E of size greater than $\aleph_1$.

Usually in geometry and analysis, "curve" *does* imply continuity, so it is natural to ask whether the f_i can all be continuous, or even C^∞:

Definition 1.1 *For $n \in \omega \cup \{\infty\}$, call $E \subseteq \mathbb{R}$ n–*small *iff there are C^n functions $f_i : \mathbb{R} \to \mathbb{R}$ such that $E^2 \subseteq \bigcup_{i\in\omega}(f_i \cup f_i^{-1})$. Here, C^0 just means "continuous", and C^∞ means C^n for all $n \in \omega$.*

Countable sets are trivially ∞–small, and by Sierpiński, $|E| \le \aleph_1$ for every 0–small set E, so we are only interested in sets of size $\aleph_1$. Every 0–small set is of first category and measure 0 (and perfectly meager and universally null). Just in ZFC, we shall prove the following in Section 2:

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Theorem 1.2 *There is an $E \in [\mathbb{R}]^{\aleph_1}$ which is ∞–small.*

The existence of a 0–small set is due to Kubiś and Vejnar [9].

But now we can ask whether *every* $E \in [\mathbb{R}]^{\aleph_1}$ is n–small for some n. Even when $n = 0$, this would imply that every such E is of first category and measure 0 (and perfectly meager and universally null), which is a well-known consequence of MA($\aleph_1$). In fact, the following theorem follows easily from results already in the literature, as we shall point out in Section 2:

Theorem 1.3

1. MA($\aleph_1$) *implies that every set of size $\aleph_1$ is 0–small.*
2. PFA *implies that every set of size $\aleph_1$ is 1–small.*
3. MA($\aleph_1$) *does not imply that every set of size $\aleph_1$ is 1–small.*
4. *In* ZFC, *there is an $E \in [\mathbb{R}]^{\aleph_1}$ which is not 2–small.*

We remark that Sierpiński's use of "curve" is unusual in another way: Usually, we would call a subset of $\mathbb{R}^2$ a curve iff it is a continuous image of $[0, 1]$, and not necessarily the graph of a function; but with that usage, the plane is always a countable union of curves by Peano [11].

Our second issue involves the isomorphism of $\aleph_1$-dense subsets of $\mathbb{R}$.

Definition 1.4 *$E \subseteq \mathbb{R}$ is* $\aleph_1$-dense *iff $|E \cap (x, y)| = \aleph_1$ whenever $x, y \in \mathbb{R}$ and $x < y$. $\mathcal{F}$ is the set of all order-preserving bijections from $\mathbb{R}$ onto $\mathbb{R}$.*

By Baumgartner [3, 4], PFA implies that whenever D, E are $\aleph_1$-dense, there is an $f \in \mathcal{F}$ such that $f(D) = E$. By [2, 1], this cannot be proved from MA($\aleph_1$) alone. Clearly, every $f \in \mathcal{F}$ is continuous, but we can ask whether we can always get our f to be C^n.

For $n = 2$, a ZFC counter-example is apparent from Theorems 1.2 and 1.3, since we may take D to be 2–small and E to be not 2–small, and also assume that $D = D + \mathbb{Q} = \{D + q : q \in \mathbb{Q}\}$ and $E = E + \mathbb{Q}$. Note that D is 2–small iff $D + \mathbb{Q}$ is 2–small, and the latter set is also $\aleph_1$-dense.

But in fact, even $n = 1$ is impossible, since the following holds in ZFC, as we shall show in Section 3:

Theorem 1.5 *There are $\aleph_1$-dense $D, E \subset \mathbb{R}$ such that for all $f \in \mathcal{F}$ and $\aleph_1$-dense $D^* \subseteq D$ and $E^* \subseteq E$ with $f(D^*) = E^*$: If $p < q$ and $a = f(p)$ and $b = f(q)$ then:*

1. *Either f is not uniformly Lipschitz on* (p, q) or f^{-1} *is not uniformly Lipschitz on* (a, b)*; equivalently, whenever* $0 < \Lambda \in \mathbb{R}$*, there are* $x_0, x_1 \in (p, q)$ *such that either* $|f(x_1) - f(x_0)| > \Lambda|x_1 - x_0|$ or $|x_1 - x_0| > \Lambda|f(x_1) - f(x_0)|$.
2. *Either* f' *does not exist at some* $d \in D^* \cap (p, q)$ or $(f^{-1})'$ *does not exist at some* $e \in E^* \cap (a, b)$.
3. *If* $f'(d)$ *exists for all* $d \in D^* \cap (p, q)$*, then* $f'(d) = 0$ *for all but countably many* $d \in D^* \cap (p, q)$.

In particular, f cannot be in $C^1(\mathbb{R})$, since f' cannot be 0 everywhere, so if f' were continuous, there would be an interval on which $f' > 0$, contradicting (3).

On the other hand, f' can exist everywhere and be 0 on a dense set if f' is not required to be continuous:

Theorem 1.6 *Assume* PFA*, and let* $D, E \subset \mathbb{R}$ *be* $\aleph_1$*-dense. Then there exist* $f \in \mathcal{F}$ *and* $D^* \subseteq D$ *such that* D^* *is* $\aleph_1$*-dense and* $f(D^*) = E$ *and*

1. *For all* $x \in \mathbb{R}$*,* $f'(x)$ *exists and* $0 \le f'(x) \le 2$.
2. $f'(d) = 0$ *for all* $d \in D^*$.

By (1), f satisfies a uniform Lipschitz condition with Lipschitz constant 2. The "2" is an artifact of the proof, and may be replaced by an arbitrarily small number; if $\varepsilon > 0$, we can get our f with $f'(x) \le 2$ so that $f(D^*) = (1/\varepsilon)E$; then $\varepsilon f'(x) \le 2\varepsilon$ and $\varepsilon f(D^*) = E$. In (2), the $f'(d) = 0$ is to be expected, in view of Theorem 1.5(3). We do not know whether we can make D^* equal D.

The proof of Theorem 1.6 in Sections 4 and 5 actually shows that one can force the result to hold in an appropriate ccc extension of any model of ZFC + $2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} = \aleph_2$. Then the result follows from PFA using the same forcing plus the "collapsing the continuum" trick.

We remark that Theorem 1.6 contradicts Proposition 9.4 in the paper [1] of Abraham, Rubin, and Shelah, which produces a ZFC example of $\aleph_1$-dense $D, E \subset \mathbb{R}$ such that every $f \in \mathcal{F}$ with $f \cap (D \times E)$ uncountable fails to be differentiable at uncountably many elements of D. Their "proof" uses ideas similar to our proof of Theorem 1.5, but insufficient details are given to be able to locate a specific error. Burke [5] also noticed a problem with this result from [1] and gave a correct proof of a result similar to our Theorem 1.5; see Proposition 1.2 and Remark 1.4 of his paper.

2 On Smallness

We first point out that Theorem 1.3 follows easily from known results:

Proof of Theorem 1.3. For (1), fix $E \in [\mathbb{R}]^{\aleph_1}$. By Sierpiński, $E^2 \subseteq \bigcup_{i\in\omega}(f_i \cup f_i^{-1})$, where each f_i is the graph of a function and $|f_i| = \aleph_1$. Then, assuming MA($\aleph_1$), a standard forcing shows that for each i, there are Cantor sets $P_{i,n}$ for $n \in \omega$ with each $P_{i,n}$ the graph of a function and $f_i \subseteq \bigcup_n P_{i,n}$. Now each $P_{i,n}$ extends to a function $g_{i,n} \in C(\mathbb{R}, \mathbb{R})$, so that $E^2 \subseteq \bigcup_{i,n}(g_{i,n} \cup g_{i,n}^{-1})$.

For (2), use the fact from [6] that under PFA, every $A \in [\mathbb{R}^2]^{\aleph_1}$ is a subset of a countable union of C^1 arcs. Now apply this with $A = E \times E$, and note that every C^1 arc is contained in a finite union of (graphs of) C^1 functions and inverse functions.

(4) also follows from [6], which shows in ZFC that there is an $A \in [\mathbb{R}^2]^{\aleph_1}$ which is not a subset of a countable union of C^2 arcs. So, choose E such that $A \subseteq E \times E$.

Likewise, (3) follows from [10], which shows that it is consistent with MA($\aleph_1$) to have an $A \in [\mathbb{R}^2]^{\aleph_1}$ which is a weakly Luzin set; and such a set is not a subset of a countable union of C^1 arcs. ☕

Next, to prove Theorem 1.2, we first state an abstract version of the argument involved:

Lemma 2.1 *Suppose that T is an uncountable set with functions f_i on T for $i \in \omega$ such that for all countable $Q \subset T$, there is an $x \in T$ such that $Q \subseteq \{f_i(x) : i \in \omega\}$. Then there is an $E \subseteq T$ of size $\aleph_1$ such that $E \times E \subseteq \Delta \cup \bigcup_i(f_i \cup f_i^{-1})$, where Δ is the identity function.*

Proof. Note, by considering supersets of Q, that there must be uncountably many such x. Now, let $E = \{e_\alpha : \alpha < \omega_1\}$ where e_α is chosen recursively so that $e_\alpha \notin \{e_\xi : \xi < \alpha\} \subseteq \{f_i(e_\alpha) : i \in \omega\}$. ☕

To illustrate the idea of our argument, we first produce an $E \in [\mathbb{R}]^{\aleph_1}$ which is 0–small, in which case T can be any Cantor set.

Lemma 2.2 *There are $f_i \in C(2^\omega, 2^\omega)$ for $i < \omega$ such that for all countable non-empty $Q \subseteq 2^\omega$, there is an $x \in 2^\omega$ such that $Q = \{f_i(x) : i < \omega\}$.*

Proof. Let φ map $\omega \times \omega$ 1-1 into ω, and let $(f_i(x))(j) = x(\varphi(i, j))$. Now, let $Q = \{y_i : i \in \omega\}$. Since φ is 1-1, we may choose $x \in 2^\omega$ such that $x(\varphi(i, j)) = y_i(j)$ for all i, j; then $f_i(x) = y_i$. ☕

So, if $T \subseteq \mathbb{R}$ is a Cantor set, then $T \cong 2^\omega$, and the existence of an $E \in [T]^{\aleph_1}$ which is 0–small follows from Lemmas 2.1 and 2.2, and the observation that every function in $C(T, T)$ extends to a function in $C(\mathbb{R}, \mathbb{R})$.

Now, if we want our functions to be smooth, as required by Theorem 1.2, we must be a bit more careful. The f_i will be defined on the standard middle-third Cantor set H, but they will only satisfy the hypothesis of Lemma 2.1 on a thin subset $T \subset H$.

To simplify notation, H will be a subset of $[0,3]$ rather than $[0,1]$. For $x \in [0,3]$, $x \in H$ iff x has only 0s and 2s in its ternary expansion, so that $x = \sum_{n\in\omega} x(n)3^{-n}$, where each $x(n) \in \{0,2\}$, and we write x in ternary as $x(0).x(1)x(2)x(3)x(4)\cdots$. If $x, y \in H$ with $x \neq y$, let $\delta(x,y)$ be the least n such that $x(n) \neq y(n)$, and note that $3^{-n} \le |x-y| \le 3^{-n+1}$.

Fix any $\Gamma : \omega \to \omega$ such that $\Gamma(0) = 0$, Γ is strictly increasing, and $\Gamma(k+1) \ge (\Gamma(k))^2$ for each k. The minimum such Γ is the sequence $0, 1, 2, 4, 16, 256, \ldots$, but any other such Γ will do.

We view x in H as coding an ω–sequence of *blocks*, where the k^{th} block is a sequence of length $\Gamma(k+1)-\Gamma(k)$. Note that $\Gamma(k+2)-\Gamma(k+1) \ge \Gamma(k+1)-\Gamma(k)$ for each k, so the blocks get longer as $k \nearrow$.

More formally, for $x \in H$ and $k \ge 0$, we define $B^x_k : \omega \to \{0,2\}$ so that $B^x_k(j) = x(\Gamma(k)+j)$ when $j < \Gamma(k+1)-\Gamma(k)$ and $B^x_k(j) = 0$ for $j \ge \Gamma(k+1)-\Gamma(k)$. Note that x is determined by $\langle B^x_k : k \in \omega\rangle$. Let $B^x_k(j) = 0$ when $k < 0$.

Now, we wish each $x \in H$ to encode a sequence of ω elements of H, $\langle f_i(x) : i \in \omega\rangle$. We do this using a bijection φ from $\omega \times \omega$ onto ω. We assume that $\max(i,j) < \max(i',j') \to \varphi(i,j) < \varphi(i',j')$ for all i, j, i', j', which implies that $\max(i,j)^2 \le \varphi(i,j) < (\max(i,j)+1)^2$.

In the "standard" encoding, as in the proof of Lemma 2.2, an $x \in \{0,2\}^\omega$ encodes ω elements of $\{0,2\}^\omega$, where the i^{th} element is $j \mapsto x(\varphi(i,j))$. But here, for $x \in K$, we apply this separately to each of the ω blocks of x, *and* we shift right two places to ensure that the functions are smooth. Define $f_i : H \to H$ so that for $x \in H$, $f_i(x)$ is the $z \in H$ such that $B^z_k(j) = B^x_{k-2}(\varphi(i,j))$ for all j; so $B^z_k(j) = 0$ when $k < 2$. There is such a z because

$$j \ge \Gamma(k+1)-\Gamma(k) \;\Rightarrow\; \varphi(i,j) \ge j \ge \Gamma(k-1)-\Gamma(k-2) \;\Rightarrow\; B^z_k(j) = 0 \quad .$$

Let $S = \{0,2\}^{<\omega}$. For $i \in \omega$ and $s \in S$, define $f^s_i : H \to H$ so that for $x \in H$, $f^s_i(x)$ is the $z \in H$ such that $z(n)$ is $s(n)$ for $n < \text{lh}(s)$ and $f_i(x)(n)$ for $n \ge \text{lh}(s)$.

Note that most elements of H are not in $\bigcup\{f^s_i(H) : i \in \omega \;\&\; s \in S\}$, but the T of Lemma 2.1 will be a proper subset of H.

First, we verify that we get C^∞ functions. Following [6], call $f : H \to H$ *flat* iff for all $q \in \omega$, there is a bound M_q such that for all $u, t \in H$, $|f(u)-f(t)| \le M_q|u-t|^q$. By Lemma 6.4 of [6], this implies that f can be extended to a C^∞ function defined on all of $\mathbb{R}$, all of whose derivatives vanish on H.

Lemma 2.3 *Each f^s_i is flat.*

Proof. Fix x, y in H with $x \neq y$. Let $n = \delta(x, y)$. Fix $k \in \omega$ so that $\Gamma(k) \leq n < \Gamma(k+1)$. Assume that $f_i^s(x) \neq f_i^s(y)$. Then $\Gamma(k+2) \leq \delta(f_i^s(x), f_i^s(y))$. Now $|x - y| \geq 3^{-n} \geq 3^{-\Gamma(k+1)}$, and $|f_i^s(x) - f_i^s(y)| \leq 3^{-\Gamma(k+2)+1}$, so

$$|f_i^s(x) - f_i^s(y)|/|x-y|^q \leq 3^{-\Gamma(k+2)+1+q\Gamma(k+1)} \leq \quad 3^{-(\Gamma(k+1))^2+1+q\Gamma(k+1)} \quad ,$$

which is bounded, and in fact goes to 0 as $k \nearrow \infty$.

Now, we define $T \subset H$: For $x \in H$ and $k \in \omega$, let ℓ_k^x be the least $\ell \in \omega$ such that $\forall j \geq \ell\, [B_k^x(j) = 0]$. So, $\ell_k^x \leq \Gamma(k+1) - \Gamma(k)$.

Call $\psi : \omega \to \omega$ *tiny* iff $\lim_{k\to\infty}(\psi(k)^n)/k = 0$ for all $n \in \omega$. Note that tininess is preserved by powers and shifts. That is, if ψ is tiny, then so is $k \mapsto \psi(k)^r$ and $k \mapsto r + \psi(k+r)$ for each $r > 0$.

Proof of Theorem 1.2. Let T be the set of $x \in H$ such that $k \mapsto \ell_k^x$ is tiny. Then T is an uncountable Borel set, and we are done by Lemma 2.4:

Lemma 2.4 *If $y_i \in T$ for $i \in \omega$, then there is an $x \in T$ and $s_i \in S$ for $i \in \omega$ such that $f_i^{s_i}(x) = y_i$ for all i.*

Proof. Fix any $\psi : \omega \to \omega$ such that $\psi(k) \leq \Gamma(k+1) - \Gamma(k)$ for all k. Then we can define $x \in H$ so that $B_k^x(\varphi(i,j)) = B_{k+2}^{y_i}(j)$ whenever $\varphi(i,j) < \psi(k)$; let $B_x^k(m) = 0$ for $m \geq \psi(k)$. Then $x \in T$ provided that ψ is tiny.

For each i, the function $k \mapsto (i + \ell_{k+2}^{y_i})^2$ is tiny. Now, fix a tiny ψ such that $\psi(k) \leq \Gamma(k+1) - \Gamma(k)$ for all $k \in \omega$ and $\psi \geq^* (k \mapsto (i + \ell_{k+2}^{y_i})^2)$ for each i; this is possible by a standard diagonal argument.

Now fix i. Then fix $r \in \omega$ such that $\psi(k) \geq (i + \ell_{k+2}^{y_i})^2$ for all $k \geq r$. Let $s_i = y_i \restriction \Gamma(r+2)$. Let $z = f_i^{s_i}(x)$. We shall show that $z = y_i$. So, fix $n \in \omega$, and we show that $z(n) = y_i(n)$. This is obvious if $n < \Gamma(r+2)$, so assume that $n \geq \Gamma(r+2)$. Then fix $k \geq r+2$ and $j < \Gamma(k+1) - \Gamma(k)$ with $n = \Gamma(k) + j$. We must show that $B_k^z(j) = B_k^{y_i}(j)$.

By definition of $f_i^{s_i}$, $B_k^z(j) = B_{k-2}^x(\varphi(i,j))$, whereas we only know that $B_k^{y_i}(j) = B_{k-2}^x(\varphi(i,j))$ when $\varphi(i,j) < \psi(k-2)$. So, assume that $\varphi(i,j) \geq \psi(k-2)$; we show that $B_k^z(j) = 0$ and $B_k^{y_i}(j) = 0$.

Now $B_k^{y_i}(j) = 0$ because otherwise $j < \ell_k^{y_i}$, and then $\varphi(i,j) \leq (i+j)^2 < (i + \ell_k^{y_i})^2 \leq \psi(k-2)$, a contradiction.

Also, $B_k^z(j) = B_{k-2}^x(\varphi(i,j)) = 0$ by the definition of x, since $\varphi(i,j) \geq \psi(k-2)$.

3 Non-Isomorphisms

Here we prove Theorem 1.5. First,

Lemma 3.1 *There are Cantor sets $H, K \subset \mathbb{R}$ such that*

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x_0, x_1 \in H \, \forall y_0, y_1 \in K$$
$$\big[0 < |x_1 - x_0| < \delta \ \wedge \ 0 < |y_1 - y_0| < \delta \ \longrightarrow$$
$$(y_1 - y_0)/(x_1 - x_0) \in (-\varepsilon, \varepsilon) \cup (1/\varepsilon, \infty) \cup (-\infty, -1/\varepsilon)\big] \ .$$

Proof. We obtain H, K by the usual trees of closed intervals:

1. $H = \bigcap_{n\in\omega} \bigcup \{I_\sigma : \sigma \in {}^n 2\}$ and $K = \bigcap_{n\in\omega} \bigcup \{J_\tau : \tau \in {}^n 2\}$.
2. $I_\sigma = [a_\sigma, b_\sigma]$ and $J_\tau = [c_\tau, d_\tau]$.
3. $a_\sigma = a_{\sigma \frown 0} < b_{\sigma \frown 0} < a_{\sigma \frown 1} < b_{\sigma \frown 1} = b_\sigma$.
4. $c_\tau = c_{\tau \frown 0} < d_{\tau \frown 0} < c_{\tau \frown 1} < d_{\tau \frown 1} = d_\tau$
5. Whenever $\mathrm{lh}(\sigma) = \mathrm{lh}(\tau) = n$: $b_\sigma - a_\sigma = p_n$ and $b_\tau - a_\tau = q_n$.

Informally, assume that $\mathrm{lh}(\sigma) = \mathrm{lh}(\tau) = n$. Then $I_\sigma \times J_\tau$ is a box of dimensions $p_n \times q_n$. It will be very long and skinny ($p_n \gg q_n$). Inside this box will be four little boxes, of dimensions $p_{n+1} \times q_{n+1}$, situated at the corners of the $p_n \times q_n$ box. These little ones are much smaller; that is, $p_n \gg q_n \gg p_{n+1} \gg q_{n+1}$. Now suppose that the two points (x_0, y_0) and (x_1, y_1) both lie in $I_\sigma \times J_\tau$, but lie in different smaller boxes $I_{\sigma\frown\mu} \times J_{\tau\frown\nu}$. So, there are $\binom{4}{2} = 6$ possibilities. For two of them, between $I_{\sigma\frown\mu} \times J_{\tau\frown 0}$ and $I_{\sigma\frown\mu} \times J_{\tau\frown 1}$ ($\mu \in \{0, 1\}$), the slope $|\Delta y/\Delta x|$ is very large. For the other four, between $I_{\sigma\frown 0} \times J_{\tau\frown\nu}$ and $I_{\sigma\frown 1} \times J_{\tau\frown\nu}$ ($\nu \in \{0, 1\}$), or between $I_{\sigma\frown 0} \times J_{\tau\frown 0}$ and $I_{\sigma\frown 1} \times J_{\tau\frown 1}$ or between $I_{\sigma\frown 0} \times J_{\tau\frown 1}$ and $I_{\sigma\frown 1} \times J_{\tau\frown 0}$, $|\Delta y/\Delta x|$ is very small.

More formally, assume that $p_0 > q_0 > p_1 > q_1 > \cdots$ and $q_n/p_n \to 0$ and $p_{n+1}/q_n \to 0$ as $n \to \infty$. Fix (x_0, y_0) and (x_1, y_1) in $H \times K$, and then fix n such that for some $\sigma, \tau \in {}^n 2$, $(x_0, y_0), (x_1, y_1) \in I_\sigma \times J_\tau$, but $(x_0, y_0), (x_1, y_1)$ are in two different smaller boxes $I_{\sigma\frown\mu} \times J_{\tau\frown\nu}$. Note that this $n \to \infty$ as $\delta \to 0$. In the two large slope cases, $|\Delta y/\Delta x| \geq (q_n - 2q_{n+1})/p_{n+1} \to \infty$ as $n \to \infty$, since $q_n/p_{n+1} \to \infty$ and $q_{n+1}/p_{n+1} \to 0$. In the four small slope cases, $|\Delta y/\Delta x| \leq q_n/(p_n - 2p_{n+1}) \to 0$, since $p_n/q_n \to \infty$ and $p_{n+1}/q_n \to 0$.

Proof of Theorem 1.5. Fix H, K as in Lemma 3.1, and then fix $\tilde{H} \in [H]^{\aleph_1}$ and $\tilde{K} \in [K]^{\aleph_1}$. Let $D = \bigcup\{\tilde{H} + s : s \in \mathbb{Q}\}$ and $E = \bigcup\{\tilde{K} + t : t \in \mathbb{Q}\}$.

Now, fix f, D^*, E^*, p, q, a, b as in Theorem 1.5. Then the function $f^* := f \cap \big((D^* \cap (p,q)) \times (E^* \cap (a,b))\big)$ is uncountable, and is an order-preserving bijection from $D^* \cap (p,q)$ onto $E^* \cap (a,b)$.

Now fix $s, t \in \mathbb{Q}$ so that $f^* \cap (\tilde{H}+s) \times (\tilde{K}+t)$ is uncountable, so in particular it contains a convergent sequence. So, we have $(x_n, y_n) \in f^*$ for $n \le \omega$, with $x_n \to x_\omega$ and $y_n \to y_\omega$ as $n \nearrow \omega$, and $x_n \in \tilde{H}+s$ and $y_n \in \tilde{K}+t$ for all $n \le \omega$. We may assume that all the x_n are distinct and that all the y_n are distinct. Since f^* is order-preserving and the property of H, K in Lemma 3.1 is preserved by translation, $\forall \varepsilon > 0\ \exists n \in \omega\ [(y_\omega - y_n)/(x_\omega - x_n) \in (0, \varepsilon) \cup (1/\varepsilon, \infty)]$. Passing to a subsequence, we may assume that either $\forall n \in \omega\ [(y_\omega - y_n)/(x_\omega - x_n) \in (2^n, \infty)]$ or $\forall n \in \omega\ [(y_\omega - y_n)/(x_\omega - x_n) \in (0, 2^{-n})]$. In the first case, $f'(x_\omega)$ doesn't exist and f is not Lipschitz on (p, q). In the second case, $(f^{-1})'(y_\omega)$ doesn't exist and f^{-1} is not Lipschitz on (a, b).

For (3), repeat the argument, now letting f^* be the set of all $(d, f(d))$ such that $p < d < q$ and $f'(d)$ exists and $f'(d) \ne 0$.

4 Everywhere Differentiable Functions

We prove here some lemmas to be used in the proof of Theorem 1.6, where we shall construct the isomorphism f along with its derivative g.

Definition 4.1 *For $g : \mathbb{R} \to \mathbb{R}$, let $\|g\| = \sup\{|g(x)| : x \in \mathbb{R}\} \in [0, \infty]$.*

Definition 4.2 *For bounded measurable $\psi : \mathbb{R} \to \mathbb{R}$ and $a \ne b$:*

$$\mathsf{AV}_a^b \psi = \frac{1}{b-a} \int_a^b \psi(x)\, dx \quad .$$

Definition 4.3 *$\mathcal{D}$ is the set of all measurable $g : \mathbb{R} \to \mathbb{R}$ such that $\|g\| < \infty$ and $g(x) = \lim_{\eta \to 0} \mathsf{AV}_x^{x+\eta} g\, dt$ for all x.*

By this last condition, if $f(x) = \int_0^x g(t)\, dt$, then $f'(x) = g(x)$ for all x.

Note that $\mathcal{D}$ is a Banach space with the sup norm $\| \cdot \|$. Also, $\mathcal{D}$ contains all bounded continuous functions, and every function in $\mathcal{D}$ is of Baire class 1; that is, a pointwise limit of continuous functions. However, many Baire 1 functions, such as $\chi_{\{0\}}$, fail to be in $\mathcal{D}$. A function in $\mathcal{D}$ can be everywhere discontinuous; this has been known since the 1890s; see pp. 412–421 of Hobson [7] for references. Katznelson and Stromberg [8] describe a method for constructing such functions which we can embed into our forcing construction. Here we summarize their method and make some minor additions to it.

Definition 4.4 *Fix $C > 1$. $\psi : \mathbb{R} \to \mathbb{R}$ has the* C–average property *iff ψ is bounded and continuous, and $\psi(x) \ge 0$ for all x, and $\mathsf{AV}_a^b \psi \le C \min(\psi(a), \psi(b))$ whenever $a \ne b$. Let $\mathcal{AP}_C$ be the set of all functions with the C–average property.*

So, the average value of ψ on an interval is bounded by C times the value at either endpoint. Note that either $\psi(x) > 0$ for all x or $\psi = 0$ for all x. Also, $\mathcal{AP}_C$ is closed under finite sums and uniform limits, and if $\psi \in \mathcal{AP}_C$ then $(x \mapsto \alpha\psi(\beta x + \gamma)) \in \mathcal{AP}_C$ for all $\alpha, \beta, \gamma \in \mathbb{R}$ with $\alpha \geq 0$. $\mathcal{AP}_C$ clearly contains all non-negative constant functions, but also, by [8], the function $(1 + |x|)^{-1/2}$ has the 4–average property; see also Lemma 4.7 below. Functions in $\mathcal{AP}_C$ can be used to build functions in $\mathcal{D}$ by:

Lemma 4.5 *Fix $C > 1$. Assume that all $\psi_j \in \mathcal{AP}_C$. Let $g(x) = \sum_{j \in \omega} \psi_j(x)$, and assume that $g(x) < \infty$ for all x and $\|g\| < \infty$. Then $g \in \mathcal{D}$.*

Proof. Fix $x \in \mathbb{R}$ and $\varepsilon > 0$. It is sufficient to produce a $\delta > 0$ such that:

$$\forall \eta \in (-\delta, \delta) \backslash \{0\} \ : \qquad g(x) - 2\varepsilon \leq \mathsf{AV}_x^{x+\eta} g \leq g(x) + (C+1)\varepsilon \quad . \qquad (*)$$

Let $g_m(x) = \sum_{j<m} \psi_j(x)$. First fix m such that $g_m(x) \geq g(x) - \varepsilon$. Then fix $\delta > 0$ such that $|g_m(x) - g_m(x+\eta)| \leq \varepsilon$ for all $\eta \in (-\delta, \delta)\backslash\{0\}$. Then, fix such an η, and we verify $(*)$. For the first $\leq$, use $g(x) - 2\varepsilon \leq g_m(x) - \varepsilon \leq \mathsf{AV}_x^{x+\eta} g_m \leq \mathsf{AV}_x^{x+\eta} g$. For the second $\leq$, note that for each $n \geq m$, $(g_n - g_m) \in \mathcal{AP}_C$, and hence $\mathsf{AV}_x^{x+\eta}(g_n - g_m) \leq C(g_n(x) - g_m(x)) \leq C\varepsilon$. Letting $n \nearrow \infty$, we get $\mathsf{AV}_x^{x+\eta}(g - g_m) \leq C\varepsilon$, so that $\mathsf{AV}_x^{x+\eta} g \leq \mathsf{AV}_x^{x+\eta} g_m + C\varepsilon \leq g_m(x) + (C+1)\varepsilon \leq g(x) + (C+1)\varepsilon$.

To verify that the function $(1 + |x|)^{-1/2}$ has the 4–average property:

Lemma 4.6 *Suppose that $\psi : \mathbb{R} \to [0, \infty)$ is a bounded continuous function such that $\psi(x) = \psi(-x)$ for all x, ψ is decreasing for $x > 0$, and $\mathsf{AV}_0^b \psi \leq C\psi(b)$ for all $b > 0$. Then $\psi \in \mathcal{AP}_{2C}$.*

Proof. We must show that $\mathsf{AV}_a^b \psi \leq 2C \min(\psi(a), \psi(b))$ whenever $a \neq b$. By symmetry, there are only two cases:

Case I: $a < 0 < b$, where $0 < \hat{a} := -a \leq b$ (so $\psi(a) \geq \psi(b)$):

$$\mathsf{AV}_a^b \psi = \frac{1}{b + \hat{a}} \left[\int_0^b \psi + \int_0^{\hat{a}} \psi \right] \leq \frac{1}{b} \cdot 2 \int_0^b \psi \leq 2C\psi(b) \quad .$$

Case II: $0 \leq a < b$: Then, since ψ is decreasing, $\mathsf{AV}_a^b \psi \leq \mathsf{AV}_0^b \psi \leq C\psi(b)$.

Lemma 4.7 *If $\psi(x) = (1 + |x|)^{-1/2}$ then $\psi \in \mathcal{AP}_4$.*

Proof. For $b > 0$,

$$\frac{1}{\psi(b)} \mathsf{AV}_0^b \psi = \frac{\sqrt{1+b}}{b} \left[2\sqrt{1+b} - 2 \right] = \frac{2}{b} \left[b + 1 - \sqrt{1+b} \right] < 2 \quad ,$$

so apply Lemma 4.6.

Then, by taking translates and finite sums:

Corollary 4.8 *If* $\psi(x) = \sum_{\ell<L} \gamma_\ell (1 + r_\ell |x - \delta_\ell|)^{-1/2}$*, where* $L \in \omega$*, all* $\gamma_\ell, r_\ell, \delta_\ell \in \mathbb{R}$*, and all* $\gamma_\ell, r_\ell \geq 0$*, then* $\psi \in \mathcal{AP}_4$*.*

5 Isomorphisms

This entire section is devoted to the proof of Theorem 1.6. We plan to construct f along with $g = f'$, which will be in $\mathcal{D}$; so $f(x) = \int_0^x g(t)\,dt$. We shall construct g as a limit of an ω–sequence, using the following modification of Lemma 4.5:

Lemma 5.1 *Assume that we have* g_n, ψ_n, θ_n *for* $n \in \omega$ *such that:*

1. $g_0 \in C(\mathbb{R}, [0, \infty)\,)$ *and* $\|g_0\| < \infty$*.*
2. $\theta_n \in C(\mathbb{R}, \mathbb{R})$*, and* $\sum_n \|\theta_n\| < \infty$*.*
3. *Each* $\psi_n \in \mathcal{AP}_4$*.*
4. $g_{n+1} = g_n - \psi_n + \theta_n$ *and* $g_{n+1}(x) \geq 0$ *for all* x*.*

Then $\langle g_n : n \in \omega \rangle$ *converges pointwise to some* $g : \mathbb{R} \to [0, \infty)$*, and* $g \in \mathcal{D}$*.*

Proof. Since all $\psi_i \geq 0$ and all $g_n \geq 0$, all sums $h_n := \sum_{i<n} \psi_i$, and hence also $h := \sum_{i<\omega} \psi_i$, are bounded by $\|g_0\| + \sum_i \|\theta_i\|$. It follows that the sequence $\langle g_n : n \in \omega \rangle$ converges pointwise, and $h \in \mathcal{D}$ by Lemma 4.5. Then $g \in \mathcal{D}$ because $g = g_0 + \sum_n \theta_n - h$ and $g_0 + \sum_n \theta_n \in \mathcal{D}$ (since it is bounded and continuous).

We plan to build the ψ_n and θ_n by forcing, and the forcing conditions will guarantee that each $g_n(x) \geq 0$ for all x. Besides $f(x) := \int_0^x g(t)\,dt$, we also have $f_n(x) := \int_0^x g_n(t)\,dt$, and the f_n will converge pointwise to f. Since $f(0)$ must be 0, we shall assume WLOG that $0 \in D \cap E$. The proof applies the "collapsing the continuum" trick; so we assume CH, and we describe a ccc poset which forces the ψ_n and θ_n.

To construct ccc posets, we use the standard setup with elementary submodels:

Definition 5.2 *Fix* κ*, a suitably large regular cardinal. Let* $\langle M_\xi : 0 < \xi < \omega_1 \rangle$ *be a continuous chain of countable elementary submodels of* $H(\kappa)$*, with* $D, E \in M_1$ *and each* $M_\xi \in M_{\xi+1}$*. Let* $M_0 = \emptyset$*. For* $x \in \bigcup_\xi M_\xi$*, let* $\mathrm{ht}(x)$*, the* height *of* x*, be the* ξ *such that* $x \in M_{\xi+1} \backslash M_\xi$*.*

By setting $M_0 = \emptyset$, we ensure that under CH, $\mathrm{ht}(x)$ is defined whenever $x \in \mathbb{R}$ or x is a Borel subset of $\mathbb{R}$. Observe that $\{d \in D : \mathrm{ht}(d) = \xi\}$ and $\{e \in E : \mathrm{ht}(e) = \xi\}$ are both countable and dense for each $\xi < \omega_1$.

We now state the basic combinatorial lemma behind the proof of ccc.

Lemma 5.3 *Assume* CH. *Say we have* $2n$*–tuples*

$$p^\alpha = \big((d_0^\alpha, e_0^\alpha), \ldots, (d_{n-1}^\alpha, e_{n-1}^\alpha)\big) \in \mathbb{R}^{2n}$$

for $\alpha < \omega_1$. *Fix* $\varphi \in C((0,\infty),(0,\infty))$. *Assume that:*

a. $d_i^\alpha \neq d_i^\beta$ *and* $e_i^\alpha \neq e_i^\beta$ *for all* α, β, i *with* $\alpha \neq \beta$.

b. $\mathrm{ht}(d_i^\alpha) > \mathrm{ht}(e_i^\alpha)$ *for all* α, i.

c. $\mathrm{ht}(d_i^\alpha) \neq \mathrm{ht}(d_j^\alpha)$ *for all* α, i, j *with* $i \neq j$.

Then there exist $\alpha \neq \beta$ *such that* p^α, q^β *are* compatible $(p^\alpha \not\perp q^\beta)$ *in the sense that for all* $i < n$, *the slope* $(e_i^\beta - e_i^\alpha)/(d_i^\beta - d_i^\alpha) > 0$ *and also* $|e_i^\beta - e_i^\alpha| < \varphi(|d_i^\beta - d_i^\alpha|)$.

Here, $p^\alpha \not\perp q^\beta$ asserts that each two-element partial function $\{(d_i^\alpha, e_i^\alpha), (d_i^\beta, e_i^\beta)\}$ is order-preserving, and also has growth rate bounded by a "small" function φ.

We shall use the usual symbol $p \not\perp q$ to denote compatibility in a forcing poset. This lemma does not mention forcing explicitly, but the $p \not\perp q$ used here will be part of a proof of ccc for a poset later (Lemma 5.11).

Proof. Induct on n. The case $n = 0$ is trivial, so assume the result for n and we prove it for $n+1$, so now $p^\alpha = ((d_0^\alpha, e_0^\alpha), \ldots, (d_n^\alpha, e_n^\alpha)) \in \mathbb{R}^{2n+2}$. Applying (b)(c), WLOG, each sequence is arranged so that $\mathrm{ht}(p^\alpha) = \mathrm{ht}(d_n^\alpha)$, and hence $\mathrm{ht}(p^\alpha) > \mathrm{ht}(d_i^\alpha)$ for all $i < n$ and $\mathrm{ht}(p^\alpha) > \mathrm{ht}(e_i^\alpha)$ for all $i \leq n$. Also, by (a), WLOG, $\alpha < \beta \to \mathrm{ht}(p^\alpha) < \mathrm{ht}(p^\beta)$, which implies that $\mathrm{ht}(p^\alpha) \geq \alpha$.

Let $F = \mathrm{cl}\{p^\alpha : \alpha < \omega_1\} \subseteq \mathbb{R}^{2n+2}$. For each α and each $x \in \mathbb{R}$, obtain $q_x^\alpha \in \mathbb{R}^{2n+2}$ by replacing the d_n^α by x in p^α. Let $F^\alpha = \{x \in \mathbb{R} : q_x^\alpha \in F\}$. Fix ζ such that $F \in M_\zeta$. For $\alpha \geq \zeta$, F^α is uncountable because $d_n^\alpha \in F^\alpha$ and $F^\alpha \in M_{\mathrm{ht}(p^\alpha)}$ while $d_n^\alpha \notin M_{\mathrm{ht}(p^\alpha)}$. So, choose any $u^\alpha, v^\alpha \in F^\alpha$ with $u^\alpha < v^\alpha$. Then, get an uncountable $S \subseteq \omega_1 \backslash \zeta$, along with rational open intervals U, V such that $\sup U < \inf V$ and $u^\alpha \in U$ and $v^\alpha \in V$ for all $\alpha \in S$. Let $\Xi = \inf\{\varphi(y - x) : y \in V \ \&\ x \in U\}$. Thinning S, we assume also that for $\alpha, \beta \in S$, $|e_n^\alpha - e_n^\beta| < \Xi$.

Let ${}^*p^\alpha = p^\alpha \restriction (2n)$ (delete the last pair). Applying induction, fix $\alpha, \beta \in S$ such that ${}^*p^\alpha \not\perp {}^*p^\beta$ and $e_n^\alpha < e_n^\beta$. Now, $q_{u^\alpha}^\alpha, q_{v^\beta}^\beta \in F$, so we may choose p^δ, p^ϵ sufficiently close to $q_{u^\alpha}^\alpha, q_{v^\beta}^\beta \in F$, respectively, such that ${}^*p^\delta \not\perp {}^*p^\epsilon$ and also so that $d_n^\delta \in U$ and $d_n^\epsilon \in V$, and also so that $0 < e_n^\epsilon - e_n^\delta < \Xi$. Then $(e_n^\epsilon - e_n^\delta)/(d_n^\epsilon - d_n^\delta) > 0$ and also $|e_n^\epsilon - e_n^\delta| < \varphi(|d_n^\epsilon - d_n^\delta|)$, so $p^\delta \not\perp p^\epsilon$. ☕

Our forcing conditions will contain, among other things, a finite $\sigma \subseteq D \times E$ which is a partial isomorphism; this σ will be a sub-function of the f of Theorem 1.6. We let $g_0(x) = x^2/(x^2+1)$, so that $f_0(x) = x - \arctan(x)$. The forcing conditions will determine successively $\psi_0, \theta_0, \psi_1, \theta_1, \ldots$, and hence also $g_1, f_1, g_2, f_2, \ldots$. We shall demand that all $\psi_n, \theta_n \in M_1$ (and hence also all $g_n, f_n \in M_1$), so that there are only countably many possibilities for them; this will facilitate the proof that the poset is ccc. Then, $\lim_n f_n = f \supset \sigma$; the f_n will not actually extend σ; rather, they will approximate σ in the sense of the following definition:

Definition 5.4 *(τ, g, f, ι) is* correctable *iff:*

- $\breve{P}1.$ *$(0,0) \in \tau$ and $\tau \in [\mathbb{R} \times \mathbb{R}]^{<\omega}$.*
- $\breve{P}2.$ *τ is an order-preserving bijection.*
- $\breve{P}7.$ *$g \in C(\mathbb{R}, [0, \infty))$ and $g^{-1}\{0\} = \{0\}$.*
- $\breve{P}8.$ *$f(x) = \int_0^x g(t)\,dt$.*
- $\breve{P}13.$ *$\iota > 0$ and whenever $(d_0, e_0), (d_1, e_1) \in \tau$ and $d_0 < d_1$:*

$$0 \ < \ \frac{e_1 - e_0}{d_1 - d_0} - \frac{f(d_1) - f(d_0)}{d_1 - d_0} \ < \ \iota \quad .$$

The labels on these items correspond to the labels in Definition 5.6 (of $\mathbb{P}$). In $\mathbb{P}$, the f, g will be replaced by suitable f_n, g_n.

Think of ι as being "very small". So, our hypotheses ($\breve{P}2$)($\breve{P}3$)($\breve{P}7$)($\breve{P}13$) imply that f and τ are strictly increasing, and between $d_0, d_1 \in \mathrm{dom}(\tau)$, the slope of f is very slightly less than the slope of τ.

We remark that it is sufficient to assume that ($\breve{P}13$) holds between adjacent elements of $\mathrm{dom}(\tau)$; that implies the full ($\breve{P}13$), since if $d_0 < d_1 < d_2$ we have

$$\left\{ \begin{array}{l} 0 \ < \ \big(\tau(d_1) - \tau(d_0)\big) - \big(f(d_1) - f(d_0)\big) \ < \ \iota(d_1 - d_0) \ \ \& \\ 0 \ < \ \big(\tau(d_2) - \tau(d_1)\big) - \big(f(d_2) - f(d_1)\big) \ < \ \iota(d_2 - d_1) \end{array} \right\} \Longrightarrow$$
$$0 \ < \ \big(\tau(d_2) - \tau(d_0)\big) - \big(f(d_2) - f(d_0)\big) \ < \ \iota(d_2 - d_0) \quad .$$

Since $f(0) = \tau(0) = 0$, we can set $d_0 = 0$ or $d_1 = 0$ in ($\breve{P}13$) to obtain, for $(d, e) \in \tau$:

$\breve{P}12.$ $d, e > 0 \ \to \ 0 < (e - f(d)) < \iota d \quad ; \quad d, e < 0 \ \to \ 0 > (e - f(d)) > \iota d \quad .$

That is, if $(d, e) \in \tau$, then $f(d)$ is a slight under-estimate of e when $d > 0$ and a slight over-estimate of e when $d < 0$. The next lemma says that this "error" can be corrected by adding a small positive function θ to g:

Lemma 5.5 *Assume that (τ, g, f, ι) is correctable and $J \subset \mathbb{R}$ is finite. Then for some $\theta : \mathbb{R} \to \mathbb{R}$:*

- *a. $\theta(x) \geq 0$ for all x, and $\|\theta\| < \iota$, and $\theta(x) \to 0$ as $x \to \pm\infty$.*
- *b. θ is continuous, and $\theta^{-1}\{0\} = J$.*
- *c. If $g^* = g + \theta$ and $f^*(x) = \int_0^x g^*(t)\,dt$, then $f^*(d) = e$ for each $(d, e) \in \tau$.*

Proof. Since $\tau(0) = f(0) = 0$, item (c) will hold if we have, for adjacent $d_0, d_1 \in \mathrm{dom}(\tau)$ with $d_0 < d_1$:

$$\int_{d_0}^{d_1} \theta(t)\,dt = (\tau(d_1) - \tau(d_0)) - (f(d_1) - f(d_0)) \quad ,$$

and this quantity is assumed to lie in $(0, \iota(d_1 - d_0))$. It is now easy to construct a C^∞ function θ which satisfies this, along with (a)(b). ☕

Definition 5.6 *$\mathbb{P}$ is the set of all tuples $p = (\sigma^p, N^p, g^p_{n+1}, f^p_{n+1}, \psi^p_n, \theta^p_n)_{n<N^p}$, satisfying the following conditions. We drop the superscript p when it is clear from context. Let $g_0(x) = x^2/(x^2+1)$ and $f_0(x) = x - \arctan(x)$.*

- *P1.* $\sigma^p \in [D \times E]^{<\omega}$ *and* $(0,0) \in \sigma^p$.
- *P2.* σ^p *is an order-preserving bijection.*
- *P3. For* $(d,e) \in \sigma^p$ *and* $(d,e) \neq (0,0)$: $\mathrm{ht}(e) < \mathrm{ht}(d) < \mathrm{ht}(e) + \omega$.
- *P4. If* $(d_0,e_0),(d_1,e_1) \in \sigma^p$ *and* $d_0 \neq d_1$ *then* $\mathrm{ht}(d_0) \neq \mathrm{ht}(d_1)$.
- *P5.* $N^p \in \omega$.
- *P6.* $\mathrm{ht}(g_n) = \mathrm{ht}(f_n) = \mathrm{ht}(\psi_n) = \mathrm{ht}(\theta_n) = 0$.
- *P7.* $g_n \in C(\mathbb{R}, [0, 2 - 2^{-n}))$ *and* $g_n^{-1}\{0\} = \{0\}$ *and* $\lim_{x\to\pm\infty} g_n(x) = 1$.
- *P8.* $f_n(x) = \int_0^x g_n(t)\,dt$.
- *P9.* $g_{n+1} = g_n - \psi_n + \theta_n$ *when* $n < N^p$.
- *P10.* $\psi_n \in \mathcal{AP}_4$.
- *P11.* $\theta_n \in C(\mathbb{R})$ *and* $\|\theta_n\| \leq 2^{-n-1}$.
- *P12. For* $(d,e) \in \sigma$: $d, e > 0 \;\to\; 0 < (e - f_{N^p}(d)) < 2^{-N^p-2} d$ *and*
  $d, e < 0 \;\to\; 0 > (e - f_{N^p}(d)) > 2^{-N^p-2} d$.
- *P13. Whenever* $(d_0,e_0),(d_1,e_1) \in \sigma$ *and* $d_0 < d_1$:

$$0 \;<\; \frac{e_1 - e_0}{d_1 - d_0} - \frac{f_{N^p}(d_1) - f_{N^p}(d_0)}{d_1 - d_0} \;<\; 2^{-N^p-2} \quad .$$

Define $q \leq p$ iff

- *Q1.* $\sigma^q \supseteq \sigma^p$ *and* $N^q \geq N^p$.
- *Q2.* $(g^p_{n+1}, f^p_{n+1}, \psi^p_n, \theta^p_n) = (g^q_{n+1}, f^q_{n+1}, \psi^q_n, \theta^q_n)$ *for all* $n < N^p$.
- *Q3. Whenever* $(0,0) \neq (d,e) \in \sigma^p$ *and* $N^p < n \leq N^q$: $g^q_n(d) \in (0, 2^{-n})$.

Then $\mathbb{1} = (\{(0,0)\}, 0)$*; that is, when* $N^p = 0$*, the rest of the tuple is empty.*

We shall now prove a sequence of lemmas leading up to Theorem 1.6, at the same time explaining some of the clauses in Definition 5.6.

The restriction on heights in (P3)(P4) will be important in the proof of ccc, and are analogous to the restrictions in Lemma 5.3.

If G is a generic filter on $\mathbb{P}$, then in $\mathbf{V}[G]$ we can define $\widehat{\sigma} = \bigcup\{\sigma^p : p \in G\}$. Then $\widehat{\sigma}$ is an order-preserving function from a subset of D to a subset of E, and the f of Theorem 1.6 will extend $\widehat{\sigma}$ (Lemma 5.10 below).

We shall apply Lemma 5.1 in $\mathbf{V}[G]$ to obtain f, g, and (Q3) will let us prove that $g(d) = 0$ for all $d \in \mathrm{dom}(\widehat{\sigma})$. Note that (Q3) is vacuous when $N^p = N^q$.

By (P1)(P2)(P7)(P8)(P13), each $(\sigma, g_N, f_N, 2^{-N-2})$ is correctable. Then, as noted above, (P12) follows, but we state it separately for emphasis, since it is used to prove that f extends $\widehat{\sigma}$. Also, the $g_n(x) < 2 - 2^{-n}$ asserted by (P7) follows by induction from the other assumptions; specifically, $g_0(x) < 1$, $g_{n+1} = g_n - \psi_n + \theta_n$, $\theta_n(x) \le 2^{-n-1}$, and $\psi_n(x) \ge 0$.

Definition 5.7 $\mu(p) = \min\big(\{1\} \cup \{|d_0 - d_1| : d_0, d_1 \in \mathrm{dom}(\sigma_p)\ \&\ d_0 \neq d_1\}\big)$. *Call a map ζ from $\mathbb{P}$ into the rationals a $\mathbb{P}$–function iff $\zeta(p) \in (0, \mu(p)/2)$ for all p. For such a ζ, say $p, q \in \mathbb{P}$ are ζ–close iff $N^p = N^q$ and $|\sigma^p| = |\sigma^q|$ and $\zeta(p) = \zeta(q)$ and all elements of $\mathrm{dom}(\sigma^p) \cup \mathrm{dom}(\sigma^q)$ have different heights and*

$$(g^p_{n+1}, f^p_{n+1}, \psi^p_n, \theta^p_n)_{n<N^p} = (g^q_{n+1}, f^q_{n+1}, \psi^q_n, \theta^q_n)_{n<N^q} \quad ,$$

and, setting $\zeta = \zeta(p) = \zeta(q)$: For all $d \in \mathrm{dom}(\sigma^p)$ there is a $d' \in \mathrm{dom}(\sigma^q)$ such that $|d - d'| < \zeta$; furthermore, if $(d, e) \in \sigma^p$ and $(d', e') \in \sigma^q$, then $d = d'$ implies $e = e'$, and $d \neq d'$ implies $0 < (e - e')/(d - d') < \zeta$.

Note that p is always ζ–close to itself. Inserting the "$\{1\}\cup$" in the definition of $\mu(p)$ makes it well-defined in the case that $\sigma_p = \{(0, 0)\}$, but also, it will be useful to think of ζ as being "small", so that, e.g., $\zeta^2 < \zeta < \sqrt{\zeta}$.

The requirement that $\zeta(p) < \mu(p)/2$ implies that the d' above is uniquely determined from d. The actual $\zeta(p)$ used in Lemma 5.8 will be *much* smaller than $\mu(p)/2$. The requirement that all the slopes $(e - e')/(d - d')$ be small but positive will be fulfilled in the proof of ccc using Lemma 5.3.

If p, q are ζ–close, then they are "close" to being compatible, with the tuple $(\sigma^p \cup \sigma^q, N^p, g^p_{n+1}, f^p_{n+1}, \psi^p_n, \theta^p_n)_{n<N^p}$ being a common extension, *except* that this may fail (P2)(P12)(P13).

Lemma 5.8 *There is a $\mathbb{P}$–function ζ such that for all $p, q \in \mathbb{P}$: If p, q are ζ–close then $p \not\perp q$ and there is an $s \in \mathbb{P}$ such that $s \le p$ and $s \le q$ and $N^s = N^p + 1$.*

We shall prove this later, after listing some of its consequences. First, when $p = q$, we get:

Corollary 5.9 *For each $p \in \mathbb{P}$, there is an $s \leq p$ with $N^s = N^p + 1$. Hence, $\{q : N^q > i\}$ is dense in $\mathbb{P}$ for each i.*

So, in $\mathbf{V}[G]$, we have $g_n, f_n, \psi_n, \theta_n$ for each $n \in \omega$; e.g., $g_n = g_n^p$ for some (any) $p \in G$ such that $N^p \geq n$. Then Lemma 5.1 applies: (1) is obvious, (2) follows from (P11), (3) follows from (P10), and (4) follows from (P7)(P9), So, by Lemma 5.1, $\langle g_n : n \in \omega \rangle$ converges pointwise to some $g : \mathbb{R} \to [0, \infty)$, and $g \in \mathcal{D}$; also, $\|g\| \leq 2$ by (P7). Then, since the g_n are uniformly bounded, $\langle f_n : n \in \omega \rangle$ converges pointwise to f, where $f(x) = \int_0^x g(t)\,dt$. Note that we are applying Lemma 5.1 in $\mathbf{V}[G]$ to the natural extensions of $g_n, f_n, \psi_n, \theta_n$ (which were, in $\mathbf{V}$, functions from $\mathbb{R}^{\mathbf{V}}$ to $\mathbb{R}^{\mathbf{V}}$).

Regarding (Q3): By not requiring $g_n(d) \approx 0$ for *all* $n \leq N$, we make it easier to add new pairs (d, e) into extensions of p (see the proof of Lemma 5.13). Likewise, we only require (P12)(P13) for $n = N^p$, so that when proving Lemma 5.13, we do not need to consider (P12)(P13) for $n < N^p$. But still,

Lemma 5.10 *For $(d, e) \in \widehat{\sigma}$: $g(d) = 0$ and $f(d) = e$.*

Proof. Since $\langle g_n : n \in \omega \rangle$ and $\langle f_n : n \in \omega \rangle$ converge pointwise, it is sufficient to show that some subsequence of $\langle g_n(d) : n \in \omega \rangle$ converges to 0 and some subsequence of $\langle f_n(d) : n \in \omega \rangle$ converges to e. Say $(d, e) \in p \in G$. Then by Corollary 5.9, $S := \{N^q : q \in G \wedge (d, e) \in q\}$ is infinite. Then, applying (Q3)(P12), $\langle g_n(d) : n \in S \rangle$ converges to 0 and $\langle f_n(d) : n \in S \rangle$ converges to e.

Another consequence of Lemma 5.8:

Lemma 5.11 *$\mathbb{P}$ has the ccc.*

Proof. Let $A \subseteq \mathbb{P}$ be uncountable; we prove that A cannot be an antichain. Let $\zeta(p)$ be as in Lemma 5.8. We may assume that $\zeta(p)$ is the same rational ζ for all $p \in A$. Furthermore, by a delta system argument, we may assume that $A = \{p^\alpha : \alpha < \omega_1\}$ and $\sigma^{p^\alpha} = \sigma^\alpha \cup \tau$, where τ is the root of the delta system. We may also assume (applying (P2)(P3)(P6)) that the σ^α satisfy the hypotheses of Lemma 5.3, and that all p^α, p^β satisfy everything in Definition 5.7 (of "ζ–close") except possibly for the requirement "$d \neq d'$ implies $0 < (e - e')/(d - d') < \zeta$". But now Lemma 5.3 (applied with $\varphi(t) = \zeta \cdot t$) implies that there is *some* pair p^α, p^β with $\alpha \neq \beta$ satisfying this requirement (since $\sigma^\alpha \not\perp \sigma^\beta$), so that $p^\alpha \not\perp p^\beta$ by Lemma 5.8.

By applying Lemma 5.5 with $J = \emptyset$ to $\mathbb{P}$ we get:

Lemma 5.12 *Fix $p \in \mathbb{P}$. Let $N = N^p$ and $\sigma = \sigma^p$. Then for some $\theta : \mathbb{R} \to \mathbb{R}$:*

a. $\theta(x) > 0$ *for all* x, *and* $\|\theta\| < 2^{-N-2}$, *and* $\theta(x) \to 0$ *as* $x \to \pm\infty$.

b. θ *is continuous.*

c. If $g^* = g_N + \theta$ *and* $f^*(x) = \int_0^x g^*(t)\,dt$, *then* $f^*(d) = e$ *for each* $(d,e) \in \sigma$.

Lemma 5.13 $\mathrm{ran}(\widehat{\sigma}) = E$.

Proof. It is sufficient to prove that for each $e \in E$, $\{q : e \in \mathrm{ran}(\sigma^q)\}$ is dense. So fix $p \in \mathbb{P}$ with $e \notin \mathrm{ran}(\sigma^p)$, and we find a $q \le p$ with $e \in \mathrm{ran}(\sigma^q)$; q will be exactly like p, except that $\sigma^q = \sigma^p \cup \{(d,e)\}$, where $d \in D_\xi := \{d \in D : \mathrm{ht}(d) = \xi\}$ and $\mathrm{ht}(e) < \xi < \mathrm{ht}(e) + \omega$ and ξ is different from $\mathrm{ht}(d')$ for all $d' \in \mathrm{dom}(\sigma^p)$. Then $q \le p$ is clear, but we must make sure that $q \in \mathbb{P}$.

Let f^* be as in Lemma 5.12. Then f^* is a continuous increasing function, and, using the $\lim_{x\to\pm\infty} g_N(x) = 1$ from (P7), $f^*(x) \to \infty$ as $x \to \infty$ and $f^*(x) \to -\infty$ as $x \to -\infty$. There is thus a unique $\hat{d}$ such that $f^*(\hat{d}) = e$; and $\hat{d} \notin \mathrm{dom}(\sigma^p)$. Setting $\sigma^q = \sigma^p \cup \{(\hat{d}, e)\}$ would satisfy (P2)(P12)(P13); to verify this: (P2) is clear and (P12) follows from (P13). Also, (P13) is clear unless one of (d_0, e_0), (d_1, e_1) is $(\hat{d}, e)$. Assume that $(d_1, e_1) = (\hat{d}, e)$, since the other case is similar; so $d_0 < \hat{d} = d_1$. For the "$< 2^{-N-2}$" in (P13), use $\|\theta\| < 2^{-N-2}$ and the fact that $f^* \supset \{(d_0, e_0), (\hat{d}, e)\}$. For the "$0 <$" in (P13), observe that $\int_{d_0}^{\hat{d}} \theta > 0$.

For all d sufficiently close to $\hat{d}$, setting $\sigma^q = \sigma^p \cup \{(d,e)\}$ will also satisfy (P2)(P12)(P13), so choose such a d in D_ξ, which is possible because D_ξ is dense.

Although $\mathrm{dom}(\widehat{\sigma}) \ne D$ (by (P3)(P4)), we do have:

Lemma 5.14 *In* $\mathbf{V}[G]$, $\mathrm{dom}(\widehat{\sigma})$ *is an* $\aleph_1$*–dense subset of* D.

Proof. Use the facts that f is strictly increasing and continuous, $\|f'\| < \infty$ (by *P7*), $f \supset \widehat{\sigma}$ (by Lemma 5.10), and $\mathbf{V}$, $\mathbf{V}[G]$ have the same $\aleph_1$ (by the ccc).

We are now done if we prove Lemma 5.8. First, a few remarks.

As noted above, to prove that $p \not\perp q$ whenever p, q are ζ–close, we need to make sure that the common extension satisfies (P2)(P12)(P13). But (P12) is a special case of (P13), and it is easy to satisfy (P2); that is, if the function ζ is small enough then $\sigma^p \cup \sigma^q$ will be order-preserving. A more serious issue is that the natural extension $(\sigma^p \cup \sigma^q, N, g_{n+1}, f_{n+1}, \psi_n, \theta_n)_{n<N}$ may fail condition (P13); that is, $(\sigma^p \cup \sigma^q, g_N, f_N, 2^{-N-2})$ may not be correctable, since correctability puts a *lower* bound on the slopes between adjacent elements of σ in terms of the slope of f_N. But "ζ–close" implies that the slopes between neighboring pairs (d,e) and (d', e') are *small* (bounded above by ζ).

The common extension s will have $\sigma^s = \sigma^p \cup \sigma^q$ but $N^s = N + 1$. Then ψ^s_N will be a linear combinations of functions of the form $(1 + r|x - \bar{d}|)^{-1/2}$ for $\bar{d}$ close to a d, d' pair and suitably large r. Also, $r \approx 1/\sqrt{\zeta}$, so $r\zeta \ll 1$, so that for x near $d, d', \bar{d}$: $g_N(x)$ and $\psi_N^s(x)$ will be approximately constant; and, we shall make $g_N(x) - \psi_N^s(x)$ very slightly negative for these x. But r will be large enough that $\int_0^d \psi(t)\, dt$ will be negligible for each d.

Now, we need to define $\psi_N = \psi_N^s$ and $\theta_N = \theta_N^s$, which will determine $g_{N+1} = g_{N+1}^s$ and $f_{N+1} = f_{N+1}^s$. We do not know a "simple" definition of ζ which "works", so rather than defining ζ right away, we shall simply define ψ_N and θ_N and check that they have the right properties, assuming that ζ is small enough. ψ_n and θ_N will determine $g_{N+1} = g_{N+1}^s$ and $f_{N+1} = f_{N+1}^s$ by $g_{N+1} = g_N - \psi_N + \theta_N$ and $f_{N+1}(x) = \int_0^x g_{N+1}(t)\, dt$. We shall also have $\theta_N = \theta_N^\dagger + \theta_N^*$ because there are two tasks for θ_N: to make sure that g_{N+1} is positive (the task of $\theta_N^\dagger$), and to correct f_{N+1} to come close to σ^s, so as to satisfy (P13) (the task of θ_N^*). Both $\theta_N^\dagger$ and θ_N^* will be positive functions.

First, some notation: Applying the definition of "close", let $L = |\sigma^p| = |\sigma^q|$ and let $\sigma^p = \{(d_\ell^p, e_\ell^p) : \ell < L\}$ and $\sigma^q = \{(d_\ell^q, e_\ell^q) : \ell < L\}$, where $|d_\ell^p - d_\ell^q| < \zeta$, which implies also $|e_\ell^p - e_\ell^q| < \zeta^2$.

Before defining anything, we must make sure that σ^s satisfies (P2); that is, that $\sigma^p \cup \sigma^q$ is an order-preserving bijection. In view of the definition of "close", the problem is to show that whenever $d_\ell^p < d_j^p$ (and hence also $d_\ell^q < d_j^q$), we have both $e_\ell^p < e_j^q$ and $e_\ell^q < e_j^p$. Since $|e_\ell^p - e_\ell^q| < \zeta^2$ and $|e_j^p - e_j^q| < \zeta^2$, it is sufficient that $\zeta^2 < |e_j^p - e_\ell^p|/3$ and $\zeta^2 < |e_j^q - e_\ell^q|/3$; but this follows if we assume that $3(\zeta(p))^2 < |e - e'|$ whenever $e, e' \in \mathrm{ran}(\sigma^p)$ and $e \neq e'$.

Next, we define ψ_N so that for each ℓ, the function $g_N - \psi_N$ is slightly negative near d_ℓ^p and d_ℓ^q. To make sure that $\psi_N \in M_1$: Choose rational $\bar{d}_\ell$ such that $|\bar{d}_\ell - d_\ell^p|, |\bar{d}_\ell - d_\ell^q| < \zeta$. Let $\bar{\gamma}_\ell = \max(g_N(d_\ell^p), g_N(d_\ell^q), g_N(\bar{d}_\ell))$. By (P7), $0 < \bar{\gamma}_\ell < 2 - 2^{-N}$. Then choose a rational r such that $1/\sqrt{\zeta} < r < 2/\sqrt{\zeta}$ and rational γ_ℓ so that $\bar{\gamma}_\ell < \gamma_\ell < 2 - 2^{-N}$ and $\gamma_\ell - \bar{\gamma}_\ell < 2^{-N}/256$, and define:

$$\psi_N(x) = \sum_{\ell < L} \big(\gamma_\ell + 2^{-N}/16\big)\big(1 + r|x - \bar{d}_\ell|\big)^{-1/2} \quad .$$

Then $\psi_N \in \mathcal{AP}_4$ by Corollary 4.8. Clearly, $g_N(\bar{d}_\ell) - \psi_N(\bar{d}_\ell) < \gamma_\ell - (\gamma_\ell + 2^{-N}/16) < 0$, but we wish to assert also that $g_N(x) - \psi_N(x) < 0$ whenever $|x - \bar{d}_\ell| < \zeta$; in particular, for $x = d_\ell^p, d_\ell^q$. Note that always $\zeta(p) < 1$; then, for $|x - \bar{d}_\ell| < \zeta$:

$$\big(1 + r|x - \bar{d}_\ell|\big)^{-1/2} \geq (1 + \big(2/\sqrt{\zeta}\big) \cdot \zeta)^{-1/2} = (1 + 2\sqrt{\zeta})^{-1/2} > 1 - \sqrt{\zeta} \quad .$$

Now, assume that our function $\zeta(p)$ satisfies $\forall d \in \mathrm{dom}(\sigma^p)\, \forall x\, [|x - d| < 2\zeta(p) \to |g_N(x) - g_N(d)| < 2^{-N}/256]$. Then, using $\bar{\gamma}_\ell < \gamma_\ell$:

$$g_N(x) - \psi_N(x) \leq \big(\gamma_\ell + 2^{-N}/256\big) - \big(\gamma_\ell + 2^{-N}/16\big)\big(1 - \sqrt{\zeta}\big)$$

when $|x - \bar{d}_\ell| < \zeta$, so that

$$g_N(x) - \psi_N(x) \le 2^{-N}/256 - 2^{-N}/16 + \big(\gamma_\ell + 2^{-N}/16\big)\sqrt{\zeta} < 0 \quad ;$$

This last $<$ holds if we assume that always $\sqrt{\zeta(p)} < 2^{-N}/256$.

Now, we define $\theta_N^\dagger(x) = \max(0, \psi_N(x) - g_N(x)) + \varepsilon x^2/(x^2+1)$, where ε is a positive rational which is small enough to make the rest of the argument work. Let $g_N^\dagger(x) = g_N(x) - \psi_N(x) + \theta_N^\dagger(x)$, which is positive everywhere except at 0. Let $f_N^\dagger(x) = \int_0^x g_N^\dagger(t)\,dt$. We shall eventually show that $(\sigma^p \cup \sigma^q, g_N^\dagger, f_N^\dagger, 2^{-N-2})$ is correctable.

(P11) requires $\|\theta_N\| \le 2^{-N-1}$. To accomplish this, we first verify that $\|\theta_N^\dagger\| \le 2^{-N-2}$, and later we shall verify that $\|\theta_N^*\| \le 2^{-N-2}$. As long as $\varepsilon \le 2^{-N-2}$, $\theta_N^\dagger(x) \le 2^{-N-2}$ whenever $\psi_N(x) \le g_N(x)$, which holds as $x \to \pm\infty$ since $g_N(x) \to 1$ and $\psi_N(x) \to 0$. Also, $\theta_N^\dagger(x) \le \psi_N(x) + \varepsilon$, so that if $\varepsilon \le 2^{-N-3}$, then $\theta_N^\dagger(x) \le 2^{-N-2}$ whenever $\psi_N(x) \le 2^{-N-3}$, and if ζ is large enough, this will hold unless x is very close to one of the $\bar{d}_\ell$. More precisely, if $|x - \bar{d}_\ell| \ge c$ for all ℓ, then $\psi_N(x) \le 2L\big(1 + rc\big)^{-1/2} < 2Lr^{-1/2}c^{-1/2}$. Then, using $1/\sqrt{\zeta} < r < 2/\sqrt{\zeta}$, if $|x - \bar{d}_\ell| \ge \zeta^{1/4}$ for all ℓ then

$$\psi_N(x) < 2L\zeta^{1/4}\zeta^{-1/8} = 2L\zeta^{1/8} \quad .$$

Then $\theta_N^\dagger(x) \le 2^{-N-2}$ for these x provided we assume that our ζ function satisfies $2|\sigma^p| \cdot (\zeta(p))^{1/8} \le 2^{-N^p-3}$.

Now, fix x and assume that $|x - \bar{d}_m| \le \zeta^{1/4}$ for some m; this m will be unique if we assume that $(\zeta(p))^{1/4} < \mu(p)/4$ for all p. We need to show that $\theta_N^\dagger(x) \le 2^{-N-2}$. Assume that $\psi_N(x) > g_N(x)$, since we have already covered the case that $\psi_N(x) \le g_N(x)$. So,

$$\begin{aligned}
&\theta_N^\dagger(x) = \psi_N(x) - g_N(x) + \varepsilon x^2/(x^2+1) \le \\
&\qquad \varepsilon + \big(\gamma_m + 2^{-N}/16\big)\big(1 + r|x - \bar{d}_m|\big)^{-1/2} + \\
&\qquad \sum_{\ell \ne m} \big(\gamma_\ell + 2^{-N}/16\big)\big(1 + r|x - \bar{d}_\ell|\big)^{-1/2} - g_N(x) \le \\
&\varepsilon + \big(\gamma_m + 2^{-N}/16\big) + 2^{-N-4} - g_N(x) = \varepsilon + (\gamma_m - g_N(x)) + 2^{-N-2}/2 \quad ;
\end{aligned}$$

for the last $\le$, use the previous argument, but now assuming that our ζ function satisfies $2|\sigma^p| \cdot (\zeta(p))^{1/8} \le 2^{-N^p-4}$. Assuming that $\varepsilon \le 2^{-N-2}/4$ and $\gamma_m - g_N(x) \le 2^{-N-2}/4$ we have $\theta_N^\dagger(x) \le 2^{-N-2}$. Since $\bar{\gamma}_m \in \{g_N(d_m^p), g_N(d_m^q), g_N(\bar{d}_m)\}$ and $\bar{\gamma}_m < \gamma_m < \bar{\gamma}_m + 2^{-N}/256$, and x is within $2\zeta^{1/4}$ of each of $d_m^p, d_m^q, \bar{d}_m$, we obtain $\gamma_m - g_N(x) \le 2^{-N-2}/4$ if we assume that $\forall x\, \forall d \in \mathrm{dom}(\sigma^p)\, [|x - d| \le 2(\zeta(p))^{1/4} \to |g_N(x) - g_N(d)| \le 2^{-N-2}/16$.

We next show that $(\sigma^p, g_N^\dagger, f_N^\dagger, 2^{-N-2})$ and $(\sigma^q, g_N^\dagger, f_N^\dagger, 2^{-N-2})$ are correctable. To do this, we bound the change in $f_N(d)$ caused by replacing g_N by $g_N - \psi_N + \theta_N^\dagger$; this change is $\int_0^d (\psi_N(t) - \theta_N^\dagger(t))\, dt$. Let Δ be the diameter of $\mathrm{dom}(\sigma^p)$. Then, since $\gamma_\ell + 2^{-N}/16 < 2$ and $r > 1/\sqrt{\zeta}$

$$\int_0^d \psi_N(t)\, dt < 2L \int_0^\Delta (1 + rt)^{-1/2}\, dt = \frac{4L}{r}\left[\sqrt{1 + r\Delta} - 1\right] \leq \frac{4L}{r}\sqrt{r\Delta} \leq 4L\sqrt{\Delta}\sqrt[4]{\zeta}\ .$$

This can be made arbitrarily small by requiring the ζ function to be small enough. Likewise, $\int_0^d \theta_N^\dagger(t)\, dt$ can be made arbitrarily small using $0 \leq \theta_N^\dagger(t) \leq \psi_N(t) + \varepsilon$. So, the correctability of $(\sigma^p, g_N^\dagger, f_N^\dagger, 2^{-N-2})$ and $(\sigma^q, g_N^\dagger, f_N^\dagger, 2^{-N-2})$ follows from the correctability of $(\sigma^p, g_N, f_N, 2^{-N-2})$ and $(\sigma^q, g_N, f_N, 2^{-N-2})$ if ζ makes $f_N^\dagger$ close enough to f_N.

Now, to verify that $(\sigma^p \cup \sigma^q, g_N^\dagger, f_N^\dagger, 2^{-N-2})$ is correctable, we must show that ($\check{\text{P}}$13) holds between any two elements of $\mathrm{dom}(\sigma^p \cup \sigma^q)$. There are two cases not already covered by the above:

Case I: Between d_m^p and d_ℓ^q where $m \neq \ell$: We need

$$0 \ < \ \frac{e_m^p - e_\ell^q}{d_m^p - d_\ell^q} - \frac{f_N^\dagger(d_m^p) - f_N^\dagger(d_\ell^q)}{d_m^p - d_\ell^q} \ < \ 2^{-N-2} \quad .$$

This is handled by making ζ small enough, since the inequality holds if we replace d_ℓ^q, e_ℓ^q by d_ℓ^p, e_ℓ^p.

Case II: Between d_ℓ^p and d_ℓ^q, when $d_\ell^p \neq d_\ell^q$. WLOG, $d_\ell^p < d_\ell^q$, and we need

$$0 \ < \ \frac{e_\ell^q - e_\ell^p}{d_\ell^q - d_\ell^p} - \frac{f_N^\dagger(d_\ell^q) - f_N^\dagger(d_\ell^p)}{d_\ell^q - d_\ell^p} \ < \ 2^{-N-2} \quad .$$

By the definition of "close", we have $0 < (e_\ell^q - e_\ell^p)/(d_\ell^q - d_\ell^p) < \zeta$, and our assumptions above about ζ already imply that $\zeta < 2^{-N-2}$. Thus, it is sufficient to have $f_N^\dagger(d_\ell^q) - f_N^\dagger(d_\ell^p) < e_\ell^q - e_\ell^p$. Now we have already checked that $g_N(x) - \psi_N(x) < 0$ for $x \in [d_\ell^p, d_\ell^q]$, so that $g_N^\dagger(x) = \varepsilon x^2/(x^2+1)$ for these x. Then $f_N^\dagger(d_\ell^q) - f_N^\dagger(d_\ell^p) = \varepsilon \int_{d_\ell^p}^{d_\ell^q} x^2/(x^2+1)\, dx < \varepsilon(d_\ell^q - d_\ell^p)$, which will be less than $e_\ell^q - e_\ell^p$ if we have chosen a small enough ε.

Since $(\sigma^p \cup \sigma^q, g_N^\dagger, f_N^\dagger, 2^{-N-2})$ is correctable, Lemma 5.5 gives us a positive function $\theta_N^\#$ such that $\|\theta_N^\#\| < 2^{-N-2}$ and such that, setting $g_{N+1}^\# = g_N^\dagger + \theta_N^\#$ and integrating, we get $f_{N+1}^\# \supset \sigma^s := \sigma^p \cup \sigma^q$; so, instead of (P13) for s we have, for $(d_0, e_0), (d_1, e_1) \in \sigma^s$ and $d_0 < d_1$:

$$(e_1 - e_0) - (f_{N+1}^\#(d_1) - f_{N+1}^\#(d_0)) = 0 \quad .$$

This is not exactly what we want, and this $\theta_N^{\#}$ need not be in M_1, but by modifying our $\theta_N^{\#}$ slightly, we can get $\theta_N^* \in M_1$ so that setting $g_{N+1}^s = g_N^\dagger + \theta_N^*$ and integrating we get f_{N+1}^s satisfying

$$0 < (e_1 - e_0) - (f_{N+1}^s(d_1) - f_{N+1}^s(d_0)) < 2^{-N-3}(d_1 - d_0) \quad ,$$

which is (P13) for the forcing condition s, so that $s \in \mathbb{P}$.

Of course, we also need to verify that $s \leq p$ and $s \leq q$. (Q1) and (Q2) are trivial, but (Q3) requires $g_{N+1}^s(d) \in (0, 2^{-N-1})$ for $d \in \mathrm{dom}(\sigma^p) \cup \mathrm{dom}(\sigma^q) \backslash \{0\}$. Now $g_{N+1}^s = g_N^\dagger + \theta_N^*$, and we already know that $g_N^\dagger(d) = \varepsilon d^2/(d^2+1) < \varepsilon$, and we already assumed that $\varepsilon \leq 2^{-N-2}$. So, when we apply Lemma 5.5, get $\theta_N^{\#}(d) = 0$ for these d. Then, when we modify $\theta_N^{\#}$ slightly to get θ_N^*, make sure that $\theta_N^*(d) \in (0, 2^{-N-2})$. This, plus $g_N^\dagger(d) \in (0, 2^{-N-2})$ implies that $g_{N+1}^s(d) \in (0, 2^{-N-1})$.

We remark on the relationship between this proof and Baumgartner's proof in [3, 4], which forced an order-preserving bijection f from D onto E. Since f^{-1} is also an order-preserving bijection, there is a symmetry between D and E in the forcing. Specifically, in his forcing, our (P3) is replaced by the requirement that $\mathrm{ht}(e) \neq \mathrm{ht}(d)$ and that they differ by a finite ordinal. Our (P4) is replaced by the requirement that $\max(\mathrm{ht}(d_0), \mathrm{ht}(e_0)) \neq \max(\mathrm{ht}(d_1), \mathrm{ht}(e_1))$. The forcing condition is just the σ alone; there is no need for an (N, g, f, ψ, θ) part.

But requiring f to be differentiable breaks the symmetry, since $f'(d)$ must be 0 at many places (see Theorem 1.5), so that f^{-1} is not differentiable. Then, getting the derivative to be small seems to require that $\mathrm{ht}(e) < \mathrm{ht}(d)$ in (P3) so that the proof of ccc (via Lemma 5.3) works, which leads to the domain of the order-preserving map being not all of D.

References

[1] U. Abraham, M. Rubin, and S. Shelah, On the consistency of some partition theorems for continuous colorings, and the structure of $\aleph_1$-dense real order types, *Ann. Pure Appl. Logic* 29 (1985) 123-206.

[2] U. Avraham and S. Shelah, Martin's axiom does not imply that every two $\aleph_1$-dense sets of reals are isomorphic, *Israel J. Math.* 38 (1981) 161-176.

[3] J. Baumgartner, All $\aleph_1$-dense sets of reals can be isomorphic. *Fund. Math.* 79 (1973) 101-106.

[4] J. Baumgartner, Applications of the proper forcing axiom, in *Handbook of Set-Theoretic Topology*, North-Holland, 1984, pp. 913-959.

[5] M. R. Burke, Entire functions mapping uncountable dense sets of reals onto each other monotonically. *Trans. Amer. Math. Soc.* 361 (2009) 2871-2911.

[6] J. Hart and K. Kunen, Arcs in the plane, to appear, *Topology Appl.*, or see arXiv:0906.2753v1 .

[7] E. W. Hobson, *The Theory of Functions of a Real Variable and the Theory of Fourier's Series*, Vol. 2, 2$^{\text{nd}}$ Edition, The University Press, 1926.

[8] Y. Katznelson and K. Stromberg, Everywhere differentiable, nowhere monotone, functions, *Amer. Math. Monthly* 81 (1974) 349-354.

[9] W. Kubiś and B. Vejnar, Covering an uncountable square by countably many continuous functions, arXiv:0710.1402v3, 2009.

[10] K. Kunen, Locally connected hereditarily Lindelöf compacta, to appear, *Topology Appl.*, or see arXiv:0808.2795v1 .

[11] G. Peano, Sur une courbe, qui remplit toute une aire plane, *Math. Ann.* 36 (1890) 157-160.

[12] W. Sierpiński, *Hypothèse du Continu*, Chelsea Publishing Company, 1956.