Some Points in Spaces of Small Weight *

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Abstract

There is a compact 0-dimensional Hausdorff space X of weight $\aleph_1$ with an $x \in X$ which is a weak P-point and not a P-point. There is a zero-dimensional $L_{\aleph_1}$ space X of weight and cardinality $\aleph_2$, with a non-isolated weak $P_{\aleph_2}$-point to which no discrete subset of X accumulates.

1 Introduction

In this paper, we obtain two examples of spaces of weight κ^+ where the known example from the literature has weight 2^κ. Both examples involve weak P_{κ^+}-points that are not P_{κ^+}-points:

Definition 1.1 *For a point x in the topological space X:*

1. *$x \in X$ is a* P_κ-point *in X iff the intersection of any family of* fewer than *κ neighbourhoods of x is also a neighbourhood of x.*
2. *$x \in X$ is a* weak P_κ-point *in X iff x is not a limit point of any subset of $X \backslash \{x\}$ of size* less than κ.
3. *"P-point" and "weak P-point" mean "$P_{\aleph_1}$-point" and "weak $P_{\aleph_1}$-point", respectively.*

So, in any T_1 space, every P_κ-point is a weak P_κ-point. If $w(X) = \aleph_0$, then every weak P-point is isolated, whereas the ordinal $\omega_1 + 1$ is an example of a space of weight $\aleph_1$ with a non-isolated P-point. In Section 2, we shall show:

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Theorem 1.2 *There is a compact 0-dimensional Hausdorff space X of weight $\aleph_1$ with an $x \in X$ which is a weak P-point and not a P-point.*

By [7], there is an example of weight $2^{\aleph_0}$, taking $X = \mathbb{N}^* = \beta\mathbb{N}\backslash\mathbb{N}$. To prove Theorem 1.2 in *ZFC*, we shall apply an elementary submodel argument to this $x \in \mathbb{N}^*$; see Dow [3] for more on such arguments. The point in [7] was a weak P-point *because* it was ω_1-OK (see Definition 2.2). After applying the elementary submodel, the x from Theorem 1.2 will be ω_1-soso, a weaker property which still implies "weak P-point". The strengthening of Theorem 1.2 in which x is actually ω_1-OK is independent of $ZFC + \neg CH$ (see Theorems 2.7 and 2.8).

The following is easy to prove (see, e.g., [4]):

Proposition 1.3 *If X is compact Hausdorff and $x \in X$ is not isolated, then x is the accumulation point of some discrete subset.*

So, the x of Theorem 1.2 must be a limit of a discrete subset of size $\aleph_1$. However, Proposition 1.3 fails in non-compact spaces:

Theorem 1.4 (van Douwen [1]) *There is a countable 0-dimensional Hausdorff space X of weight $2^{\aleph_0}$ with a non-isolated point p to which no discrete subset of X accumulates.*

In fact, in this example, X was countable and dense in itself, but every discrete subspace of X was closed, so p could be any point of X.

Again, one can ask if the X of Theorem 1.4 can have weight $\aleph_1$. It can, assuming an L space:

Definition 1.5 *X is an L_κ space iff X is T_3 and hereditarily κ-Lindelöf but not hereditarily κ-separable. An L space is an L_ω space.*

So, an L space is hereditarily Lindelöf but not hereditarily separable. In Section 3, we shall show:

Theorem 1.6 *If there is a 0-dimensional L_κ space, then there is a 0-dimensional L_κ space X of weight and cardinality κ^+, with a non-isolated point p to which no discrete subset of X accumulates. Furthermore, p is a weak P_{κ^+}-point.*

Some remarks:

The p in Theorem 1.6 cannot be a P_{κ^+}-point, since in a T_3 hereditarily κ-Lindelöf space, every point is the intersection of at most κ of its neighbourhoods.

For $\kappa = \omega$, the X in Theorem 1.6 cannot be countable, as it is in Theorem 1.4, since under MA, every non-isolated point in a countable T_2 space of weight less that $2^{\aleph_0}$ is a limit of a discrete ω-sequence.

For $\kappa = \omega$, it is still unknown whether there is an L space in ZFC, although there is one in every known model of set-theory. Theorem 1.6 for $\kappa = \omega$ was proved in [4] by a different method which does not seem to generalise to arbitrary κ. As is well known (see e.g. [5] or [8]), the existence of an L space implies that of a 0-dimensional one of weight ω_1. It is not clear whether this generalises to arbitrary L_κ spaces, although once one has a 0-dimensional L_κ space, one easily gets one of weight κ^+ (see Section 3).

For $\kappa = \omega_1$: The existence of a 0-dimensional L_{ω_1} space is provable in ZFC, using Shelah's colouring theorem; see [9] and Theorem 1.11 of [6]. Thus:

Corollary 1.7 *There is a 0-dimensional L_{ω_1} space X of weight and cardinality ω_2, with a non-isolated point p to which no discrete subset of X accumulates. Furthermore, p is a weak P_{ω_2}-point.*

2 Some Flavours of Weak P-Points

As stated in the Introduction, we plan to start with an $x \in \mathbb{N}^*$ which is a weak P-point and not a P-point, and take an elementary submodel. To compare x in the universe, V, with x in the submodel, it is simpler to view $\mathbb{N}^*$ as a Stone space. If $\mathcal{A}$ is a boolean algebra, let $\mathrm{st}(\mathcal{A})$ denotes its Stone space; so $x \in \mathrm{st}(\mathcal{A})$ iff x is an ultrafilter on $\mathcal{A}$. The clopen sets of $\mathrm{st}(\mathcal{A})$ are all of the form $N_a = \{x \in \mathrm{st}(\mathcal{A}) : a \in x\}$, for $a \in \mathcal{A}$, so $w(\mathrm{st}(\mathcal{A})) = |\mathcal{A}|$ whenever $\mathcal{A}$ is infinite. $\mathbb{N}^* = \mathrm{st}(\mathcal{P}(\omega)/\mathit{fin})$, where $\mathit{fin} \subset \mathcal{P}(\omega)$ denotes the ideal of finite sets.

Suppose that $x \in \mathrm{st}(\mathcal{A})$ and $x, \mathcal{A} \in M \prec H(\theta)$. Then $x \cap M$ is an ultrafilter on the boolean algebra $\mathcal{A} \cap M$; that is $(x \cap M) \in \mathrm{st}(\mathcal{A} \cap M)$. If $|M| = \aleph_1$, then $w(\mathrm{st}(\mathcal{A} \cap M)) \le \aleph_1$. Now, we need to relate properties of $x \in \mathrm{st}(\mathcal{A})$ to properties of $(x \cap M) \in \mathrm{st}(\mathcal{A} \cap M)$. The property "not a P-point" is easy; M must contain an ω-sequence $\langle N_{a_n} : n \in \omega \rangle$ which refutes "P-point", so:

Lemma 2.1 *If $x \in \mathrm{st}(\mathcal{A})$ is not a P-point in $\mathrm{st}(\mathcal{A})$ and $x, \mathcal{A} \in M \prec H(\theta)$, then $x \cap M$ is not a P-point in $\mathrm{st}(\mathcal{A} \cap M)$.*

However, the property "weak P-point" is trickier. Suppose that $x \in \mathrm{st}(\mathcal{A})$ is a weak P-point in $\mathrm{st}(\mathcal{A})$ and is not isolated (i.e., is not a principal ultrafilter generated by an atom). If M is countable, then $\mathrm{st}(\mathcal{A} \cap M)$ will be second countable and hence separable, so that $x \cap M$ will not be a weak P-point in $\mathrm{st}(\mathcal{A} \cap M)$. Even when $|M| = \aleph_1$, if $MA + \neg CH$ holds and $\mathcal{A}$ has the countable chain condition (ccc), then $\mathrm{st}(\mathcal{A} \cap M)$ will still be separable, so that $x \cap M$ will again fail to be a weak P-point. Furthermore, there are many examples of such $x, \mathcal{A}$, since under MA there are weak P-points in $\mathrm{st}(\mathcal{A})$ whenever $\mathcal{A}$ is complete and ccc and $\mathrm{st}(\mathcal{A})$ is not separable (see Dow [2], Theorems 2.3 and 3.2).

Thus, if this plan for proving Theorem 1.2 is to work, we must use a property of x which implies weak P-point and which is incompatible with the ccc. So, we turn to OK points:

Definition 2.2 *For a point x in a space X:*

1. *A sequence of neighbourhoods of x, $\langle U_n : n \in \omega \rangle$, is an* ω_1-OK sequence *iff there are neighbourhoods V_α of x for $\alpha < \omega_1$ such that for all $n \geq 1$ and all $\alpha_1 < \cdots < \alpha_n < \omega_1$, we have $V_{\alpha_1} \cap \cdots \cap V_{\alpha_n} \subseteq U_n$.*
2. *x is* ω_1-OK *in X iff every ω-sequence of neighbourhoods of x is ω_1-OK.*
3. *x is* ω_1-soso *in X iff for every countable family $\mathcal{W}$ of neighbourhoods of X, there is an ω_1-OK sequence of neighbourhoods of x, $\langle U_n : n \in \omega \rangle$, such that $\mathcal{W} \subseteq \{U_n : n \in \omega\}$.*

Clearly, ω_1-OK implies ω_1-soso. The notion of "ω_1-OK" is from [7], and was used there to produce weak P-points in $\mathbb{N}^*$. Unfortunately (see Theorem 2.8), it is consistent with ZFC that for all compact X of weight $\aleph_1$, every ω_1-OK point in X is already a P-point. Thus, we turn to the more complicated notion of "ω_1-soso" to prove Theorem 1.2. We remark that no ccc T_3 space can have a non-isolated ω_1-soso point; the proof is the same as the one in [7] for OK points.

Lemma 2.3 *If x is ω_1-soso in X and H is a G_δ set containing x, then there are neighbourhoods V_α of x for $\alpha < \omega_1$ such that $\bigcap_{n\in\omega} V_{\alpha_n} \subseteq H$ whenever the $\alpha_n < \omega_1$ are distinct.*

Proof. Apply the definition, 2.2.3, to any $\mathcal{W}$ such that $\bigcap \mathcal{W} = H$. □

Lemma 2.4 *If X is a T_1 space and $x \in X$ is ω_1-soso, then x is a weak P-point.*

Proof. If Y is a countable subset of $X \backslash \{x\}$, let $H = X \backslash Y$. If the V_α are neighbourhoods of x as in Lemma 2.3, then all but countably many V_α are disjoint from Y, so $x \notin \overline{Y}$. □

We call $M \prec H(\theta)$ *ω-covering* iff for all countable $E \subseteq M$, there is a countable $F \in M$ such that $E \subseteq F$. Such an M of size $\aleph_1$ is easily produced as a union of an elementary chain (see [3], §3).

Lemma 2.5 *Assume that $x \in \mathrm{st}(\mathcal{A})$ is ω_1-soso in $\mathrm{st}(\mathcal{A})$ and that $x, \mathcal{A} \in M \prec H(\theta)$, where M is ω-covering. Then $x \cap M$ is ω_1-soso in $\mathrm{st}(\mathcal{A} \cap M)$.*

Proof. Let $\mathcal{W} = \{W_i : i \in \omega\}$ be a family of neighbourhoods of $x \cap M$ in $\mathrm{st}(\mathcal{A} \cap M)$. Choose $e_i \in x \cap M$ such that $N_{e_i} \subseteq W_i$. Then, fix a countable $F \in M$ such that $\{e_i : i \in \omega\} \subseteq F$. Since $x \in M$, we may assume (intersecting with x) that $F \subseteq x \cap M$. Now, apply the definition of "soso" to $\mathcal{W}' = \{N_a : a \in F\}$. $\square$

Proof of Theorem 1.2. Apply Lemmas 2.1 and 2.5 with $\mathcal{A} = \mathcal{P}(\omega)/\mathit{fin}$ and x any ω_1-OK point in $\mathrm{st}(\mathcal{A})$ which is not a P-point (see [7]). Then in $\mathrm{st}(\mathcal{A} \cap M)$, the point $x \cap M$ is not a P-point, but is ω_1-soso, and hence a weak P-point. $\square$

Now, whether Theorem 1.2 can hold with x an ω_1-OK point depends on the model of set theory. As usual, $\mathfrak{d}$ denotes the least size of a dominating family in ω^ω, and $\mathfrak{b}$ denotes the least size of an unbounded family; so $\aleph_1 \leq \mathfrak{b} \leq \mathfrak{d} \leq 2^{\aleph_0}$. We can modify the proof of Lemma 2.5 to get:

Lemma 2.6 *Assume that $x \in \mathrm{st}(\mathcal{A})$ is ω_1-OK and $x, \mathcal{A} \in M \prec H(\theta)$, where M is ω-covering and $(\omega^\omega) \cap M$ is cofinal in ω^ω. Then $x \cap M$ is ω_1-OK in $\mathrm{st}(\mathcal{A} \cap M)$.*

Proof. Now, we start with a sequence, $\langle N_{a_n} : n \in \omega \rangle$, of neighbourhoods of $x \cap M$; so each $a_n \in x \cap M$. We need to get $b_\alpha \in x \cap M$ for $\alpha < \omega_1$ such that $b_{\alpha_1} \wedge \cdots \wedge b_{\alpha_n} \leq a_n$ whenever $n \geq 1$ and $\alpha_1 < \cdots < \alpha_n < \omega_1$.

Since M is ω-covering, we can get a sequence $\langle c_n : n \in \omega \rangle \in M$ such that each $c_n \in x \cap M$ and each $a_n = c_{\varphi(n)}$ for some $\varphi : \omega \to \omega$. Fix $\psi \in \omega^\omega \cap M$ such that $\varphi(n) \leq \psi(n)$ for all n. Note that $\omega_1 \subset M$ since M is ω-covering. Since $\psi \in M$, we can, in M, apply the definition of ω_1-OK to the sequence $\langle c_0 \wedge c_1 \wedge \cdots \wedge c_{\psi(n)} : n \in \omega \rangle$ to get $b_\alpha \in x \cap M$ for $\alpha < \omega_1$ such that for all $n \geq 1$ and all $\alpha_1 < \cdots < \alpha_n < \omega_1$, we have $b_{\alpha_1} \wedge \cdots \wedge b_{\alpha_n} \leq c_0 \wedge c_1 \wedge \cdots \wedge c_{\psi(n)}$, and hence $b_{\alpha_1} \wedge \cdots \wedge b_{\alpha_n} \leq c_{\varphi(n)} = a_n$. $\square$

In particular, if $\mathfrak{d} = \aleph_1$ then we can get $|M| = \aleph_1$. Hence, analogously to Theorem 1.2, we have:

Theorem 2.7 *If $\mathfrak{d} = \aleph_1$, then there is a compact Hausdorff space X of weight $\aleph_1$ with an ω_1-OK point which is not a P-point.*

We do not know if the converse to this theorem holds, but the hypothesis cannot be weakened to "$\mathfrak{b} = \aleph_1$":

Theorem 2.8 *Assume that $V[G]$ is an extension of V by $\geq \aleph_2$ Cohen reals. Then in $V[G]$:*

1. $\mathfrak{b} = \omega_1$.
2. *In every compact Hausdorff space X of weight $\aleph_1$ every ω_1-OK point is a P-point.*

Proof.(1) holds because the first $\aleph_1$ Cohen reals yield an unbounded family of size $\aleph_1$. For (2):

First, work in $V[G]$: Assume that $z \in X$ is not a P-point. We must show that it is not ω_1-OK. Following Tychonov, we may assume that X is a closed subspace of $[-1,1]^{\omega_1}$, that $z = \vec{0}$ (the identically 0 sequence), and that "P-point" is refuted by the neighbourhoods $U_n = \{x \in X : |x_0| < 2^{-n}\}$; that is, $\vec{0}$ is a boundary point of the set $\{x \in X : x_0 = 0\}$ in X. Let $D \subseteq X$ be dense in X, with $|D| \leq \aleph_1$.

Now, since $|D| \leq \aleph_1$, it depends on at most $\aleph_1$ of the Cohen reals, so by the usual splitting argument, we may (and shall) assume that $D \in V$.

In V: Let $\{B_\alpha : \alpha < \omega_1\}$ enumerate a local base for $\vec{0}$ in $[-1,1]^{\omega_1}$. Assume that each B_α is a finitely supported product of rational intervals of the form $(-r, r)$, and that $B_n = \{x \in [-1,1] : |x_0| < 2^{-n}\}$ for $n < \omega$. Then, in $V[G]$, and hence also in V, $\forall \beta \, \exists i < \omega \, [B_\beta \cap D \not\subseteq B_i]$.

Again by splitting, $V[G] = V[f][H]$, where $f \in \omega^\omega$ is generic over V using the partial order $\mathbb{P} = Fn(\omega, \omega)$, and H adds the rest of the Cohen reals, via some $\mathbb{Q} = Fn(\kappa, \omega)$. We shall show that in $V[G]$, the neighbourhoods $U_n = B_{f(n)} \cap X$ establish that $\vec{0}$ is not ω_1-OK. To do this, it is sufficient to show that there is no S such that $\Phi(f, S)$ holds, where $\Phi(f, S)$ asserts:

$$S \subseteq \omega_1 \ \& \ |S| = \aleph_1 \ \& \\ \forall n \geq 1 \, \forall \alpha_1 < \cdots < \alpha_n \, [\{\alpha_1, \ldots, \alpha_n\} \subseteq S \to B_{\alpha_1} \cap \cdots \cap B_{\alpha_n} \cap D \subseteq B_{f(n)}]$$

If, in $V[G]$, there is an S satisfying $\Phi(f, S)$, then, working in $V[f]$, there is a $\mathbb{Q}$-name $\dot{S}$ and a $q \in \mathbb{Q}$ such that $q \Vdash \Phi(f, \dot{S})$. Then, in $V[f]$, we can find an uncountable $T \subseteq \omega_1$ and $q_\alpha \leq q$ for $\alpha \in T$ such that each $q_\alpha \Vdash [\alpha \in \dot{S}]$. Furthermore, shrinking T, we may assume that $\{q_\alpha : \alpha \in T\}$ is centred, which implies that $\Phi(f, T)$ holds in $V[f]$. Furthermore, since $\mathbb{P}$ is countable, there will be an uncountable subset of T in V. Thus, shrinking T again, we may assume that $T \in V$.

Retreating to V, we have a $\mathbb{P}$-name $\dot{f}$ for the generic function and a $p \in \mathbb{P}$ such that $p \Vdash \Phi(\dot{f}, T)$. Assume that $\mathrm{dom}(p) \subseteq n$, fix $\alpha_1 < \cdots < \alpha_n \in T$, and fix β with $B_\beta \subseteq B_{\alpha_1} \cap \cdots \cap B_{\alpha_n}$. Then $p \Vdash [B_\beta \cap D \subseteq B_{\dot{f}(n)}]$, but p does not determine the value of $\dot{f}(n)$. Fix i so that $B_\beta \cap D \not\subseteq B_i$, and let $p' \leq p$ with $p' \Vdash \dot{f}(n) = i$. Then $p' \Vdash [B_\beta \cap D \subseteq B_i]$, a contradiction. $\square$

3 Proof of Theorem 1.6

Let Z be our L_κ space. Since Z is not hereditarily κ-separable, it contains a sequence which is left separated in type κ^+. But then we may assume without loss of generality that this sequence is all of Z, so that $Z = \langle \kappa^+, \tau \rangle$, where τ is some topology on κ^+. "Left separated" means that every initial segment $\alpha \in \kappa^+$ is closed in Z, so each final segment $\kappa^+ \setminus \alpha$ is open. Since Z is 0-dimensional and hereditarily κ-Lindelöf, we can write:

$$\kappa^+ \setminus \alpha = \bigcup \{U^\alpha_\xi : \xi \in \kappa\} \quad ,$$

where U^α_ξ is clopen in Z for every $\xi \in \kappa$. Let τ_0 be the coarser topology with a base consisting of all finite boolean combinations from $\mathcal{U} = \{U^\alpha_\xi : \xi \in \kappa \;\&\; \alpha \in \kappa^+\}$. Then τ_0 is Hausdorff (because $\mathcal{U}$ separates points), hereditarily κ-Lindelöf (because it is coarser than τ), and not hereditarily κ-separable (because it is still left separated). But then, we may assume that $\tau_0 = \tau$, so that Z has weight only κ^+.

Let $Y = [\kappa]^{<\omega} \times Z$, where $[\kappa]^{<\omega}$ is discrete, so that Y is a topological sum of κ copies of Z. For $E \subseteq Y$ and $a \in [\kappa]^{<\omega}$, let $E_a = \{\alpha : (a, \alpha) \in E\}$. Our space X will be $Y \cup \{p\}$ where $p \notin Y$ and Y is an open subspace of X. So, the topology on X is defined once we define the neighbourhoods of p in X. To this end, for any $\alpha \in \kappa^+$ define $W^\alpha \subseteq Y$ such that for each $a \in [\kappa]^{<\omega}$

$$(W^\alpha)_a = \bigcup \{U^\alpha_\xi : \xi \in a\} \quad .$$

Now let $\{W^\alpha \cup \{p\} : \alpha \in \kappa^+\}$ be a neighbourhood subbase of p in X. Each W^α is clopen in Y (because each $(W^\alpha)_a$ is clopen in Z), so that X is 0-dimensional. Also, it is easy to see that Y and X are both L_κ spaces of weight κ^+.

Next to show that p is non-isolated in X, we fix any $\alpha_1, \ldots, \alpha_n \in \kappa^+$ and show that $|W^{\alpha_1} \cap \ldots \cap W^{\alpha_n}| = \kappa^+$. To do this, fix any $\beta \in \kappa^+ \setminus \max\{\alpha_1, \ldots, \alpha_n\}$. Then, for every $1 \le i \le n$, choose $\xi_i \in \kappa$ with $\beta \in U^{\alpha_i}_{\xi_i}$. Let $a = \{\xi_1, \ldots, \xi_n\}$. Then, by definition, we have $\beta \in (W^{\alpha_i})_a$ for every i, so $(a, \beta) \in W^{\alpha_1} \cap \ldots \cap W^{\alpha_n}$.

Finally, p is a weak P_{κ^+}-point in X because for every set $S \in [Y]^{\le \kappa}$ there is some $\alpha \in \kappa^+$ with $S \subseteq [\kappa]^{<\omega} \times \alpha$, so that $S \cap W^\alpha = \emptyset$. Then, since Y is hereditarily κ-Lindelöf, every discrete $D \subseteq Y$ has size $\le \kappa$, so that $p \notin \overline{D}$. $\square$

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