# SMALL LOCALLY COMPACT LINEARLY LINDELÖF SPACES

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Abstract. There is a locally compact Hausdorff space of weight $\aleph_\omega$ which is linearly Lindelöf and not Lindelöf.

We shall prove:

**Theorem 1.** *There is a compact Hausdorff space $X$ and a point $p$ in $X$ such that:*

1. $\chi(p, X) = w(X) = \aleph_\omega$.
2. *For all regular $\kappa > \omega$, no $\kappa$-sequence of points distinct from $p$ converges to $p$.*

As usual, $\chi(p, X)$, the *character* of $p$ in $X$, is the least size of a local base at $p$, and $w(X)$, the *weight* of $X$, is the least size of a base for $X$. This theorem with "$\beth_\omega$" replacing "$\aleph_\omega$" was proved in [11]. Arhangel'skii and Buzyakova [1] point out that if $X, p$ satisfy (2) of the theorem, then the space $X\backslash\{p\}$ is linearly Lindelöf and locally compact; if in addition $\chi(p, X) > \aleph_0$, then $X\backslash\{p\}$ is not Lindelöf. (2) requires $\mathrm{cf}(\chi(p, X)) = \omega$, because there must be a sequence of type $\mathrm{cf}(\chi(p, X))$ converging to $p$. Thus, in (1) of the theorem, $\aleph_\omega$ is the smallest possible uncountable value for $\chi(p, X)$ and $w(X)$.

As in [11], the $X$ of the theorem will be constructed as an inverse limit, using the following terminology:

**Definition 2.** *An* inverse system *is a sequence $\langle X_n, \pi_n^{n+1} : n \in \omega\rangle$, where each $X_n$ is a compact Hausdorff space, and each $\pi_n^{n+1}$ is a continuous map from $X_{n+1}$ onto $X_n$.*

Such an inverse systems yields a compact Hausdorff space, $X_\omega = \overleftarrow{\lim}_n X_n$, and maps $\pi_m^\omega : X_\omega \twoheadrightarrow X_m$ for $m < \omega$ and $\pi_m^n : X_n \twoheadrightarrow X_m$ for $m \le n < \omega$. Exactly as in [11], one easily proves:

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**Lemma 3.** *Suppose that* $\langle X_n, \pi_n^{n+1} : n \in \omega\rangle$ *is an inverse system and* $p \in X = X_\omega$*, with the* $p_n = \pi_n^\omega(p) \in X_n$ *satisfying:*

A. *Each* $p_n$ *is a weak* $P_{\aleph_n}$*-point in* $X_n$*.*
B. *Each* $w(X_n) < \aleph_\omega$*.*
C. *Each* $(\pi_0^n)^{-1}\{p_0\}$ *is nowhere dense in* $X_n$*.*

*Then* $X, p$ *satisfies Theorem 1.*

As usual, $y \in Y$ is a *weak* $P_\kappa$-point iff $y$ is not in the closure of any subset of $Y\backslash\{y\}$ of size less than $\kappa$, and $y$ is a $P_\kappa$-point iff the intersection of fewer than $\kappa$ neighborhoods of $y$ is always a neighborhood of $y$. These properties are trivial for $\kappa = \aleph_0$. The terms "$P$-point" and "weak $P$-point" denote "$P_{\aleph_1}$-point" and "weak $P_{\aleph_1}$-point", respectively.

Every $P_\kappa$-point is a weak $P_\kappa$-point, but as pointed out in [11], one cannot have each $p_n$ being a $P_{\aleph_n}$-point, as that would contradict (C). In the construction we describe, it will be natural to make every $p_n$ fail to be a $P$-point in $X_n$.

We shall build the $X_n$ and $p_n$ inductively using the following:

**Lemma 4.** *Assume that* $y \in F \subseteq Y$*, where* $Y$ *is compact Hausdorff,* $w(Y) \le \aleph_n$*, and* $\mathrm{int}(F) = \emptyset$*. Then there is a compact Hausdorff space* $X$*, a point* $x \in X$*, and a continuous* $g : X \to Y$ *such that:*

1. $g(X) = Y$ *and* $g(x) = y$*.*
2. $g^{-1}(F)$ *is nowhere dense in* $X$*.*
3. $w(X) = \aleph_n$*.*
4. *In* $X$*,* $x$ *is a weak* $P_{\aleph_n}$*-point and not a* $P$*-point.*

**Proof of Theorem 1.** Inductively build an inverse system as in Lemma 3, with each $w(X_n) = \aleph_n$. $X_0$ can be the Cantor set. When $n > 0$ and we are given $X_{n-1}, p_{n-1}$, we apply Lemma 4 with $F = (\pi_0^{n-1})^{-1}\{p_0\}$. □

Of course, we still need to prove Lemma 4. We remark that we do not assume that $F$ is closed, although that was true in our proof of Theorem 1. Even if $F$ is dense in $Y$ in Lemma 4, we still get (2) — that is, $\mathrm{int}(\mathrm{cl}(g^{-1}(F))) = \emptyset$.

In Lemma 4, $n$ can be 0, although this case is not used in the proof of Theorem 1. For this case, the "weak $P_{\aleph_0}$-point" is trivial, and the lemma is easily proved by an Aleksandrov duplicate construction. A more convoluted proof is: Let $D \subseteq Y\backslash F$ be dense in $Y$ and countable. Let $g$ map $\omega$ onto $D$ and extend $g$ to a map $\beta g : \beta\omega \twoheadrightarrow Y$. Choosing $x$ to be any point in $(\beta g)^{-1}(\{y\})$ yields (1)(2)(4), but $\beta\omega$ has weight $2^{\aleph_0}$. Now, we can take a countable elementary submodel of the whole construction to get an $X$ of weight $\aleph_0$. Our proof for a general $n$ will follow this pattern.

As usual, $\beta\kappa$ denotes the Čech compactification of a discrete $\kappa$, and $\kappa^* = \beta\kappa\backslash\kappa$. Equivalently, $\beta\kappa$ is the space of ultrafilters on $\kappa$, and $\kappa^*$ is the space of nonprincipal ultrafilters. If $g : \kappa \to Y$, where $Y$ is compact Hausdorff, then $\beta g$ denotes the unique extension of $g$ to a continuous map from $\beta\kappa$ to $Y$. Our weak $P_\kappa$-point in Lemma 4 will be a *good* ultrafilter in the sense of Keisler [9]:

**Definition 5.** *An ultrafilter $x$ on $\kappa$ is* good *iff for all $H : [\kappa]^{<\omega} \to x$, there is a $K : \kappa \to x$ such that $K(\alpha_1) \cap \cdots \cap K(\alpha_n) \subseteq H(s)$ for each $s = \{\alpha_1, \ldots, \alpha_n\} \in [\kappa]^{<\omega}$.*

The following is well-known:

**Lemma 6.** *Let $\kappa$ be any infinite cardinal.*

1. *There are ultrafilters $x$ on $\kappa$ which are both good and countably incomplete.*
2. *Any $x$ as in (1) is a weak $P_\kappa$ point and not a $P$-point in $\beta\kappa$.*

In (2), $x$ is not a $P$-point by countable incompleteness, and proofs that it is a weak $P_\kappa$ point can be found in [2, 3, 5]. For (1), see [4], Theorem 6.1.4; also, [2, 3] construct good ultrafilters with various additional properties.

We first point out (Lemma 9) that taking $x$ to be a good ultrafilter on $\omega_n$ will give us (1)(2)(4) of Lemma 4. Unfortunately, $w(\beta\omega_n) = 2^{\aleph_n}$, so we shall take an elementary submodel to bring the weight down. Omitting the elementary submodel, our argument is as in [11], which obtained the $X$ of Theorem 1 with $w(X) = \beth_\omega$, rather than $\aleph_\omega$. A related use of elementary submodels to reduce the weight occurs in [7].

Before we consider the weight problem, we explain how to map the good ultrafilter onto the given point $y$. This part of the argument works for any regular ultrafilter.

**Definition 7.** *An ultrafilter $x$ on $\kappa$ is* regular *iff there are $E_\alpha \in x$ for $\alpha < \kappa$ such that $\{\alpha : \xi \in E_\alpha\}$ is finite for all $\xi < \kappa$.*

Such an $x$ is countably incomplete because $\bigcap_{n<\omega} E_n = \emptyset$. For the following, see Exercise 6.1.3 of [4] or the proof of Lemma 2.1 in Keisler [10]:

**Lemma 8.** *If $x$ is a countably incomplete good ultrafilter on $\kappa$, then $x$ is regular.*

**Lemma 9.** *Let $x$ be a regular ultrafilter on $\kappa$. Assume that $y \in F \subseteq Y$, where $Y$ is compact Hausdorff, $w(Y) \le \kappa$, and $\mathrm{int}(F) = \emptyset$. Then there is a map $g : \kappa \to Y$ such that:*

A. *$\beta g$ maps $\beta\kappa$ onto $Y$.*
B. *$(\beta g)(x) = y$.*
C. *$g(\xi) \notin F$ for all $\xi \in \kappa$.*
D. *$g^{-1}(F)$ is nowhere dense in $\beta\kappa$.*

**Proof**. Of course, (D) follows from (C) because $g^{-1}(F) \subseteq \kappa^*$. Fix $A \subseteq \kappa$ with $A \notin x$ and $|A| = \kappa$. Let $\{E_\alpha : \alpha < \kappa\}$ be as in Definition 7, with each $E_\alpha \cap A = \emptyset$. Let $\{U_\alpha : \alpha < \kappa\}$ be an open base at $y$ in $Y$. Let $D \subseteq Y \backslash F$ be dense in $Y$ with $|D| \leq \kappa$. Choose $g : \kappa \to Y$ such that $g$ maps $A$ onto $D$ (ensuring (A)) and each $g(\xi) \in \bigcap\{U_\alpha : \xi \in E_\alpha\} \setminus F$ (ensuring (B)(C)). $\square$

To apply the elementary submodel technique (as in Dow [6]), we put the construction of Lemma 9 inside an $H(\theta)$, where $\theta$ is a suitably large regular cardinal. Let $M \prec H(\theta)$, with $\kappa \subset M$ and $|M| = \kappa$, such that $M$ contains $Y$ and its topology $\mathcal{T}$, along with $F, g, x, y$. Let $\mathcal{B} = \mathcal{P}(\kappa) \cap M$, let $\mathrm{st}(\mathcal{B})$ denote its Stone space, and let $\Gamma : \beta\kappa \twoheadrightarrow \mathrm{st}(\mathcal{B})$ be the natural map; so $\Gamma(x) = x \cap \mathcal{B} = x \cap M$. Since $\mathcal{T} \cap M$ is a base for $Y$ (by $w(Y) \leq \kappa$), we have $\Gamma(z_1) = \Gamma(z_2) \to (\beta g)(z_1) = (\beta g)(z_2)$, so that $\beta g$ yields a map $\widetilde{g} : \mathrm{st}(\mathcal{B}) \to Y$ with $\beta g = \widetilde{g} \circ \Gamma$. Note that $\mathcal{B}$ contains all finite subsets of $\kappa$, so that $\mathrm{st}(\mathcal{B})$ is some compactification of a discrete $\kappa$. It is easily seen that we still have (A–D), replacing $\beta g$ by $\widetilde{g}$, $\beta\kappa$ by $\mathrm{st}(\mathcal{B})$, and $x$ by $\Gamma(x)$. Note that $\Gamma(x)$ must be countably incomplete by $M \prec H(\theta)$, so that $\Gamma(x)$ will not be a $P$-point in $\mathrm{st}(\mathcal{B})$. But to prove Lemma 4 (letting $\kappa = \aleph_n$), we also need $\Gamma(x)$ to be a weak $P_\kappa$-point in $\mathrm{st}(\mathcal{B})$. We may assume that $x \in \beta\kappa$ is good, so it is a weak $P_\kappa$-point there. But we need to show that in $\mathrm{st}(\mathcal{B})$, $\Gamma(x)$ is not a limit point of any set of size $\lambda < \kappa$. Our argument here needs to assume that $M$ is $\lambda$-covering and that $\lambda^+$ is not a Jónsson cardinal. These two assumptions will cause no problems when $\lambda < \aleph_\omega$.

As usual, $M \prec H(\theta)$ is *$\lambda$-covering* iff for all $E \in [M]^\lambda$, there is an $F \in [M]^\lambda$ such that $E \subseteq F$ and $F \in M$. By taking a union of an elementary chain of type $\lambda^+$ (see [6], §3), we see that there is an $M \prec H(\theta)$ with $|M| = \lambda^+$ such that $M$ is $\lambda$-covering.

$\kappa$ is called a *Jónsson cardinal* iff for all $\psi : [\kappa]^{<\omega} \to \kappa$, there is a $W \in [\kappa]^\kappa$ such that $\psi([W]^{<\omega})$ is a proper subset of $\kappa$. By Tryba [12] (or see [8]):

**Lemma 10.** *No successor to a regular cardinal is Jónsson.*

In particular, each $\aleph_n$ is not a Jónsson cardinal; this fact is much older and is easily proved by induction on $n$.

**Lemma 11.** *Let $\kappa$ be infinite and $x \in \beta\kappa$ a good ultrafilter on $\kappa$. Fix an infinite $\lambda < \kappa$ and let $\theta > 2^\kappa$ be regular. Let $M \prec H(\theta)$, with $x, \kappa \in M$ and $\kappa \subset M$. Assume that $M$ is $\lambda$-covering and $\lambda^+$ is not a Jónsson cardinal. Let $\mathcal{B} = \mathcal{P}(\kappa) \cap M$, and let $\Gamma : \beta\kappa \twoheadrightarrow \mathrm{st}(\mathcal{B})$ be the natural map. Then $\Gamma(x)$ is a weak $P_{\lambda^+}$-point of $\mathrm{st}(\mathcal{B})$.*

**Proof**. Fix $Z \subseteq \mathrm{st}(\mathcal{B}) \backslash \{\Gamma(x)\}$ with $|Z| \leq \lambda$. We shall show that $\Gamma(x)$ is not in the closure of $Z$. For each $z \in Z$, choose $F_z \in \Gamma(x) = x \cap \mathcal{B} = x \cap M$ such that $F_z \notin z$. Since $M$ is $\lambda$-covering, we can get $\langle G_\xi : \xi < \lambda \rangle \in M$ such that each $G_\xi \in x$ and $\forall z \in Z \, \exists \xi < \lambda \, [G_\xi = F_z]$. Since $\lambda^+$ is not Jónsson and $\lambda^+ \in M$, we can fix $\psi \in M$ such that $\psi : [\lambda^+]^{<\omega} \to \lambda$ and such that $\psi([W]^{<\omega}) = \lambda$ for all $W \in [\lambda^+]^{\lambda^+}$. Define $H(s) = G_{\psi(s)}$. Then $H \in M$ and $H : [\lambda^+]^{<\omega} \to \Gamma(x)$. Since $x$ is good, we can find $\langle K_\alpha : \alpha < \lambda^+ \rangle \in M$ such that each $K_\alpha$ is in $x$ (and hence in $\Gamma(x) = x \cap M$), and such that $K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} \subseteq H(\{\alpha_1, \ldots, \alpha_n\})$ for each $n$ and each $\alpha_1, \ldots, \alpha_n \in \lambda^+$.

Now (in $V$), we claim that $\exists \alpha < \lambda^+ \, \forall z \in Z \, [K_\alpha \notin z]$ (so that $\Gamma(x) \notin \mathrm{cl}(Z)$). If not, then we can fix $W \in [\lambda^+]^{\lambda^+}$ and $z \in Z$ such that $K_\alpha \in z$ for all $\alpha \in W$. Fix $\xi < \lambda$ such that $G_\xi = F_z$. Since $\psi([W]^{<\omega}) = \lambda$, fix $s \in [W]^{<\omega}$ such that $\psi(s) = \xi$. Say $s = \{\alpha_1, \ldots, \alpha_n\}$. Then $G_\xi = G_{\psi(s)} = H(s) \supseteq K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} \in z$, a contradiction, since $F_z \notin z$. □

**Proof of Lemma 4.** Use Lemmas 11 and 9, with $\kappa = \lambda^+ = \aleph_n$. □

In view of Lemma 10, we can also prove Theorem 1 replacing $\aleph_\omega$ with any other singular cardinal of cofinality $\omega$, since we can replace $\aleph_n$ in Lemma 4 by any successor to a regular cardinal.

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