Continuous Maps on Aronszajn Trees*

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Abstract

Assuming $\diamondsuit$: Whenever B is a totally imperfect set of real numbers, there is special Aronszajn tree with no continuous order preserving map into B.

1 Introduction

We use the following notation: If $\sqsubset$ is a relation on T and $x \in T$, then $x{\uparrow}$ denotes $\{y \in T : x \sqsubset y\}$ and $x{\downarrow}$ denotes $\{y \in T : y \sqsubset x\}$. Then a *tree* is a set T with a strict partial order $\sqsubset$ such that each $x{\downarrow}$ is well-ordered by $\sqsubset$. In a tree T, height(x) is the order type of $x{\downarrow}$ and $\mathcal{L}_\alpha = \mathcal{L}_\alpha(T) = \{x \in T : \text{height}(x) = \alpha\}$. T is an ω_1*–tree* iff $|T| = \aleph_1$, each $\mathcal{L}_\alpha(T)$ is countable, and $\mathcal{L}_{\omega_1}(T) = \emptyset$. An *Aronszajn tree* is an ω_1–tree T with no uncountable chains; then, T is *special* iff T is a countable union of antichains.

We give a tree T its natural *tree topology*, in which $U \subseteq T$ is open iff for all $y \in U$ with height(y) a limit ordinal, there is an $x \sqsubset y$ such that $x{\uparrow} \cap y{\downarrow} \subseteq U$. Then the elements whose heights are successor ordinals or 0 are isolated points. Note that T need not be Hausdorff, although any tree that we construct explicitly will be Hausdorff (equivalently, $y{\downarrow} = z{\downarrow} \to y = z$).

Let T be an ω_1–tree. A map $\varphi : T \to \mathbb{R}$ is called *order preserving* iff $x \sqsubset y \to \varphi(x) < \varphi(y)$ for all $x, y \in T$. The existence of such a φ clearly implies that T is Aronszajn, but not necessarily special; there is a counter-example [2] under $\diamondsuit$. However, it is easy to see (first noted by Kurepa [3]) that T is special iff there is an order preserving $\varphi : T \to \mathbb{Q}$.

Let T be an Aronszajn tree. If there is an order preserving $\varphi : T \to \mathbb{R}$, then there is also a *continuous* order preserving $\psi : T \to \mathbb{R}$, where $\psi(y) = \varphi(y)$ unless height(y) is a limit ordinal, in which case $\psi(y) = \sup\{\varphi(x) : x \sqsubset y\}$. If we assume $MA(\aleph_1)$, then every Aronszajn tree is special, as Baumgartner [1]

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proved by forcing with finite order preserving maps into $\mathbb{Q}$. Note that this same forcing also produces a *continuous* order preserving $\psi : T \to \mathbb{Q}$. We show here that this cannot be done in *ZFC*, since assuming $\diamondsuit$, there is an Aronszajn tree T with an order preserving map into $\mathbb{Q}$ (so T is special), but no continuous order preserving $\psi : T \to \mathbb{Q}$.[1]

This last result can be generalized somewhat. First, we can replace "order preserving" by the weaker requirement that each $\psi^{-1}\{q\}$ is discrete in the tree topology; observe that when ψ is order preserving, each $\psi^{-1}\{q\}$ is an antichain, and hence closed and discrete. Then, we can replace $\mathbb{Q}$ by any metric space which has no Cantor subsets (that is, subsets homeomorphic to 2^ω):

Theorem 1.1 *Assume $\diamondsuit$, and fix a metric space B with no Cantor subsets such that $|B| \le \aleph_1$. Then there is a special Aronszajn tree T which has no continuous map $\psi : T \to B$ such that each $\psi^{-1}\{b\}$ is discrete.*

By *CH* (which follows from $\diamondsuit$), $|B| \le \aleph_1$ holds whenever B is separable, as well as when B has a dense subset of size $\aleph_1$.

Observe that if T is special and $B \subseteq \mathbb{R}$ does have a Cantor subset F, then there must be a continuous order preserving $\psi : T \to B$. Just let $D \subseteq F$ be countable and order-isomorphic to $\mathbb{Q}$, let $\varphi : T \to D$ be order preserving, and then construct a continuous $\psi : T \to F$ as described above.

In Theorem 1.1, T depends on B. There is no one tree which works for all B by the following, which holds in *ZFC* (although it is trivial unless *CH* is true):

Theorem 1.2 *Let T be any special Aronszajn tree. Then there is a $B \subseteq \mathbb{R}$ with no Cantor subsets and a continuous order preserving map $\psi : T \to B$ such that for all $x, y \in T$, $\psi(x) \ne \psi(y)$ unless $x{\downarrow} = y{\downarrow}$.*

So, ψ is actually 1-1 if T is Hausdorff. Theorem 1.1 is proved in Section 2, and Theorem 1.2 is proved in Section 3.

By Theorem 1.2, the "$|B| \le \aleph_1$" cannot be removed in Theorem 1.1, since B could be the direct sum of all totally imperfect subspaces of $\mathbb{R}$.

2 Killing Continuous Maps

Throughout, T always denotes an ω_1–tree and B denotes a metric space. We begin with some remarks on pruning open $U \subseteq T$. In the special case when U is a subtree (that is, $x{\downarrow} \subseteq U$ for all $x \in U$), the pruning reduces to the standard procedure of removing all $x \in U$ with $x{\uparrow} \cap U$ countable. For a general U, we replace "countable" by "non-stationary" (which is the same when U is a subtree).

[1] A continuous order preserving map ψ from an Aronszajn tree T into the rationals is a nice thing to have. Todorčević [4, Remark 4.3.(d) on page 429] proved that a combination of such a map with his osc map can be used to color the 2-element chains of T with countably many colors so that every chain of order type ω^ω receives all the colors.

Definition 2.1 *For $U \subseteq T$: U is* stationary *iff $\{\text{height}(x) : x \in U\}$ is stationary, and U^p is the set of all $x \in U$ such that $x\uparrow \cap U$ is stationary.*

Clearly $U^p \subseteq U$. If U is open then U^p is open, since $x \in U^p \to x\downarrow \cap U \subseteq U^p$.

Lemma 2.2 *If $U \subseteq T$ is open, then $(U^p)^p = U^p$.*

Proof. Fix $a \in U^p$; so $a\uparrow \cap U$ is stationary. We need to show: $\{x \in a\uparrow \cap U : x\uparrow \cap U \text{ is stationary}\}$ is stationary. So, we fix a club $C \subseteq \omega_1$, and we shall find an x such that height$(x) \in C$ and $a \sqsubset x$ and $x \in U$ and $x\uparrow \cap U$ is stationary.

Since $a \in U^p$, fix a stationary S such that for all $\beta \in S$: $a\uparrow \cap U \cap \mathcal{L}_\beta(T) \neq \emptyset$ and β is a limit point of C. For each $\beta \in S$: Choose $y_\beta \in a\uparrow \cap U \cap \mathcal{L}_\beta(T)$; then, since U is open, choose $x_\beta \sqsubset y_\beta$ such that $x_\beta \in a\uparrow \cap U$ and height$(x_\beta) \in C$.

By the Pressing Down Lemma, fix x and a stationary $S' \subseteq S$ such that $x_\beta = x$ for all $x \in S'$. Then $x\uparrow \cap U$ is stationary (since it contains $\{y_\beta : \beta \in S'\}$) and height$(x) \in C$ and $a \sqsubset x$ and $x \in U$. ☺

Lemma 2.3 *If $A \subseteq T$ is discrete in the tree topology and U is a stationary open set, then the set $S := \{\alpha : U \cap \mathcal{L}_\alpha \neq \emptyset \wedge U \cap \mathcal{L}_\alpha \subseteq A\}$ is non-stationary. Hence, $U \backslash A$ is stationary.*

Proof. In fact, S is discrete in the ordinal (= tree) topology on ω_1. To see this, suppose that $\alpha \in S$ is a limit ordinal. Then fix $y \in U \cap \mathcal{L}_\alpha$. Note that $y \in A$ since $U \cap \mathcal{L}_\alpha \subseteq A$. Since U is open and A is discrete, we may fix $x \sqsubset y$ such that $x\uparrow \cap y\downarrow \subseteq U$ and $x\uparrow \cap y\downarrow \cap A = \emptyset$. Let $\xi = \text{height}(x)$. Then $\xi < \alpha$, and S contains no ordinals between ξ and α. ☺

The next lemma has a much simpler proof when B is separable (then, each $\mathcal{W}_n$ can be a singleton). For $b \in B$ and $\varepsilon > 0$, let $N_\varepsilon(b) = \{z \in B : d(b, z) < \varepsilon\}$ (where d is the metric on B).

Lemma 2.4 *Suppose that $U \subseteq T$ is a stationary open set, B is any metric space, and $\psi : U \to B$ is continuous, with each $\psi^{-1}\{b\}$ discrete. Then there are infinitely many $b \in B$ such that $\psi^{-1}(N_\varepsilon(b))$ is stationary for all $\varepsilon > 0$.*

Proof. Since each $U \setminus \psi^{-1}\{b\}$ is also stationary open by Lemma 2.3, it is sufficient to prove that there is one such b. If there are no such b, then B is covered by the open sets W such that $\psi^{-1}(W)$ is non-stationary. By paracompactness of B, this cover has a σ–discrete open refinement, $\{\mathcal{W}_n : n \in \omega\}$. So, each $\mathcal{W}_n$ is a discrete (and hence disjoint) family of open sets W such that $\psi^{-1}(W)$ is non-stationary, and $B = \bigcup_{n\in\omega}(\bigcup \mathcal{W}_n)$.

Fix n such that $\psi^{-1}(\bigcup \mathcal{W}_n)$ is stationary. We may assume that $|\mathcal{W}_n| \geq \aleph_1$, since $|\mathcal{W}_n| \leq \aleph_0$ yields an obvious contradiction. Also, we may assume that $|B| \leq \aleph_1$ (replacing B by $\psi(U)$), so that $|\mathcal{W}_n| = \aleph_1$. Let $\mathcal{W}_n = \{W_\xi : \xi < \omega_1\}$.

For each ξ, let C_ξ be a club disjoint from $\{\text{height}(y) : y \in \psi^{-1}(W_\xi)\}$. Let D be the diagonal intersection; so D is club and $\xi < \alpha \in D \to \alpha \in C_\xi$. Let S be the set of limit $\alpha \in D$ such that $\mathcal{L}_\alpha(T) \cap \psi^{-1}(\bigcup \mathcal{W}_n) \neq \emptyset$; then S is stationary. For $\alpha \in S$, choose $y_\alpha \in \mathcal{L}_\alpha(T) \cap \psi^{-1}(\bigcup \mathcal{W}_n)$. Then $y_\alpha \in \psi^{-1}(W_{\xi_\alpha})$ for some (unique) ξ_α, and $\xi_\alpha \geq \alpha$ since $\alpha \in D$. Then fix $x_\alpha \sqsubset y_\alpha$ with $x_\alpha\!\uparrow \cap\, y_\alpha\!\downarrow \subseteq \psi^{-1}(W_{\xi_\alpha})$. By the Pressing Down Lemma, fix x and a stationary $S' \subseteq S$ such that $x_\alpha = x$ for all $\alpha \in S'$. Then, using $\xi_\alpha \geq \alpha$, fix stationary $S'' \subseteq S'$ such that the ξ_α, for $\alpha \in S''$, are all different. Then the sets $x\!\uparrow \cap\, y_\alpha\!\downarrow$, for $\alpha \in S''$ are pairwise disjoint, which is impossible because $\mathcal{L}_{\mathrm{height}(x)+1}(T)$ is countable. ☺

Proof of Theorem 1.1. Call $\psi : T \to B$ a *DP* map iff ψ is continuous and each $\psi^{-1}\{b\}$ is discrete.

We build T, along with an order–preserving $\varphi : T \to \mathbb{Q}$, and use $\diamondsuit$ to defeat all DP maps $\psi : T \to B$.

As a set, T will be the ordinal ω_1, and the root will be 0. We shall define the tree order $\sqsubset$ so that $\mathcal{L}_0(T) = \{0\}$, $\mathcal{L}_1(T) = \omega\backslash\{0\}$, $\mathcal{L}_{n+1}(T) = \{\omega\cdot n + k : k \in \omega\}$ for $0 < n < \omega$, and $\mathcal{L}_\alpha(T) = \{\omega \cdot \alpha + k : k \in \omega\}$ when $\omega \leq \alpha < \omega_1$. As in the usual construction of a special Aronszajn tree, we construct $\varphi : T \to \mathbb{Q}$ and $\sqsubset$ recursively so that $\varphi(0) = 0$ and

$$\begin{aligned} &\forall x \in T\, \forall \alpha < \omega_1\, \forall q \in \mathbb{Q}\, [\alpha > \mathrm{height}(x) \wedge q > \varphi(x) \to \\ &\qquad \exists y \in \mathcal{L}_\alpha(T)\, [x \sqsubset y \wedge \varphi(y) = q]] \quad . \end{aligned} \tag{$*$}$$

This implies, in particular, that each node has $\aleph_0$ immediate successors.

Let $\langle \psi_\alpha : \alpha < \omega_1 \rangle$ be a $\diamondsuit$ sequence, where each $\psi_\alpha : \alpha \to B$. Such a sequence exists by $\diamondsuit$ because $|B| \leq \aleph_1$.

In the recursive construction of $\sqsubset$ and φ, do the usual thing in building each $\mathcal{L}_\gamma(T)$ to preserve $(*)$. But in addition, whenever $\omega \cdot \gamma = \gamma > 0$ (so $T_\gamma = \gamma$ as a set, and $\psi_\gamma : T_\gamma \to B$): *if* ψ_γ is a DP map, then *if* it is possible, extend $\sqsubset$ so that the node $\gamma \in \mathcal{L}_\gamma(T)$ satisfies:

$$\sup\{\varphi(x) : x \sqsubset \gamma\} \leq 1 \text{ and } \langle \psi_\gamma(x) : x \sqsubset \gamma \rangle \text{ does not converge in } B \quad . \tag{$\dagger$}$$

This implies that ψ_γ could not extend to a continuous map into B. Use the nodes $\gamma + 1, \gamma + 2, \ldots$ to preserve $(*)$, so if $(\dagger)$ is possible, we may let $\varphi(\gamma) = 1$. If $(\dagger)$ is impossible, then ignore it and just preserve $(*)$. To ensure that the tree will be Hausdorff, make sure that if $j \neq k$ then $\gamma + j$ and $\gamma + k$ are limits of distinct branches.

Lemma 2.5 (Main Lemma) *Suppose that $\psi : T \to B$ is a DP map. Then there is a club $C \subseteq \omega_1$ so that for all limit points γ of C: $\omega \cdot \gamma = \gamma$, and* if *$\psi_\gamma = \psi \restriction \gamma$*, then $(\dagger)$ *is possible at level γ.*

The theorem follows immediately, since choosing such a γ for which $\psi_\gamma = \psi\restriction\gamma$, we see that ψ cannot be continuous at node $\gamma \in \mathcal{L}_\gamma(T)$.

So, we proceed to prove the Main Lemma. We use a standard definition of C — namely, let $\langle M_\xi : \xi < \omega_1 \rangle$ be a continuous chain of countable elementary submodels of $H(\theta)$ (for a suitably large regular θ), such that $\varphi, \psi, \sqsubset, B \in M_0$ and each $M_\xi \in M_{\xi+1}$. Let $C = \{M_\xi \cap \omega_1 : \xi < \omega_1\}$.

Now, fix a limit point γ of C, with $\psi_\gamma = \psi\restriction\gamma$. Let $\alpha_n \nearrow \gamma$, with all $\alpha_n \in C$. We shall build a Cantor tree of candidates for the path satisfying (†), and then prove that one of these works by using the fact that B does not have a Cantor subset. For $s \in 2^{<\omega}$, construct W_s, U_s, x_s with the following properties; here, $|s|$ denotes the length of s.

1. $W_s \subseteq B$ is open and non-empty, and $\mathrm{diam}(W_s) \leq 1/|s|$.
2. $W_\emptyset = B$.
3. $\overline{W_{s\frown 0}}, \overline{W_{s\frown 1}} \subseteq W_s$ and $\overline{W_{s\frown 0}} \cap \overline{W_{s\frown 1}} = \emptyset$.
4. U_s is a stationary open subset of T, with $(U_s)^p = U_s$.
5. $U_\emptyset = \{x \in T : \varphi(x) < 1\}$,
6. $U_{s\frown 0}, U_{s\frown 1} \subseteq U_s$ and $U_s \subseteq \psi^{-1}(W_s)$.
7. $x_s \in U_s$ and $U_{s\frown i} \subseteq x_s\!\uparrow$ for $i = 0, 1$.
8. $x_\emptyset = 0$, the root node of T.
9. For $n = |s|$: $\mathrm{height}(x_s) < \alpha_n$ and, when $n > 0$, $\mathrm{height}(x_s) \geq \alpha_{n-1}$.
10. For $n = |s|$ and $\alpha_n = M_{\xi_n} \cap \omega_1$: $W_s, U_s, x_s \in M_{\xi_n}$.

For each $f \in 2^\omega$, conditions (7) and (9) guarantee that $P_f := \bigcup\{x_{f\restriction n}\!\downarrow : n \in \omega\}$ is a cofinal path through T_γ. Now, fix f so that $\bigcap_{n\in\omega} W_{f\restriction n} = \emptyset$. There is such an f because otherwise, by conditions (1)(3), $\bigcup\{\bigcap_{n\in\omega} W_{f\restriction n} : f \in 2^\omega\}$ would be a Cantor subset of B. Then, (†) will hold if we place node γ above the path P_f; note that condition (5) guarantees that $\sup\{\varphi(x) : x \sqsubset \gamma\} \leq 1$, and every limit point of $\langle \psi_\gamma(x) : x \sqsubset \gamma\rangle$ must lie in $\bigcap_{n\in\omega} W_{f\restriction n}$, which is empty.

Of course, we need to verify that the W_s, U_s, x_s can be constructed. Fix s, with $n = |s|$, and assume that we have W_s, U_s, x_s. Note that $U_s \cap x_s\!\uparrow$ is stationary by $(U_s)^p = U_s$. Applying Lemma 2.4 $\big($to $\psi\restriction(U_s \cap x_s\!\uparrow) : (U_s \cap x_s\!\uparrow) \to W_s\big)$, there exist $b_0 \neq b_1$ in W_s such that $\psi^{-1}(N_\varepsilon(b_i)) \cap U_s \cap x_s\!\uparrow$ is stationary for all $\varepsilon > 0$; applying condition (10), choose such $b_0, b_1 \in M_{\xi_n}$. Then fix ε to be the smallest of $1/(n+1)$, $d(b_0, b_1)/3$, $d(b_0, B\backslash W_s)/2$, and $d(b_1, B\backslash W_s)/2$. Let $W_{s\frown i} = N_\varepsilon(b_i)$ and $U_{s\frown i} = (\psi^{-1}(W_{s\frown i}) \cap U_s \cap x_s\!\uparrow)^p$.

Then choose $x_{s\frown i} \in U_{s\frown i}$ with $\alpha_n \leq \mathrm{height}(x_{s\frown i})$; such an $x_{s\frown i}$ exists by $(U_{s\frown i})^p = U_{s\frown i}$. Also, make sure that $x_{s\frown i} \in M_{\xi_{n+1}}$ (using $M_{\xi_{n+1}} \prec H(\theta)$), which guarantees that $\mathrm{height}(x_{s\frown i}) < \alpha_{n+1}$ and that condition (10) will continue to hold. ☺

3 Constructing Continuous Maps

Proof of Theorem 1.2. Let $H = \{1, 4, 16, \ldots\} = \{2^{2i} : i \in \omega\}$ and $K = \{2, 8, 32, \ldots\} = \{2^{2i+1} : i \in \omega\}$. Observe that $H \cap K = \emptyset$ and

$$\forall n_1, n_2 \in H\, \forall j_1, j_2 \in K\, [n_1 + j_1 = n_2 + j_2 \;\to\; n_1 = n_2 \wedge j_1 = j_2] \enspace .$$

Now, let P be the set of all real numbers of the form $\sum_{j\in K} \varepsilon_j 2^{-j}$, where each $\varepsilon_j \in \{0,1\}$. Then P is a Cantor set and $0 \in P \subset [0,1]$.

Let S be the set of all sums of the form $\sum_{n\in H} z_n 2^{-n}$, where each $z_n \in P$. Then S is compact, since it is the range of the continuous map $\Gamma : P^H \to \mathbb{R}$ defined by $\Gamma(\vec{z}) = \sum_{n\in H} z_n 2^{-n}$. Also, Γ is 1-1; that is,

$$\sum_{n\in H} z_n 2^{-n} = \sum_{n\in H} w_n 2^{-n} \;\Rightarrow\; \forall n \in H\, [z_n = w_n] \qquad (\text{all } z_n, w_n \in P\ ) \ . \qquad (❀)$$

To see this, let $z_n = \sum_{j\in K} \varepsilon_{j,n} 2^{-j}$ and $w_n = \sum_{j\in K} \delta_{j,n} 2^{-j}$. We then have $\sum\{\varepsilon_{j,n} 2^{-(j+n)} : j \in K \wedge n \in H\} = \sum\{\delta_{j,n} 2^{-(j+n)} : j \in K \wedge n \in H\}$. Since the values $j+n$ are all different, each $\varepsilon_{j,n} = \delta_{j,n}$.

For $n \in H$, define the "coordinate projection" $\pi_n : S \to P$ so that we have $\pi_n(\sum_{n\in H} z_n 2^{-n}) = z_n$. So, $\pi_n = \hat{\pi}_n \circ \Gamma^{-1}$, where $\hat{\pi}_n : P^H \to P$ is the usual coordinate projection.

Since T is special, fix $a : T \to H$ such that each $A_n := a^{-1}\{n\}$ is antichain. Also, fix a 1-1 function $\zeta : T \to P\backslash\{0\}$ such that $\zeta(T)$ has no perfect subsets. Then, define

$$\psi(x) = \sum\{\zeta(t) \cdot 2^{-a(t)} : t \in x{\downarrow}\} \ .$$

Let B be the range of ψ; then $\psi : T \to B$ is clearly continuous and order preserving.

Note that $\psi(x) = \sum_{n\in H} z_n 2^{-n}$, where $z_n = \zeta(t)$ if $t \in A_n \cap x{\downarrow}$, and $z_n = 0$ if $A_n \cap x{\downarrow} = \emptyset$. Then, $x{\downarrow} \neq y{\downarrow} \to \psi(x) \neq \psi(y)$ follows from (❀) and the fact that ζ is 1-1.

Suppose that $C \subseteq B$ is a Cantor set. Then each $\pi_n(C)$ is a compact subset of $\mathrm{ran}(\zeta) \cup \{0\}$, and is hence countable. There is then a countable α such that $\pi_n(C) \subseteq \zeta(T_\alpha) \cup \{0\}$ for all $n \in H$. So, fix $x \in T$ with $\psi(x) \in C$ and $\mathrm{height}(x) > \alpha$, let $x{\downarrow} \cap \mathcal{L}_\alpha(T) = \{t\}$, and let $n = a(t)$. Then $\zeta(t) = \pi_n(\psi(x)) \in \pi_n(C)$ and $\zeta(t) \notin \zeta(T_\alpha) \cup \{0\}$, a contradiction. ☺

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