LIMITS IN THE UNIFORM ULTRAFILTERS

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ABSTRACT. Let $u(\kappa)$ be the space of uniform ultrafilters on κ. If κ is regular, then there is an $\mathbf{x} \in u(\kappa)$ which is not an accumulation point of any subset of $u(\kappa)$ of size κ or less. $\mathbf{x}$ is also good, in the sense of Keisler.

1. INTRODUCTION

If κ is any infinite cardinal, we have $u(\kappa) \subseteq \kappa^* \subseteq \beta\kappa$. Here, κ has the discrete topology, so its Čech compactification, $\beta\kappa$, is the space of ultrafilters on κ, and $\kappa^* = \beta\kappa \backslash \kappa$ is the the space of nonprincipal ultrafilters on κ. Then, we use $u(\kappa)$ to denote the space of uniform ultrafilters; that is, $\mathbf{x} \in u(\kappa)$ iff every set in $\mathbf{x}$ has size κ.

In studying limits, one is naturally led to P-points and weak P-points:

Definition 1.1. *If θ is an infinite cardinal and $\mathbf{x}$ is a point in a topological space X, then:*

- $\mathbf{x}$ *is a* P_θ-point *in X iff the intersection of any family of* fewer than θ *neighborhoods of $\mathbf{x}$ is also a neighborhood of $\mathbf{x}$.*
- $\mathbf{x}$ *is a* weak P_θ-point *in X iff $\mathbf{x}$ is not a limit point of any subset of $X \backslash \{\mathbf{x}\}$ of size* less than θ.

So, a P-point is a P_{ω_1}-point, and a weak P-point is a weak P_{ω_1}-point. In any T_1 space, every P_θ-point is a weak P_θ-point. The main theorem of this paper is:

Theorem 1.2. *If κ is regular, then there is an $\mathbf{x} \in u(\kappa)$ which is a weak P_{κ^+}-point in $u(\kappa)$, and hence a weak P_κ-point in $\beta\kappa$.*

This is best possible as a *ZFC* result. Note that for $\mathbf{x} \in X \subseteq Y$, if $\mathbf{x}$ is a weak P_θ-point in Y, then $\mathbf{x}$ is a weak P_θ-point in X; but not conversely, in general. If $\mathbf{x} \in u(\kappa)$, it cannot be a weak P_{κ^+}-point in $\beta\kappa$, or even in κ^* if $2^{<\kappa} = \kappa$; and, $\mathbf{x}$ cannot be a weak $P_{\kappa^{++}}$-point in $u(\kappa)$ if $2^\kappa = \kappa^+$. The "hence" in the theorem is correct because by regularity of κ, no $\mathbf{x} \in u(\kappa)$ can be a limit of a subset of $\beta\kappa \backslash u(\kappa)$ of size less than κ.

For $\kappa = \omega$ (where $u(\omega) = \omega^*$), the theorem is already known (Kunen [5]). By a still earlier result of W. Rudin [6], it is consistent with *ZFC* that

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there is even a P-point in ω^*, although the existence of P-points is also independent (Shelah). The theorem for $\kappa = \omega$ can be improved in various models of $\neg CH$. For example, if MA holds, then Rudin's proof yields a $P_{\mathfrak{c}}$-point (and hence a weak $P_{\mathfrak{c}}$-point) in ω^*. However, it is consistent with $\mathfrak{c}$ arbitrarily large that there are no weak P_{ω_2}-points in ω^* (add random reals or Cohen reals to a model of CH).

For $\kappa > \omega$, we do not know about the consistency of possible strengthenings of this theorem (along the $\kappa = \omega$ lines), except for the following two elementary remarks: First, if κ is regular and larger than ω but less than the first (2-valued) measurable cardinal, then there are no P-points in $u(\kappa)$. Second, it is consistent with $2^\kappa \gg \kappa^+$ that there are no weak $P_{\kappa^{++}}$-points in $u(\kappa)$; just use Cohen forcing (with conditions of size less than κ) over a model of GCH.

We do not know if the theorem holds for singular κ. However, the existence of a weak P_κ-point in $\beta\kappa$ is already known, and this holds for singular κ as well. In fact, it is well-known (see Section 2) that if $\mathbf{x} \in u(\kappa)$ is κ^+-good in the sense of Keisler, then $\mathbf{x}$ is a weak P_κ-point in $\beta\kappa$, and hence also in $u(\kappa)$. Such good ultrafilters exist (Keisler [3] under GCH and Kunen [4] in ZFC). The $\mathbf{x}$ in Theorem 1.2 will be κ^+-good, and we shall add another ingredient to the inductive construction of [4] to make it also a weak P_{κ^+}-point in $u(\kappa)$. Not *every* κ^+-good ultrafilter is a weak P_{κ^+}-point in $u(\kappa)$; for a counter-example, use the product of two good ultrafilters.

In Section 2, we first observe that the notion "good" is really a topological notion which makes sense for a point $\mathbf{x}$ in an arbitrary space X. Every κ^{++}-good point is a weak P_{κ^+}-point. However, one cannot prove in ZFC that there is an $\mathbf{x} \in u(\kappa)$ which is κ^{++}-good in $u(\kappa)$ (Theorem 2.8). So, we shall weaken "κ^{++}-good" to "κ^+-mediocre". Every point which is both κ^+-mediocre and κ^+-good is a weak P_{κ^+}-point. The rest of the paper is then devoted to showing that such points exist in $u(\kappa)$.

The proof follows the standard pattern from [4, 5]; one constructs an ultrafilter in 2^κ steps with the aid of a matrix of sets. However, the matrix used here is a little more complicated than the standard ones, and it requires some argument to prove that it really exists. Section 3 proves some general results on constructing matrices, and Section 4 constructs the actual matrix we need. Then, Section 6 uses this matrix to prove Theorem 1.2. Regularity of κ is used only to prove that the matrix exists; the construction of the ultrafilter given the matrix works for any κ.

NOTATION: When discussing subsets of κ, $A \subseteq^* B$ means that $|A \backslash B| < \kappa$, $A =^* B$ means that $A \subseteq^* B \subseteq^* A$, and $A \perp B$ means that $|A \cap B| < \kappa$. $\mathcal{P}(\kappa)/{<}\kappa$ denotes the quotient algebra of $\mathcal{P}(\kappa)$ modulo the ideal $\{A \in \mathcal{P}(\kappa) : |A| < \kappa\}$. Then $[A] \in \mathcal{P}(\kappa)/{<}\kappa$ is the equivalence class of $A \in \mathcal{P}(\kappa)$. For $\kappa = \omega$, we use $\mathcal{P}(\omega)/fin$ for $\mathcal{P}(\omega)/{<}\omega$.

2. Good and Mediocre Points

The following notation for intersections will be used throughout this paper:

Definition 2.1. *Given sets X_α for $\alpha < \theta$ and $p \in [\theta]^{<\omega}$: $X_{\boxed{p}} = \bigcap_{\alpha\in p} X_\alpha$.*

Definition 2.2. *A point $\mathbf{x}$ in the topological space X is θ^+-*good *iff, given neighborhoods U_r ($r \in [\theta]^{<\omega}$) of $\mathbf{x}$, there are neighborhoods V_α ($\alpha < \theta$) of $\mathbf{x}$ such that $V_{\boxed{r}} \subseteq U_r$ for each non-empty $r \in [\theta]^{<\omega}$.*

$V_{\boxed{\emptyset}}$ will play no role in our arguments, but it would be natural to let it be X when we are discussing subsets of X.

Definition 2.3. *A point $\mathbf{x}$ in the space X is θ^+-*mediocre *iff, for some fixed collection of one-to-one functions $\varphi_\beta : \beta \to \theta$ ($\beta < \theta^+$): whenever U_ξ ($\xi < \theta$) are neighborhoods of $\mathbf{x}$, there are neighborhoods V_α ($\alpha < \theta^+$) of $\mathbf{x}$ such that $V_\alpha \cap V_\beta \subseteq U_{\varphi_\beta(\alpha)}$ whenever $\alpha < \beta < \theta^+$.*

Every ω_1-mediocre point is ω_1-OK (in the sense of [5]), but past ω_1, the notions of "mediocre" and "OK" diverge.

Lemma 2.4. *Every λ^{++}-good point is λ^+-mediocre.*

Proof. Fix a collection of one-to-one functions $\varphi_\beta : \beta \to \lambda$ ($\beta < \lambda^+$). Let U_ξ ($\xi \in \lambda$) be any neighborhoods of the point $\mathbf{x}$. Now define new neighborhoods U^*_r ($r \in [\lambda^+]^{<\omega}$) of $\mathbf{x}$ by setting $U^*_{\{\alpha,\beta\}} = U_{\varphi_\beta(\alpha)}$ whenever $\alpha < \beta < \lambda^+$, and setting all other $U^*_r = X$. Apply λ^{++}-goodness to get V_α ($\alpha < \lambda^+$). Then $V_\alpha \cap V_\beta \subseteq U^*_{\{\alpha,\beta\}} = U_{\varphi_\beta(\alpha)}$. □

It is a folklore result that every κ^+-good point in a T_1 space is a weak P_κ-point (see, e.g., Dow [1]). Examining that proof, we see that in fact mediocrity suffices:

Lemma 2.5. *If $\mathbf{x} \in X$ is a weak P_κ-point which is also κ^+-mediocre, then $\mathbf{x}$ is a weak P_{κ^+}-point.*

Proof. Let $\varphi_\beta : \beta \to \kappa$ ($\beta < \kappa^+$) be as in Definition 2.3. Given $Y = \{y_\xi : \xi \in \kappa\} \subseteq X\backslash\{\mathbf{x}\}$, we wish to show that $\mathbf{x}$ is not in the closure of Y. For $\xi \in \kappa$, let U_ξ be a neighborhood of $\mathbf{x}$ which misses $\{y_\eta : \eta \leq \xi\}$. Now, fix neighborhoods V_α ($\alpha \in \kappa^+$) of $\mathbf{x}$ such that $V_\alpha \cap V_\beta \subseteq U_{\varphi_\beta(\alpha)}$.

Then for some α, $V_\alpha \cap Y = \emptyset$: If not, then we can find $\xi \in \kappa$ and $E \subset \kappa^+$ with $|E| = \kappa^+$ such that $\forall \alpha \in E$ ($y_\xi \in V_\alpha$). Let β be any element of E such that $|E \cap \beta| = \kappa$. So, $\varphi_\beta(E \cap \beta)$ is unbounded in κ. Choose $\alpha \in E \cap \beta$ such that $\varphi_\beta(\alpha) > \xi$. Then $V_\alpha \cap V_\beta \subseteq U_{\varphi_\beta(\alpha)}$ which misses $\{y_\eta : \eta \leq \varphi_\beta(\alpha)\}$. So $y_\xi \notin V_\alpha \cap V_\beta$, but this is a contradiction, since $\alpha, \beta \in E$. □

Lemma 2.6. *If $\mathbf{x}$ is κ^+-good in the T_1 space X, then $\mathbf{x}$ is a weak P_κ-point in X.*

Proof. Observe that for all $\lambda < \kappa$: $\mathbf{x}$ is λ^{++}-good, and hence (by Lemma 2.4) λ^+-mediocre. Now, use Lemma 2.5, and show, by induction on $\lambda < \kappa$, that $\mathbf{x}$ is a weak P_{λ^+}-point for all $\lambda < \kappa$. □

Clearly, if $\mathbf{x} \in X \subseteq Y$ and $\mathbf{x}$ is κ^+-good in Y, then $\mathbf{x}$ is κ^+-good in X; and there are trivial examples to show that the converse is false. However,

Lemma 2.7. *If κ is regular and $\mathbf{x} \in u(\kappa)$ is κ^+-good in $u(\kappa)$, then $\mathbf{x}$ is κ^+-good in $\beta\kappa$.*

Proof. Viewing $\mathbf{x}$ as an ultrafilter on κ, we prove κ^+-goodness in $\beta\kappa$ by fixing $U_r \in \mathbf{x}$ for $r \in [\kappa]^{<\omega}$ and producing $V_\xi \in \mathbf{x}$ for $\xi < \kappa$ such that each $V_{\boxed{r}} \subseteq U_r$. Now, κ^+-goodness in $u(\kappa)$ only gives us $W_\xi \in \mathbf{x}$ such that each $W_{\boxed{r}} \subseteq^* U_r$. So, define $V_\xi = W_\xi \setminus \bigcup\{W_{\boxed{r}} \setminus U_r : \max(r) = \xi\}$. Then each $V_{\boxed{r}} \subseteq U_r$. Each $V_\xi =^* W_\xi$ (since κ is regular), so each $V_\xi \in \mathbf{x}$ (since $\mathbf{x}$ is uniform). □

We shall prove Theorem 1.2 by producing an $\mathbf{x} \in u(\kappa)$ which is both κ^+-good and κ^+-mediocre in $u(\kappa)$ (and then applying Lemmas 2.6 and 2.5), but then $\mathbf{x}$ will also be κ^+-good in $\beta\kappa$ (the usual sense of "good" in model theory). An obvious improvement of this result would be an ultrafilter which is κ^{++}-good in $u(\kappa)$, but one cannot produce that in *ZFC*:

Theorem 2.8. *If κ is regular, $2^\kappa = \kappa^+$, and $\kappa > \omega$, then no countably incomplete $\mathbf{x} \in u(\kappa)$ is κ^{++}-good in $u(\kappa)$.*

Proof. Choose $A^\alpha_n \in \mathbf{x}$ for $n < \omega \leq \alpha < \kappa^+$ so that every ω-sequence of elements of $\mathbf{x}$ is of the form $\langle A^\alpha_n \in \mathbf{x} : n < \omega\rangle$ for some α. Then choose $B^\alpha_n \in \mathbf{x}$ so that $B^\alpha_n \subseteq A^\alpha_n$, $\bigcap_n B^\alpha_n = \emptyset$, and $B^\alpha_0 \supseteq B^\alpha_1 \supseteq B^\alpha_2 \supseteq \cdots$. If $\mathbf{x}$ were κ^{++}-good in $u(\kappa)$, we could choose $D_\alpha \in \mathbf{x}$ for $\alpha < \kappa^+$ so that $D_\alpha \cap D_n \subseteq^* B^\alpha_{n+1}$ whenever $n < \omega \leq \alpha < \kappa^+$. Now, fix $\alpha \geq \omega$ such that each $D_n = A^\alpha_n$. Then each $D_\alpha \cap B^\alpha_n \subseteq D_\alpha \cap A^\alpha_n = D_\alpha \cap D_n \subseteq^* B^\alpha_{n+1}$, so $D_\alpha \perp B^\alpha_n \setminus B^\alpha_{n+1}$, and hence $D_\alpha \perp \bigcup_n (B^\alpha_n \setminus B^\alpha_{n+1}) = B^\alpha_0$, a contradiction, since $D_\alpha, B^\alpha_0 \in \mathbf{x}$. □

Note that this theorem fails for $\kappa = \omega$, since every P-point is ω_2-good. Likewise, if κ is measurable, then every normal ultrafilter on κ is a P_{κ^+}-point, and hence κ^{++}-good, in $u(\kappa)$.

3. Matrices

Ultrafilters on κ are often constructed with the aid of a matrix of sets consisting of 2^κ independent rows, each row being an instance of some base matrix, $\mathcal{M}$. Since $\mathcal{M}$ is sometimes a bit complicated, it may help to give an abstract discussion of such matrices:

Definition 3.1. *An* abstract matrix *is a triple, $\mathcal{M} = (\mathcal{B}, \mathcal{J}, \mathcal{P})$, such that $\mathcal{B}$ is a boolean algebra, $\mathcal{J}$ is an ideal in $\mathcal{B}$, and $\mathcal{P} \subseteq \mathcal{B}\setminus\mathcal{J}$. A $\theta \times \mathcal{M}$* independent matrix *in a boolean algebra $\mathcal{A}$ is a sequence $\langle h_\alpha : \alpha < \theta\rangle$, where each $h_\alpha : \mathcal{B} \to \mathcal{A}$ is a homomorphism, $h_\alpha(b) = 0$ whenever $b \in \mathcal{J}$, and $\bigwedge_{\ell \leq n} h_{\alpha_\ell}(b_\ell) \neq 0$ whenever $n \in \omega$, each $b_0, \ldots, b_n \in \mathcal{P}$, and the $\alpha_0, \ldots, \alpha_n < \theta$ are distinct.*

Of course, one can always reduce this to the case $\mathcal{J} = \{0\}$ by replacing $\mathcal{B}$ by $\mathcal{B}/\mathcal{J}$, but the definition as stated is closer to the way matrices are specified in practice, where it is simpler to take $\mathcal{B}$ to be a free algebra. For example, say we want to specify λ disjoint elements. Let $\mathcal{B}$ be the free

algebra generated by $\{b_\alpha : \alpha < \lambda\}$. Let $\mathcal{J}$ be generated by all $b_\alpha \wedge b_\beta$ for $\alpha < \beta < \lambda$. Let $\mathcal{P}$ (the positive elements) be $\{b_\alpha : \alpha < \lambda\}$. Then a $\theta \times \mathcal{M}$ independent matrix in the boolean algebra $\mathcal{P}(\kappa)/{<}\kappa$ consists of θ independent copies of λ almost disjoint sets. With this $\mathcal{M}$, taking $\lambda = \kappa^+$ and κ regular, Dow [1] showed that there is a $2^\kappa \times \mathcal{M}$ independent matrix in $\mathcal{P}(\kappa)/{<}\kappa$. Generalizing this, we show, for any $\mathcal{M}$, how to construct a $2^\kappa \times \mathcal{M}$ matrix if we are given $m \times \mathcal{M}$ matrices for each $m \in \omega$.

Definition 3.2. *$L = L_\kappa = \{\gamma < \kappa : \gamma = 0$ or γ is a limit$\}$. $\mathcal{F} \subseteq \omega^\kappa$ is* mildly independent *iff $f(\gamma + m) \leq m$ for all $\gamma \in L, m \in \omega, f \in \mathcal{F}$, and whenever $n < \omega$ and $f_0, \ldots, f_n \in \mathcal{F}$ are all distinct, there is a $\xi < \kappa$ such that $f_0(\xi), \ldots, f_n(\xi)$ are all distinct.*

Lemma 3.3. *For any infinite κ, there is a mildly independent $\mathcal{F} \subseteq \omega^\kappa$ with $|\mathcal{F}| = 2^\kappa$.*

Proof. Fix $\mathcal{G} \subseteq \omega^\kappa$ which is independent in the usual sense with $|\mathcal{G}| = 2^\kappa$ (Engelking and Karłowicz [2]). So, for all distinct $g_0, \ldots, g_n \in \mathcal{G}$ and all $i_0, \ldots, i_n \in \omega$, there are κ different $\alpha < \kappa$ such that $g_\ell(\alpha) = i_\ell$ for $\ell = 0, \ldots, n$.

If $\kappa > \omega$, then fix $g \in \mathcal{G}$. Each $|g^{-1}\{m\}| = \kappa = |L|$, so by permuting the index set κ, we may assume that $g(\gamma + m) = m$ for all $\gamma \in L$ and $m \in \omega$. Now, let $\mathcal{F} = \{f \wedge g : f \in \mathcal{G} \backslash \{g\}\}$, where $(f \wedge g)(\xi) = \min(f(\xi), g(\xi))$.

If $\kappa = \omega$, let $I(m) = m$ for each m, and let $\mathcal{F} = \{f \wedge I : f \in \mathcal{G}\}$. □

Theorem 3.4. *Assume that κ is regular and uncountable and let $\mathcal{M} = (\mathcal{B}, \mathcal{J}, \mathcal{P})$ be an abstract matrix. Assume that for each $m \in \omega$, there is an $m \times \mathcal{M}$ independent matrix in $\mathcal{P}(\kappa)/{<}\kappa$. Then there is a $2^\kappa \times \mathcal{M}$ independent matrix in $\mathcal{P}(\kappa)/{<}\kappa$.*

Proof. To simplify the notation, assume that $\mathcal{J} = \{0\}$ (replace $\mathcal{B}$ by $\mathcal{B}/\mathcal{J}$).

For each m, let $\langle h_j^m : j \leq m \rangle$ be an $(m+1) \times \mathcal{M}$ independent matrix. Choose $H_j^m : \mathcal{B} \to \mathcal{P}(\kappa)$ so that $h_j^m(b) = [H_j^m(b)]$ and $H_j^m(b') = \kappa \backslash H_j^m(b)$; these H_j^m need not be homomorphisms. Let $\mathcal{F} \subseteq \omega^\kappa$ be mildly independent with $|\mathcal{F}| = 2^\kappa$. Let $D = \{(\xi, \eta) : \xi \leq \eta < \kappa\}$. Then $|D| = \kappa$. Define $k_f : \mathcal{B} \to \mathcal{P}(D)/{<}\kappa$ by:

$$k_f(b) = \Big[D \cap \bigcup \Big\{ \{\gamma + m\} \times H^m_{f(\gamma+m)}(b) : \gamma \in L, m \in \omega \Big\}\Big]$$

Fix f; we shall show that k_f is a homomorphism. It preserves complements by our choice of the H_j^m, so fix $b_1, b_2, b_3 \in \mathcal{B}$ with $b_3 = b_1 \wedge b_2$. For each m and j, $H_j^m(b_3) =^* H_j^m(b_1) \cap H_j^m(b_2)$, so since $\kappa > \omega$, there is a fixed $\zeta < \kappa$ such that $H_j^m(b_3) \Delta (H_j^m(b_1) \cap H_j^m(b_2)) \subseteq \zeta$ for every m, j. It follows that in $\mathcal{P}(D)/{<}\kappa$, $k_f(b_3) \Delta (k_f(b_1) \wedge k_f(b_2)) \leq [D \cap (\kappa \times \zeta)] = 0$.

To prove independence of the k_f, fix non-zero $b_0, \ldots, b_n \in \mathcal{P}$ and distinct $f_0, \ldots, f_n \in \mathcal{F}$. By mild independence of $\mathcal{F}$, fix $\gamma \in L$ and $m \in \omega$ such that $f_0(\gamma + m), \ldots, f_n(\gamma + m)$ are all distinct. Then $\bigwedge_{\ell \leq n} k_{f_\ell}(b_\ell) \geq [\{\gamma + m\} \times \bigcap_{\ell \leq n} H^m_{f_\ell(\gamma+m)}(b_\ell)] > 0$. □

We do not know whether this theorem holds for $\kappa = \omega$, although it does hold in the case that $|\mathcal{B}| \leq \omega_1$, in which case one can drop the hypothesis on the $m \times \mathcal{M}$ independent matrices. First:

Lemma 3.5. *Suppose that $|\mathcal{B}| \leq \omega_1$ and h_j is a homomorphism from $\mathcal{B}$ into $\mathcal{P}(\omega)/fin$ for each $j < \omega$. Then one can choose $H_j : \mathcal{B} \to \mathcal{P}(\omega)$ such that each $h_j(b) = [H_j(b)]$, each $H_j(b') = \omega \backslash H_j(b)$, and for each finite subalgebra $\mathcal{A} \subseteq \mathcal{B}$, $H_j \restriction \mathcal{A}$ is a homomorphism from $\mathcal{A}$ into $\mathcal{P}(\omega)$ for all but finitely many j.*

Proof. Write $\mathcal{B}$ as a continuous increasing union of countable boolean algebras, $\mathcal{B} = \bigcup_{\alpha<\omega_1} \mathcal{B}_\alpha$. By induction on α, define all the $H_j \restriction \mathcal{B}_\alpha$. Let $\mathcal{B}_0 = \{0, 1\}$, and let $H_j(0) = \emptyset$ and $H_j(1) = \omega$. Since there is no problem at limits, we can fix α and explain how to extend the H_j to $\mathcal{B}_{\alpha+1}$. Now, write $\mathcal{B}_{\alpha+1}$ as an increasing union of finite boolean algebras, $\mathcal{B}_{\alpha+1} = \bigcup_{n<\omega} \mathcal{A}_n$, with $\mathcal{A}_0 = \{0, 1\}$. We may assume that we have $n_0 \leq n_1 \leq n_2 \leq \cdots$ with $\sup_j n_j = \omega$ and each $H_j \restriction (\mathcal{A}_{n_j} \cap \mathcal{B}_\alpha)$ a homomorphism. Since $\mathcal{A}_{n_j}$ is finite, one can extend $H_j \restriction (\mathcal{A}_{n_j} \cap \mathcal{B}_\alpha)$ to $\mathcal{A}_{n_j}$ so that it is still a homomorphism. Now that we have H_j defined on $\mathcal{B}_\alpha \cup \mathcal{A}_{n_j}$, we can extend it to all of $\mathcal{B}_{\alpha+1}$ by the Axiom of Choice. □

Theorem 3.6. *Let $\mathcal{M} = (\mathcal{B}, \mathcal{J}, \mathcal{P})$ be an abstract matrix, with $|\mathcal{B}| \leq \omega_1$. Then there is a $2^\omega \times \mathcal{M}$ independent matrix in $\mathcal{P}(\omega)/fin$.*

Proof. Let the $h^m_j : \mathcal{B} \to \mathcal{P}(\omega)/fin$ be as in the proof of Theorem 3.4. This is possible because every boolean algebra of size ω_1 (in particular, the direct sum of m copies of $\mathcal{B}/\mathcal{J}$) can be embedded into $\mathcal{P}(\omega)/fin$.

Now, follow the proof of Theorem 3.4, but choose the $H^m_j : \mathcal{B} \to \mathcal{P}(\kappa)$ as in Lemma 3.5, so that whenever $b_3 = b_1 \wedge b_2$, we have $H^m_j(b_3) = H^m_j(b_1) \cap H^m_j(b_2)$ for all but finitely many m, j. Define the k_f in precisely the same way, although D can also be $\omega \times \omega$ now. □

4. The Hat Trick

Definition 4.1. *A* hat function *is a function $\widehat{\ } : [\kappa^+]^{<\omega} \to [\kappa]^{<\omega}$ satisfying:*

- ☞ $p \subseteq q \Rightarrow \widehat{p} \subseteq \widehat{q}$.
- ☞ $\widehat{\emptyset} = \emptyset$.

If $\varphi_\beta : \beta \to \kappa$ is one-to-one for each $\beta < \kappa^+$, the derived hat function *is defined by: $\widehat{p} = \{\varphi_\beta(\alpha) : \alpha, \beta \in p \ \wedge \ \alpha < \beta\}$.*

We have three goals in introducing this notion: First, to simplify the the construction of a mediocre point by referring to $\widehat{\ }$ rather than to the sequence $\langle \varphi_\beta : \beta < \kappa^+ \rangle$. Second (Lemma 4.10), to get an $m \times \mathcal{M}$ independent matrix immediately from a $1 \times \mathcal{M}$ independent matrix. Third (Section 5), to unify the notions of "good", "mediocre", and "OK" into one kind of point.

Definition 4.2. *A* step-family *(over κ, with respect to $\widehat{\ }$) is an indexed collection of subsets of κ, $\{E_s : s \in [\kappa]^{<\omega}\} \cup \{A_\alpha : \alpha < \kappa^+\}$, satisfying:*

S1. $E_s \cap E_t = \emptyset$ *for each distinct* $s, t \in [\kappa]^{<\omega}$.
S2. $A_{\boxed{p}} \perp \bigcup\{E_s : s \not\supseteq \widehat{p}\}$ *for each* $p \in [\kappa^+]^{<\omega}$.
S3. $\widehat{p} \subseteq s \Rightarrow |A_{\boxed{p}} \cap E_s| = \kappa$ *for each* $p \in [\kappa^+]^{<\omega}$ *and* $s \in [\kappa]^{<\omega}$.

Definition 4.3. *If* $\mathcal{F}$ *is any filter on* κ, *then* $\mathcal{F}^+ = \{X \subseteq \kappa : \kappa \backslash X \notin \mathcal{F}\}$. $\mathcal{FR} = \mathcal{FR}(\kappa)$ *is the* Fréchet *filter,* $\{X \subseteq \kappa : |\kappa \backslash X| < \kappa\}$.

So, $\mathcal{FR}^+ = [\kappa]^\kappa$.

Definition 4.4. *Given any index set* I *and filter* $\mathcal{F}$ *on* κ, *the indexed collection* $\{E^i_s : s \in [\kappa]^{<\omega},\ i \in I\} \cup \{A^i_\alpha : \alpha < \kappa^+,\ i \in I\}$ *is an* independent matrix of $|I|$ step-families *(over* κ*) with respect to* $\mathcal{F}$, $\widehat{\ }$ *iff:*

1. *For each fixed* $i \in I$, $\{E^i_s : s \in [\kappa]^{<\omega}\} \cup \{A^i_\alpha : \alpha < \kappa^+\}$ *is a step-family.*
2. *Given* $n \in \omega$, $p_0, p_1, \ldots, p_{n-1} \in [\kappa^+]^{<\omega}$, $s_0, s_1, \ldots, s_{n-1} \in [\kappa]^{<\omega}$, *and distinct* $i_0, i_1, \ldots, i_{n-1} \in I$, *if each* $\widehat{p}_\ell \subseteq s_\ell$ *then*

$$\left(A^{i_0}_{\boxed{p_0}} \cap E^{i_0}_{s_0}\right) \cap \left(A^{i_1}_{\boxed{p_1}} \cap E^{i_1}_{s_1}\right) \cap \cdots \cap \left(A^{i_{n-1}}_{\boxed{p_{n-1}}} \cap E^{i_{n-1}}_{s_{n-1}}\right) \ \in\ \mathcal{F}^+. \qquad (*)$$

Theorem 4.5. *If* κ *is regular and* $\widehat{\ }$ *is any hat function, then there is an independent matrix of* 2^κ *step-families over* κ *with respect to* $\mathcal{FR}(\kappa)$, $\widehat{\ }$.

The rest of this section is devoted to proving this theorem, which will get used in the construction in Section 6. So, $\widehat{\ }$ will be fixed and the filter $\mathcal{F}$ will be understood to be $\mathcal{FR}$. We also assume that κ is regular, in which case the following lemma is easily proved:

Lemma 4.6. *Given* $A_\xi \subseteq \kappa$ *for* $\xi < \kappa$, *there are sets* $B_\xi \subseteq \kappa$ *such that:*

- $A_\xi =^* B_\xi$.
- $A_\xi \subseteq^* A_\eta \Rightarrow B_\xi \subseteq B_\eta$.
- $A_\xi \perp A_\eta \Rightarrow B_\xi \cap B_\eta = \emptyset$.

Corollary 4.7 ((κ, κ) separation property)**.** *Given subsets* S_ξ $(\xi < \kappa)$ *and* T_η $(\eta < \kappa)$ *of* κ *such that each* $S_\xi \perp T_\eta$, *there is some* $W \subseteq \kappa$ *such that* $\forall \xi\ (W \supseteq^* S_\xi)$ *and* $\forall \eta\ (W \perp T_\eta)$.

Proof. By Lemma 4.6, we can assume that actually each $S_\xi \cap T_\eta = \emptyset$, and then we can set $W = \bigcup_\xi S_\xi$. □

The following lemma does most of the work involved in getting a single step-family:

Lemma 4.8. *Assume that* Q *is a subset of* $[\kappa^+]^{<\omega}$ *which is closed downward and which contains* $\emptyset$. *Then there are* $X_\alpha \subseteq \kappa$ *for* $\alpha < \kappa^+$ *such that for all* $p \in [\kappa^+]^{<\omega}$, $|X_{\boxed{p}}| = \kappa \iff p \in Q$.

Proof. Let $(\kappa^+)^{<\omega\uparrow}$ denote all the strictly increasing finite sequences from κ^+. Choose $Y_\sigma, Z^\sigma_\alpha \in [\kappa]^\kappa$ (for $\sigma \in (\kappa^+)^{<\omega\uparrow}$, $\max\sigma < \alpha < \kappa^+$) so that $Y_{()} = \kappa$, and for each σ:

- $Z^\sigma_\beta \subseteq Y_\sigma$ whenever $\max\sigma < \beta < \kappa^+$
- $Z^\sigma_\beta \subsetneq^* Z^\sigma_\alpha$ whenever $\max\sigma < \alpha < \beta < \kappa^+$
- $Y_{\sigma^\frown\alpha} = Z^\sigma_\alpha \backslash Z^\sigma_{\alpha+1}$.

Then note that $Y_{\sigma^\frown\alpha} \subseteq Y_\sigma$, and $\alpha \neq \beta \Rightarrow Y_{\sigma^\frown\alpha} \perp Y_{\sigma^\frown\beta}$. So, the Y_σ form a tree of subsets of κ with root $Y_{()} = \kappa$. Each node Y_σ has κ^+ almost disjoint children, $Y_{\sigma^\frown\alpha}$ ($\max\sigma < \alpha < \kappa^+$). The Z^σ_β let us achieve an instance of (κ, κ^+) separation (see (5) below), when all we have is (κ, κ) separation.

Now, we choose the sets X_β by induction on β. Let $\Sigma_\beta = \{\sigma \in (\kappa^+)^{<\omega\uparrow} : \max\sigma = \beta \ \wedge\ \mathrm{ran}\,\sigma \in Q\}$. Using (κ,κ)-separation, choose X_β so that:

(1) $\tau \in \Sigma_\beta \Rightarrow X_\beta \supseteq^* Y_\tau$
(2) $(p \in [\beta]^{<\omega} \ \wedge\ p \cup \{\beta\} \notin Q) \Rightarrow X_\beta \perp X_{\boxed{p}}$
(3) $\sigma \in (\beta)^{<\omega\uparrow} \ \Rightarrow X_\beta \perp Z^\sigma_{\beta+1}$
(4) $(\sigma \in (\beta)^{<\omega\uparrow} \ \wedge\ \sigma^\frown\beta \notin \Sigma_\beta) \Rightarrow X_\beta \perp Z^\sigma_\beta$

Only (1) and (2) are needed to show that the X_α satisfy the lemma, but (3) and (4) are necessary in order to continue the induction. (3) implies:

(5) $\beta \notin \mathrm{ran}\,\tau \ \wedge\ \beta < \max\tau \Rightarrow X_\beta \perp Y_\tau$.

To see this, write $\tau = \sigma^\frown\gamma^\frown\rho$ where $\sigma \in (\beta)^{<\omega\uparrow}$, $\gamma > \beta$, and ρ fills out the rest of τ. Then $Y_\tau \subseteq Y_{\sigma^\frown\gamma} \subseteq Z^\sigma_\gamma \subseteq^* Z^\sigma_{\beta+1}$, so apply (3).

In order to see that (κ,κ)-separation really applies to get X_β as above, we need to check that the Y_τ from (1) are almost disjoint from the $X_{\boxed{p}}, Z^\sigma_{\beta+1}, Z^\sigma_\beta$ from (2,3,4). Fix $\tau = \pi^\frown\beta \in \Sigma_\beta$, and note that by the definition of Σ_β, we have $\mathrm{ran}(\pi^\frown\beta) \in Q$.

For (1) with (2), fix $p \in [\beta]^{<\omega}$ such that $p \cup \{\beta\} \notin Q$. Since Q is closed downward, this implies that $\exists\alpha \in p$ such that $\alpha \notin \mathrm{ran}\,\pi$. By (5) applied inductively to α, $X_\alpha \perp Y_\tau$ and therefore $X_{\boxed{p}} \perp Y_\tau$.

For (1) with (3) and (4), fix $\sigma \in (\beta)^{<\omega\uparrow}$. Since $\tau = \pi^\frown\beta$ cannot be an initial sequence of σ, there are three cases to consider.

Case 1: Neither of σ and τ is an initial sequence of the other: Then $Z^\sigma_{\beta+1} \subseteq^* Z^\sigma_\beta \subseteq Y_\sigma \perp Y_\tau$.

Case 2: $\sigma = \pi$: Then (4) cannot happen, and $Y_\tau = Y_{\sigma^\frown\beta} \perp Z^\sigma_{\beta+1}$.

Case 3: σ is a proper initial sequence of π. Say $\pi = \sigma^\frown\alpha^\frown\rho$, where $\alpha < \beta$. Then $Y_\tau \subseteq Y_{\sigma^\frown\alpha} = Z^\sigma_\alpha \setminus Z^\sigma_{\alpha+1} \perp Z^\sigma_\beta$.

We now prove: $|X_{\boxed{q}}| = \kappa \iff q \in Q$. Fix $q \in [\kappa^+]^{<\omega}$. If $q \notin Q$ then write $q = p \cup \{\beta\}$ where β is the maximum element of q and $p \in [\beta]^{<\omega}$; then (2) implies $|X_{\boxed{q}}| = |X_\beta \cap X_{\boxed{p}}| < \kappa$. If $q \in Q$, write $q = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ where $\alpha_0 < \alpha_1 < \cdots < \alpha_n$. Let $\sigma = (\alpha_0, \alpha_1, \ldots, \alpha_n)$. Then each $(\alpha_0, \alpha_1, \ldots, \alpha_i) \in \Sigma_{\alpha_i}$, so $Y_\sigma \subseteq Y_{(\alpha_0\cdots\alpha_i)} \subseteq^* X_{\alpha_i}$ by (1). Therefore $Y_\sigma \subseteq^* X_{\boxed{q}}$, so $|X_{\boxed{q}}| = \kappa$. □

Lemma 4.9. *Given any hat function $\widehat{\ }$, there exists a step-family, with respect to $\widehat{\ }$, $\{E_s : s \in [\kappa]^{<\omega}\} \cup \{A_\alpha : \alpha < \kappa^+\}$.*

Proof. Let $\bigcup\{E_s : s \in [\kappa]^{<\omega}\} = \kappa$ be any partition of κ into sets of size κ. For $\beta < \kappa^+$, we define A_β roughly by defining each $A_\beta \cap E_s$.

For $s \in [\kappa]^{<\omega}$, set $Q_s = \{p \in [\kappa^+]^{<\omega} : \widehat{p} \subseteq s\}$. Then Q_s is as in Lemma 4.8, so we can choose subsets X^s_α of E_s such that $|X^s_{\boxed{p}}| = \kappa \iff \widehat{p} \subseteq s$.

As a first approximation to the A_β, define $C_\beta = \bigcup\{X^s_\beta : s \in [\kappa]^{<\omega}\}$. Then $C_\beta \cap E_s = X^s_\beta$ so $|C_{\boxed{p}} \cap E_s| = \kappa \iff \widehat{p} \subseteq s$. These C_β satisfy part (S3) of Definition 4.2, but they might fail to satisfy (S2). We have $|C_{\boxed{p}} \cap E_s| < \kappa$ whenever $s \not\supseteq \widehat{p}$, but we need $|C_{\boxed{p}} \cap \bigcup\{E_s : s \not\supseteq \widehat{p}\}| < \kappa$.

Now, choose A_β (by induction on $\beta < \kappa^+$) such that:

(i) $A_\beta \supseteq^* C_\beta \cap E_s$ for all $s \in [\kappa]^{<\omega}$.
(ii) $A_\beta \perp E_s \backslash C_\beta$ for all $s \in [\kappa]^{<\omega}$.
(iii) $A_\beta \perp \bigcup\{A_{\boxed{p}} \cap E_t : t \not\supseteq \widehat{p \cup \{\beta\}}\}$ for all $p \in [\beta]^{<\omega}$.

Assuming this can be done, (i) and (ii) give us also:

(iv) $A_{\boxed{p}} \cap E_s =^* C_{\boxed{p}} \cap E_s$.

By (iv), the A_β also satisfy (S3), and (iii) implies that the A_β satisfy (S2).

To prove that such A_β can be chosen, we apply (κ, κ) separation, but (i,iii) requires us to verify that for all $s \in [\kappa]^{<\omega}$ and all $p \in [\beta]^{<\omega}$:

$$D_{\beta,s} := (C_\beta \cap E_s) \cap \bigcup\{A_{\boxed{p}} \cap E_t : t \not\supseteq \widehat{p \cup \{\beta\}}\} =^* \emptyset.$$

If $s \supseteq \widehat{p \cup \{\beta\}}$, then $D_{\beta,s} = \emptyset$, so assume that $s \not\supseteq \widehat{p \cup \{\beta\}}$. Then $D_{\beta,s} = C_\beta \cap E_s \cap A_{\boxed{p}}$. Applying (iv) inductively, $D_{\beta,s} =^* C_{\boxed{p \cup \{\beta\}}} \cap E_s$, which has size less than κ. □

Lemma 4.10. *If κ is regular, $\widehat{\ }$ is any hat function, and $n < \omega$, then there is an independent matrix of n step-families over κ with respect to $\mathcal{FR}(\kappa)$, $\widehat{\ }$: $\{E^i_s : s \in [\kappa]^{<\omega},\ i < n\} \cup \{A^i_\alpha : \alpha < \kappa^+,\ i < n\}$.*

Proof. We apply Lemma 4.9, changing the index set to $[n \times \kappa^+]^{<\omega}$. For $P \in [n \times \kappa^+]^{<\omega}$ and $i < n$, let $P_i = \{\alpha : (i, \alpha) \in P\}$, and define

$$\widetilde{P} = (\{0\} \times \widehat{P_0}) \cup (\{1\} \times \widehat{P_1}) \cup \cdots \cup (\{n-1\} \times \widehat{P_{n-1}}).$$

Then $\widetilde{\ }$ is a hat function from $[n \times \kappa^+]^{<\omega}$ to $[n \times \kappa]^{<\omega}$. So, we have a step-family with respect to $\widetilde{\ }$, $\{E_S : S \in [n \times \kappa]^{<\omega}\} \cup \{A_{(i,\alpha)} : i < n, \alpha < \kappa^+\}$.

For each $i < n$ and $s \in [\kappa]^{<\omega}$, let $E^i_s = \bigcup\{E_S : S_i = s\}$. Since all the E_S are disjoint, the E^i_s, for each fixed i, are also disjoint. Let $A^i_\alpha = A_{(i,\alpha)}$.

Fix $i < n$ and $p \in [\kappa^+]^{<\omega}$; we verify condition (S2) in the definition of step-family; that is, $A^i_{\boxed{p}} \perp \bigcup\{E^i_s : s \not\supseteq \widehat{p}\}$. Now $A^i_{\boxed{p}} = A_{\boxed{\{i\} \times p}}$, and

$$\bigcup\{E^i_s : s \not\supseteq \widehat{p}\} = \bigcup\{E_S : S \not\supseteq \widetilde{\{i\} \times p}\},$$

so we apply (S2) with respect to $\widetilde{\ }$.

Observe that for $S \in [n \times \kappa]^{<\omega}$ and $P \in [n \times \kappa^+]^{<\omega}$, we have

$$\begin{aligned} E_S &= E^0_{S_0} \cap E^1_{S_1} \cap \cdots \cap E^{n-1}_{S_{n-1}} \\ A_{\boxed{P}} &= A^0_{\boxed{P_0}} \cap A^1_{\boxed{P_1}} \cap \cdots \cap A^{n-1}_{\boxed{P_{n-1}}} \quad . \end{aligned}$$

To prove independence (which implies (S3) for each step-family), fix p_ℓ, s_ℓ for $\ell < n$ as in Definition 4.4.2 (here, $I = n$ and $i_\ell = \ell$). Assume each

$\widehat{p}_i \subseteq s_i$. Define P, S so that $P_i = p_i$ and $S_i = s_i$ for each i. Then $\widetilde{P} \subseteq S$, so $|A_{\boxed{P}} \cap E_S| = \kappa$, which implies $(*)$ in Definition 4.4. □

Proof of Theorem 4.5. We apply Theorem 3.4 (or Theorem 3.6 in the case $\kappa = \omega$), but one must use a little care because the definition of step-family involves infinite unions of the E_s. We handle this by adding a "name", b_t, to represent $B_t := \bigcup\{E_s : s \not\supseteq t\}$.

Let $\mathcal{B}$ be the free algebra generated by elements a_α for $\alpha < \kappa^+$, together with elements e_s and b_s for $s \in [\kappa]^{<\omega}$. Let $\mathcal{J}$ be generated by:

J1. $e_s \wedge e_t$ whenever s, t are distinct.
J2. $e_s \backslash b_t$ whenever $s \not\supseteq t$.
J3. $(\bigwedge_{\alpha\in p} a_\alpha \wedge b_{\widehat{p}})$ for each $p \in [\kappa^+]^{<\omega}$.

Let $\mathcal{P}$ contain:

P1. $e_s \wedge \bigwedge_{\alpha\in p} a_\alpha$ whenever $\widehat{p} \subseteq s$.

By Lemma 4.10, there is an $m \times \mathcal{M}$ independent matrix for each finite m. Hence, there is a $2^\kappa \times \mathcal{M}$ independent matrix $\langle h_\alpha : \alpha < 2^\kappa\rangle$ in $\mathcal{P}(\kappa)/{<\kappa}$.

Now, fix α. Since $|[\kappa]^{<\omega}| = \kappa$, which is regular, we can apply Lemma 4.6 to choose representatives E^α_s and B^α_t such that $h_\alpha(e_s) = [E^\alpha_s]$, $h_\alpha(b_s) = [B^\alpha_s]$, the E^α_s are really disjoint (using (J1)), and $E^\alpha_s \subseteq B^\alpha_t$ whenever $s \not\supseteq t$ (using (J2)). Then condition (J3) ensures that $A_{\boxed{p}} \perp \bigcup\{E_s : s \not\supseteq \widehat{p}\}$. Condition (P1) now ensures that $\widehat{p} \subseteq s \Rightarrow |A_{\boxed{p}} \cap E_s| = \kappa$.

Thus, the matrix gives us 2^κ independent step-families in the sense of Definition 4.4. □

5. Hatpoints

Using the hat functions, we see that "good" and "mediocre" are different flavors of the same notion. Throughout, κ is a fixed regular cardinal.

Definition 5.1. *Let* $\frown : [\kappa^+]^{<\omega} \to [\kappa]^{<\omega}$ *be a hat function and let* $\mathbf{x}$ *be a point in* X*. Then* $\mathbf{x}$ *is a* $\frown$point *in* X *iff, given neighborhoods* U_r $(r \in [\kappa]^{<\omega})$ *of* $\mathbf{x}$*, there are neighborhoods* V_α $(\alpha < \kappa^+)$ *of* $\mathbf{x}$ *such that* $V_{\boxed{r}} \subseteq U_{\widehat{r}}$ *for each non-empty* $r \in [\kappa^+]^{<\omega}$*.*

For example, if $\widehat{p} = p \cap \kappa$ for all p, then $\mathbf{x}$ is a $\frown$point iff $\mathbf{x}$ is κ^+-good. In this case, the V_α for $\alpha \geq \kappa$ are irrelevant.

Lemma 5.2. *There is a hat function* $\frown : [\kappa^+]^{<\omega} \to [\kappa]^{<\omega}$ *such that every* $\frown$ *point is both* κ^+*-mediocre and* κ^+*-good.*

Proof. Let $\langle \varphi_\beta : \beta < \kappa^+\rangle$ be as in Definition 2.3, but assume also that $\varphi_\beta(\alpha) = \alpha$ whenever $\alpha < \beta \leq \kappa$. Let $\frown$ be the derived hat function and let $\mathbf{x}$ be a $\frown$point. Applying Definition 5.1 with $r = \{\alpha, \beta\}$ shows that $\mathbf{x}$ is κ^+-mediocre. To show that $\mathbf{x}$ is κ^+-good, let $W_\alpha = V_\alpha \cap V_\kappa$. Then for $r \in [\kappa]^{<\omega}$, $\widehat{r \cup \{\kappa\}} = r$, so $W_{\boxed{r}} = V_{\boxed{r \cup \{\kappa\}}} \subseteq U_r$. □

Remark 5.3. *In the definition (5.1) of "hatpoint", it is sufficient to consider only* monotone *sequences of neigborhoods,* $\langle U_r : r \in [\kappa]^{<\omega}\rangle$ *of* $\mathbf{x}$*; that is* $r \subseteq s \Rightarrow U_r \supseteq U_s$*.*

Proof. Replace each U_r by $\bigcap\{U_s : s \subseteq r\}$. □

Keisler's original definition of "good" had "monotone", and restricting to monotone sequences will simplify the construction in Section 6.

In the definition of hat function and hatpoint, one could more generally consider $\widehat{\ } : [\theta]^{<\omega} \to [\kappa]^{<\omega}$. For example, fix any θ, let $\kappa = \omega$, and define $\widehat{r} = \{0, 1, \ldots, |r|\}$. Then the $\widehat{\ }$points are just the θ-OK points from [5].

6. Proof of Theorem 1.2

Applying Lemmas 5.2, 2.6 and 2.5, Theorem 1.2 is immediate from the following more general result:

Theorem 6.1. *If κ is regular and $\widehat{\ } : [\kappa^+]^{<\omega} \to [\kappa]^{<\omega}$ is a hat function, then there is an $\mathbf{x} \in u(\kappa)$ which is a $\widehat{\ }$point in $u(\kappa)$.*

Proof. Apply Theorem 4.5, and fix a matrix of 2^κ step-families (over κ), $\{E^i_s : s \in [\kappa]^{<\omega},\ i \in 2^\kappa\} \cup \{A^i_\alpha : \alpha < \kappa^+,\ i \in 2^\kappa\}$, which is independent with respect to $\mathcal{FR}(\kappa)$. Assume that for each i, we have, in addition to $(S1), (S2), (S3)$ in Definition 4.2:

S4. $\bigcup\{E^i_s : s \in [\kappa]^{<\omega}\} = \kappa$.
S5. $A^i_{\boxed{p}} \subseteq^* \bigcup\{E^i_s : s \supseteq \widehat{p}\}$ for each $p \in [\kappa^+]^{<\omega}$.

To get (S4), first shrink the A^i_α so that each $A^i_\alpha \subseteq \bigcup\{E^i_s : s \in [\kappa]^{<\omega}\}$, and then expand the E^i_s so that $\{E^i_s : s \in [\kappa]^{<\omega}\}$ forms a partition of κ. Then, (S5) follows from (S2,S4).

We plan to construct the ultrafilter $\mathbf{x}$ by induction, in 2^κ steps. We have two basic tasks: making $\mathbf{x}$ "ultra" and making $\mathbf{x}$ a $\widehat{\ }$point. To accomplish these, let B_μ, C^μ_r be subsets of κ, for $\mu < 2^\kappa, r \in [\kappa]^{<\omega}$, so that:

- $\mathcal{P}(\kappa) = \{B_\mu : \mu < 2^\kappa \ \wedge\ \mu \equiv 0 \bmod 2\}$.
- Each $\langle C^\mu_r : r \in [\kappa]^{<\omega}\rangle$ is monotone (see Remark 5.3), and every monotone sequence in $\mathcal{P}(\kappa)^{([\kappa]^{<\omega})}$ appears as $\langle C^\mu_r : r \in [\kappa]^{<\omega}\rangle$ for 2^κ distinct μ such that $\mu \equiv 1 \bmod 2$.

We shall construct $\mathbf{x}$ as an increasing union of filters: $\mathbf{x} = \bigcup_{\mu<2^\kappa} \mathcal{F}_\mu$. Set $\mathcal{F}_0 = \mathcal{FR}(\kappa)$ and $I_0 = 2^\kappa$. We define $\mathcal{F}_\mu$ and I_μ $(\mu < 2^\kappa)$ so that:

(1) If $\mu < \nu$ then $\mathcal{F}_\mu \subseteq \mathcal{F}_\nu$ and $I_\mu \supseteq I_\nu$.
(2) For limit ν: $\mathcal{F}_\nu = \bigcup_{\mu<\nu} \mathcal{F}_\mu$ and $I_\nu = \bigcap_{\mu<\nu} I_\mu$.
(3) Each $I_\mu \backslash I_{\mu+1}$ is finite.
(4) Each $\mathcal{F}_\mu$ is a filter on κ, and the matrix $\{E^i_s : s \in [\kappa]^{<\omega},\ i \in I_\mu\} \cup \{A^i_\alpha : \alpha < \kappa^+,\ i \in I_\mu\}$ of remaining step-families is independent w.r.t. $\mathcal{F}_\mu$.
(5) If $\mu \equiv 0$ then either $B_\mu \in \mathcal{F}_{\mu+1}$ or $\kappa \backslash B_\mu \in \mathcal{F}_{\mu+1}$.
(6) If $\mu \equiv 1$ and each $C^\mu_r \in \mathcal{F}_\mu$, then there are $D^\mu_\alpha \in \mathcal{F}_{\mu+1}$ for $\alpha < \kappa^+$ such that $\forall p \in [\kappa^+]^{<\omega}$ $\left[D^\mu_{\boxed{p}} \subseteq^* C^\mu_{\widehat{p}}\right]$.

As usual in these constructions, there is no problem at limits, so we proceed to describe the successor step. For $\mathcal{E} \subseteq \mathcal{P}(\kappa)$, let $\langle\langle \mathcal{E} \rangle\rangle$ denote the filter generated by $\mathcal{E}$.

If $\mu \equiv 0$: If $\langle\langle \mathcal{F}_\mu \cup \{B_\mu\}\rangle\rangle$ is a proper filter and the matrix of step-families $\{E^i_s : s \in [\kappa]^{<\omega},\ i \in I_\mu\} \bigcup \{A^i_\alpha : \alpha < \kappa^+,\ i \in I_\mu\}$ is independent w.r.t. $\langle\langle \mathcal{F}_\mu \cup \{B_\mu\}\rangle\rangle$, then set $\mathcal{F}_{\mu+1} = \langle\langle \mathcal{F}_\mu \cup \{B_\mu\}\rangle\rangle$ and $I_{\mu+1} = I_\mu$. Otherwise, fix $n \in \omega$, distinct $i_\ell \in I_\mu$ $(\ell < n)$, and $\widehat{p}_\ell \subseteq s_\ell$ $(\ell < n)$, such that:

$$B_\mu \cap \left(A^{i_0}_{\boxed{p_0}} \cap E^{i_0}_{s_0}\right) \cap \left(A^{i_1}_{\boxed{p_1}} \cap E^{i_1}_{s_1}\right) \cap \cdots \cap \left(A^{i_{n-1}}_{\boxed{p_{n-1}}} \cap E^{i_{n-1}}_{s_{n-1}}\right) \notin \mathcal{F}^+_\mu.$$

Set $\mathcal{F}_{\mu+1} = \langle\langle \mathcal{F}_\mu \cup \{A^{i_0}_{\boxed{p_0}}, \ldots, A^{i_{n-1}}_{\boxed{p_{n-1}}}\} \cup \{E^{i_0}_{s_0}, \ldots, E^{i_{n-1}}_{s_{n-1}}\}\rangle\rangle$ and set $I_{\mu+1} = I_\mu \backslash \{i_0, i_1, \ldots, i_{n-1}\}$. Then $\kappa \backslash B_\mu \in \mathcal{F}_{\mu+1}$, and we leave it to the reader to check that condition (4) holds for $\mathcal{F}_{\mu+1}$ and $I_{\mu+1}$.

If $\mu \equiv 1$: Assume each C^μ_s $(s \in [\kappa]^{<\omega})$ is in $\mathcal{F}_\mu$. (Otherwise, set $\mathcal{F}_{\mu+1} = \mathcal{F}_\mu$ and $I_{\mu+1} = I_\mu$). Choose $i \in I_\mu$ and set $I_{\mu+1} = I_\mu \backslash \{i\}$. We shall leave off the superscripts μ and i when they are clear from context. For $\alpha < \kappa^+$, define

$$D_\alpha = A_\alpha \cap \bigcup \Big\{ C_s \cap E_s : s \in [\kappa]^{<\omega} \Big\}.$$

Then for $p \in [\kappa^+]^{<\omega}$,

$$\begin{aligned} D_{\boxed{p}} &= A_{\boxed{p}} \cap \bigcup \Big\{ C_s \cap E_s : s \in [\kappa]^{<\omega} \Big\} \\ &\subseteq^* \bigcup \{E_s : s \supseteq \widehat{p}\} \cap \bigcup \Big\{ C_s \cap E_s : s \in [\kappa]^{<\omega} \Big\} \\ &= \bigcup \Big\{ C_s \cap E_s : s \supseteq \widehat{p} \Big\} \\ &\subseteq \bigcup \Big\{ C_s : s \supseteq \widehat{p} \Big\} \\ &= C_{\widehat{p}}\ , \end{aligned}$$

by monotonicity of $\langle C_p : p \in [\kappa]^{<\omega}\rangle$. Set $\mathcal{F}_{\mu+1} = \langle\langle \mathcal{F}_\mu \cup \{D_\alpha : \alpha < \kappa^+\}\rangle\rangle$. To verify condition (4): Any element of $\mathcal{F}_{\mu+1}$ is of the form $B \cap D_{\boxed{p}}$ for some $B \in \mathcal{F}_\mu$ and $p \in [\kappa^+]^{<\omega}$. But $D_{\boxed{p}} \supseteq A_{\boxed{p}} \cap C_{\widehat{p}} \cap E_{\widehat{p}}$, and $C_{\widehat{p}} \in \mathcal{F}_\mu$, so condition (4) for $\mathcal{F}_{\mu+1}$ now follows easily from condition (4) for $\mathcal{F}_\mu$. □

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