Logic Ph D Qualifying Exam

September, 1971

Instructions:

Majors and Minors: Do five problems.

Third area: Do four problems.

A. Model Theory. (Do at most three)

1. Let D be an ultrafilter over I. Suppose that for each $i \in I$, the model $\mathfrak{A}$ is elementarily embeddable in the model $\mathfrak{B}_i$. Prove that $\mathfrak{A}$ is elementarily embeddable in the ultraproduct $\Pi_D \mathfrak{B}_i$.

2. Let $\langle X, < \rangle$ be an infinite set of indiscernibles in a model $\mathfrak{A}$. Let $\langle Y, < \rangle$ be any linearly ordered set. Prove that there is a model $\mathfrak{B} \equiv \mathfrak{A}$ such that $Y \subset B$ and for all increasing n-tuples $x_1 < \ldots < x_n$ from X and $y_1 < \ldots < y_n$ from Y,

$$(\mathfrak{A}, x_1 \ldots x_n) \equiv (\mathfrak{B}, y_1 \ldots y_n).$$

3. Prove that every sentence which is preserved under submodels is logically equivalent to a universal sentence.

4. Show that the theory of an equivalence relation with infinitely many equivalence classes, each of which is infinite, is model-complete.

B. Set Theory. (Do at most two)

Notation: $R(\alpha)$ denotes the set of all sets of rank $< \alpha$, that is,

$R(0) = 0$, $R(\alpha) = \bigcup_{\beta<\alpha} S(R(\beta))$.

Let ZFC be Zermelo-Fraenkel set theory with the axiom of choice.

5. What, if anything, is wrong with the following argument?

Let ZFC_n be the first n axioms of ZFC.

a) By the reflection principle, for all $n < \omega$,

$ZFC \vdash (\exists\alpha)$ ($R(\alpha)$ is a model of ZFC_n).

b) For all $n < \omega$, $ZFC \vdash$ (ZFC_n has a model).

c) $ZFC \vdash$ (if every finite subset of ZFC has a model, then ZFC has a model).

d) $ZFC \vdash$ (ZFC has a model).

e) $ZFC \vdash$ (ZFC is consistent).

f) By Gödel's incompleteness theorem, ZFC is inconsistent.

6. Prove in ZFC that every well-founded model $\langle A, E \rangle$ of the axiom of extensionality is isomorphic to a model $\langle B, \epsilon \rangle$ for some transitive B.

7. Prove that if α and β are limit ordinals, $\alpha < \beta$, and $\langle R(\alpha), \epsilon \rangle$ is an elementary submodel of $\langle R(\beta), \epsilon \rangle$, then $\langle R(\alpha), \epsilon \rangle$ is a model of ZFC.

C. Recursion Theory (Continued).

C2. Show that the set defined in C1 is not recursive.

C3. Let $p(x, y)$ be an r.e. predicate such that

$$\forall x \exists y\, p(x, y) .$$

Show that there is a total recursive function f such that

$$\forall x\, p(x, f(x)) .$$

C4. Let f be a total function on the natural numbers. Show that if f is Π^1_1 then f is hyperarithmetic.

D. Special Topics.

D1. For each formula $\varphi(x)$ of set theory, prove in ZF that

$$\{x \in \omega : \varphi^L(x)\} \in L .$$

D2. Show that $Z(p^\infty) \oplus Q \cong Z(p^\infty)$. Here Q is the additive group of the rationals and $Z(p^\infty)$ is the divisible hull of $Z(p)$, the integers mod p.