Logic August 1972

A. Problems for third area students only.

A1. Show that if Fermat's last theorem is consistent with Peano arithmetic then it is true.

A2. Prove Koenig's theorem: $\kappa < \kappa^{cf(\kappa)}$, where $cf(\kappa)$ is the cofinality of κ.

A3. Prove that if a first order theory T has no infinite models then there is an integer n such that T has no models of power greater than n.

A4. Let $\varphi(P,Q)$, $\psi(P,R)$, $\theta(P,S)$ be pairwise inconsistent first order sentences in which only the relation symbols shown occur. Prove that there exist pairwise inconsistent first order sentences $\varphi'(P)$, $\psi'(P)$, $\theta'(P)$ such that

$$\varphi(P,Q) \vdash \varphi'(P), \quad \psi(P,R) \vdash \psi'(P), \quad \theta(P,S) \vdash \theta'(P).$$

You may use Church's Thesis in the following problems.

A5. Let A and B be disjoint Π^0_1 sets of integers. Show that there is a recursive set C such that $A \subseteq C$ and $B \cap C = 0$.

A6. Show that there is a recursive set whose characteristic function is not primitive recursive.

B. Model Theory.

B1. Prove that if ZF has a well-founded model then ZF has a model which is ω-standard but not well-founded. Hint: Start with a model in which no $R(\alpha)$ is a model of ZF and use the omitting types theorem.

B2. Let $\mathfrak{A}$ be a model of power 2^ω in a language with 2^{2^ω} symbols. Use ultrapowers to prove that $\mathfrak{A}$ has proper elementary extensions of power 2^ω and $(2^\omega)^+$, the successor cardinal of 2^ω.

B3. Let T be the theory with a binary relation E and axioms stating that:

E is an equivalence relation.

There is exactly one equivalence class of power n, $n = 1, 2, \ldots$.

Prove that T is complete but not categorical in any power.

B4. Let T be the theory in the previous problem. Let $\kappa > \omega$ be regular. Prove that every model of T of power κ has a set of indiscernibles of power κ.

C. Set theory

C1. Show that if Fermat's last theorem is provable in ZFC, then it is provable in ZF.

C2. Outline, in [illegible]00 words or less, a proof that if ZFC is consistent, so is ZFC + $2^{\aleph_0} > \aleph_1$.

C3. a) Show ZF $\vdash \forall x(\mathcal{P}(x) \notin x)$.

b) Show that if ZF is consistent, so is $ZF^- + \exists x(\mathcal{P}(x) \in x)$.

Hint: Try to find a model with an x, y such that $x = \{y\}$, $y = \{x, 0\}$ (so that $y = \mathcal{P}(x)$).

C4. a) Show that if $L(\alpha) = R(\alpha)$ then α is a cardinal.

b) Show that if $V = L$, then for some uncountable α, $L(\alpha) = R(\alpha)$.

D. Recursion Theory.

Also on Aug 74

D1. Let A be an r.e. set. Show that there is an e such that

$$W_e = \{n : 2^n 3^e \in A\},$$

where W_e is the domain of the recursive partial function φ_e with Gödel number e.

D2. Show that there are total functions f, g such that f is not recursive in g and g is not recursive in f.

D3. Let A be a non-recursive set of integers, P be Peano arithmetic formulated in a first order language L. Show that for each formula $\varphi(x)$ of L there is a model $\mathfrak{A}$ of P such that

$$A \neq \{n < \omega : \mathfrak{A} \models \varphi(\overline{n})\}$$

where $\overline{n}$ is the numeral for n.

D4. Let

$$A = \{\ulcorner\psi\urcorner : \psi \text{ is true in all well founded models of } ZF\}.$$

Show that A is a Π^1_2 set. (Here $\ulcorner\psi\urcorner$ is the gödel number of a sentence ψ of set theory.)