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Elementary Problems.

E1.

(a) If T is a theory in a countable language $\mathcal{L}$ and T is $\aleph_0$-categorical and has no finite models, then T is complete.

(b) Find a theory T in an uncountable language which has no finite models, is $\aleph_0$-categorical & has a countable model, but is not complete.

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E2. Call a limit ordinal γ indecomposable iff $\forall\alpha < \gamma\ (\gamma = \alpha + \gamma)$. Show γ is indecomposable iff $\exists\beta\ (\gamma = \omega^\beta)$.

E3. Let $\mathcal{L}$ be countable and have a 1-place predicate symbol, U. Suppose T has a model $\mathfrak{A}$ in which $|U_{\mathfrak{A}}| = \aleph_0$. Show T has a model in $\mathcal{B}$ in which $|U_{\mathcal{B}}| = |\mathcal{B}| = \aleph_1$.

E4. Let $R(0) = 0$, $R(n + 1) = \mathcal{P}(R(n))$. Suppose x is transitive and finite. Show $\exists n\ (x \in R(n))$.

E5. Prove that there are $2^{\aleph_0}$ complete extensions of Peano arithmetic.

E6. Let T be the complete theory of the model $(\mathbb{R}, E)$ where xEy iff $x - y$ is rational. Prove that T is decidable.

Model Theory.

M1. Let T be the theory

$$\forall x \forall y\, (f(x) = f(y) \rightarrow x = y)$$

in the language with one function symbol. Prove that every complete extension of T has a countably saturated model.

M2. Let $\mathfrak{A}$ be a model for a countable language and let $\Phi(x)$ be a set of formulas which is not realized in $\mathfrak{A}$. Let $\mathfrak{B}$ be an ultrapower of $\mathfrak{A}$. Prove that if $\Phi(x)$ is realized in $\mathfrak{B}$ then it is satisfied by infinitely many different elements of $\mathfrak{B}$.

M3. Prove that $\langle \omega_1, < \rangle$ and $\langle \omega_2, < \rangle$ are elementarily equivalent. needs hint.

M4. Assume ZF is consistent. Let X be a non-recursive set of natural numbers. Prove that ZF has a model $\mathfrak{A}$ such that for no $a \in A$ is

$$X = \{n \in \omega : \mathfrak{A} \models n \in a\}.$$

$\alpha < \omega_1$ $\alpha \cdot \omega_2$
$\alpha \hat{\ } \omega_1$ are club. ← now use elementary chain.

Recursion Theory.

R1.

(a) Show that there are disjoint Σ^0_1 sets which cannot be separated by a recursive set.

(b) Show that if B_0, B_1 are disjoint Π^0_1 sets then they can be separated by a recursive set.

R2. Let $f_n(x)$ be a recursive function of n and x and suppose that for each x the sequence $\langle f_n(x) \rangle_{n<\omega}$ is eventually constant, with value $f(x)$. Show that

$$\text{Turing} - \deg(f) \leq 0' .$$

R3. Let

$$W = \{e \mid \varphi^2_e \text{ is the characteristic function of a recursive well ordering, say } <_e\} .$$

Let $\|e\|$ be the order type of $<_e$. Let X be Σ^1_1, $X \subseteq W$. Show that

$$\{\|e\| \mid e \in X\}$$

is bounded by some recursive ordinal.

R4. Let $\mathcal{H} = \{H_a : a \in I\}$ be given by the following inductive definition.

(a) For each n, $2^n \in I$ and $H_{2^n} = \{n\}$.

(b) For each e, if $\varphi_e(n) \in I$ for all n then $3^e \in I$ and $5^e \in I$,

$$H_{3^e} = \bigcup_{n=0}^{\infty} H_{\varphi_e(n)} \quad \text{and} \quad H_{5^e} = \bigcap_{n=0}^{\infty} H_{\varphi_e(n)} .$$

Show that $\mathcal{H}$ is an effective Borel hierarchy (and hence is just the set of hyperarithmetic sets). I. e. show that there is a recursive total function $neg(x)$ such that if $x \in I$ then $neg(x) \in I$ and

$$H_{neg(x)} = N - H_x .$$

Set Theory.

S1. Assume the consistency of ZFC + SM. Prove the consistency of ZFC + SM + "∃ a standard model M for ZFC with $\bar{\bar{M}} \geq \omega_1$".

S2. Work in ZFC + SM. Prove that there are $2^{\aleph_0}$ non-isomorphic countable standard models for ZFC.

S3. Let $\kappa > \omega$ be regular.

(a) Show that $\{\alpha < \kappa : L_\alpha \models ZFC\}$ is not cub.

(b) If κ is inaccessible, show that $\{\alpha < \kappa : L_\alpha \models ZFC\}$ contains a cub subset.

S4. Assume $\diamondsuit$. Show that there is a family of $2^{\aleph_1}$ almost disjoint stationary subsets of ω_1.

Set - Theoretic Topology (special request).

T1. Let X_n be c. c. c. spaces ($n \in \omega$). Give the product, $\Pi_n X_n$, the box topology (i. e. basic open sets are of the form $\Pi_n U_n$, where each U_n is open in X_n). Show that $\Pi_n X_n$ has the $(2^{\aleph_0})^+$ - c. c. (i. e., every disjoint family of open sets has cardinality $\leq 2^{\aleph_0}$.

T2. Show that there is a function $F : \mathbb{R} \to \mathbb{R}$ such that F (as a subset of the plane) is not of first category.

T3. Show without using the continuum hypothesis that there is a normal Hausdorff space X such that:

(1) $|X| = \aleph_1$.

(2) X has no isolated points.

(3) In X, the intersection of countably many open sets is open.