A. Elementary

1. Let $\mathfrak{A} = \langle A, \sim \rangle$ be a countable equivalence relation with exactly one equivalence class of each cardinality

$$1, 2, \ldots, n, \ldots, \aleph_0 .$$

Show that there is a $\mathfrak{B} \equiv \mathfrak{A}$ which has infinitely many equivalence classes of cardinal $\aleph_0$.

2. Let T be an r.e. theory in a countable language L with no recursive complete extension. Show that there are $2^{\aleph_0}$ distinct complete extensions of T.

3. Let $cf(\kappa) > \omega$. Show that the collection of closed unbounded subsets of κ is closed under countable intersections.

4. Let $\mathfrak{A} \subset \mathfrak{B} \subset \mathfrak{C}$. Show, by proof or counterexample, which of the following hold:

i) $\mathfrak{A} \prec \mathfrak{B}$, $\mathfrak{B} \prec \mathfrak{C} \Rightarrow \mathfrak{A} \prec \mathfrak{C}$

ii) $\mathfrak{A} \prec \mathfrak{C}$, $\mathfrak{A} \prec \mathfrak{B} \Rightarrow \mathfrak{B} \prec \mathfrak{C}$

iii) $\mathfrak{A} \prec \mathfrak{C}$, $\mathfrak{B} \prec \mathfrak{C} \Rightarrow \mathfrak{A} \prec \mathfrak{B}$.

5. Let κ be the least cardinal such that $2^{\kappa} > 2^{\aleph_0}$. Prove that κ is regular.

6. A subset $A \subset \omega$ is a <u>spectrum</u> if there is a first order sentence φ such that

$$A = \{n < \omega \mid \exists \mathfrak{A}\, [\mathrm{Card}(\mathfrak{A}) = n \wedge \mathfrak{A} \models \varphi]\} .$$

Show that every spectrum is recursive.

7. Let $R(\alpha)$ be a model of ZP. Show that α is a cardinal.

B. Model Theory

1. Let T be a countable theory with infinite models. Prove that T has an elementary chain

$$\langle \mathfrak{A}_\alpha : \alpha < \omega_1 \rangle$$

of countable models such that for $\alpha < \beta < \omega_1$,

$$\mathfrak{A}_\alpha \neq \mathfrak{A}_\beta , \quad \mathfrak{A}_\alpha \cong \mathfrak{A}_\beta .$$

2. Let L be a countable language with only unary symbols. Show that every model for L is ω-homogeneous.

3. Let $\mathfrak{A} = \langle A, E \rangle$ be a model of ZF which is not wellfounded. Show that the ordinals of $\mathfrak{A}$ contain a copy of the rationals.

4. Let T be a complete theory in a countable language L. Assume that T has $\leq \aleph_0$ models of power $\aleph_1$. Prove that T has countably many 1-types.

5. Let $\mathfrak{A} = \langle X, < \rangle$ be a linear ordering of cofinality ω_1. Prove that $\mathfrak{A}$ has elementary extensions $\mathfrak{B}$ of arbitrarily large power in which X has no upper bound.

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enough to show every such $\mathfrak{A}$ has a proper extension (then iterate as much as wanted)

use $\mathfrak{A}^\omega / \mathcal{U}$ to get proper ext.

C. Recursion Theory

1. Show that

$$\{\langle e, f\rangle \mid \varphi_e \text{ is identical with } \varphi_f\}$$

is complete Π^0_2.

2. Let $P(\cdot)$ be defined by

$$P(x) \longleftrightarrow \forall f \in 2^{\omega} \exists n\, R(x, \bar{f}(n))$$

where $R(x, y)$ is recursive.

i) Prove that

$$P(x) \longleftrightarrow \exists M\, \forall f \in 2^{\omega} \exists n \leq M\, R(x, \bar{f}(x)) .$$

ii) Show that R is r.e.

3. Prove that there are three sets A, B, C such that none is recursive in any of the others.

4. Prove that the theory of algebraically closed fields is decidable.

5. Prove that if X is r.e. then there is an e such that

$$W_e = \{n \mid 2^n 3^e \in X\} .$$

D. Set Theory

1. Let $(M, \in)$ be a transitive model of ZF, $M \subsetneq R(\omega_1)$. Show that some set of integers is not in M.

2. Show that if ZF has a standard model then so does

$$ZFC + GCH + \forall a \subseteq \omega \, \exists b \subseteq \omega \;\; b \notin L[a].$$

3. Show the consistency of $ZFC + \neg CH + \neg SH$. (You may use $V = L \rightarrow \neg SH$.)

4. Show that if $ZF \vdash \exists R\, \varphi(R)$ where $\varphi(R)$ is a Σ^1_3 sentence of number theory then

$$ZF \vdash \exists R \in L\, \varphi(R).$$

5. Let M be a countable standard model of ZFC. Show that there is a countable standard model $N \supseteq M$ such that

$$N \models ZFC + \omega_2^M = \omega_1 + \omega_4^M = \omega_2.$$

E. Special Request

1. Let X_i $(i \in I)$ be completely regular. Prove that the box product $\prod_{i \in I} X_i$ is completely regular.

2. Show that if $a, b \in \mathcal{O}$ (Kleene's set of ordinal notations) and $|a| = |b|$ then $H_a \equiv_T H_b$.