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## A. Elementary Problems

1. Assume the Continuum Hypothesis. Show that there are $\omega_6$ countable subsets of $\omega_6$.

2. Supply a proof or a counter-example for each of the following:

   a) If $\mathfrak{A} \prec \mathfrak{C}$, $\mathfrak{B} \prec \mathfrak{C}$, and $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A} \prec \mathfrak{B}$.

   b) If $\mathfrak{A} \prec \mathfrak{B}$, $\mathfrak{A} \prec \mathfrak{C}$, and $\mathfrak{B} \subset \mathfrak{C}$, then $\mathfrak{B} \prec \mathfrak{C}$.

   c) If $\mathfrak{A} \subset \mathfrak{B}$ and $\mathfrak{A} \cong \mathfrak{B}$ then $\mathfrak{A} \prec \mathfrak{B}$.

3. Let $\mathfrak{A}$ be any first-order structure. Show that the following are equivalent.

   a) For all structures $\mathfrak{B}$ for the same language, $\mathfrak{A} \equiv \mathfrak{B} \longrightarrow \mathfrak{A} \cong \mathfrak{B}$.

   b) $\mathfrak{A}$ is finite.

4. Calculate

$$3 \cdot (\omega^2 + \omega^3) + (\omega^3 + \omega^2) \cdot 3$$

## B. Model Theory

1. Let $T$ be a complete theory in a countable language. Suppose that for every countable $\mathfrak{A} \models T$, $Th((\mathfrak{A}, a)_{a \in A})$ (the complete diagram of $\mathfrak{A}$) has at most $\omega$ non-isomorphic models of cardinality $\omega_3$. Prove that $T$ is $\omega$-stable.

2. Let $T$ be a complete theory in a countable language. Show that there is an $\mathfrak{A} \models T$ of cardinality $\leq 2^{\omega}$ with the following property: For every countable $\mathfrak{B} \models T$ and every $S \subseteq B$, there is an $R \subseteq A$ such that $(\mathfrak{B}, S)$ can be elementarily embedded into $(\mathfrak{A}, R)$.

3. Let $T$ be any extension of group theory which has an infinite model. Show that there is a model $\mathfrak{A} \models T$ such that $|\mathfrak{A}| = 2^{\omega}$ and not every automorphism of $\mathfrak{A}$ is inner.

4. Let $\mathcal{L}$ be a language consisting of uncountably many 1-place predicate symbols. Let $T$ be a complete theory in $\mathcal{L}$ and suppose $T$ has a countable saturated model. Show that there is a countable $\mathcal{L}' \subset \mathcal{L}$ such that for each $P \in \mathcal{L}$ there is a $Q \in \mathcal{L}'$ such that

$$T \vdash \forall x\, (P(x) \longleftrightarrow Q(x)) .$$

## C. Recursion Theory

1. Let $\mathcal{S}$ be a collection of r.e. sets. A <u>code set</u> for $\mathcal{S}$ is a set $A \subset \omega$ such that $\mathcal{S} = \{ W_e : e \in A \}$. Show that if $\mathcal{S}$ has a $\Pi^0_2$ code set and $\mathcal{S}$ contains all finite sets, then $\mathcal{S}$ has a recursive code set.

2. Let $\varphi(x, R)$ be a $\Pi^1_1$ formula, where $x$ ranges over $\omega$ and $R$ range over $P(\omega)$. Assume $R$ only occurs positively in $\varphi$. Show that the least $R \subseteq \omega$ such that

$$\forall x\, ( R(x) \longleftrightarrow \varphi(x, R))$$

is $\Pi^1_1$.

3. Prove Post's Theorem: $A \subseteq \omega$ is $\Delta^0_{k+1}$ iff $A$ is recursive in a $\Sigma^0_k$ set.

4. Let $T$ be KP plus the Power Set Axiom,

$$\forall x\, \exists y\, \forall z\, (z \in y \leftrightarrow z \subseteq x).$$

Show that for every countable admissible $\alpha$ there is a transitive model $A$ for $T$ such that the ordinal of $A$ is $\alpha$.

## D. Set Theory

1. Show that it is consistent with $ZFC + CH + 2^{\omega_1} = \omega_3$ that whenever $\mathfrak{a}$ is a family of $\omega_2$ uncountable subsets of $\omega_1$,

$$\exists X \subset \omega_1\, \forall Y \in \mathfrak{a}\, (|Y \cap X| = |Y - X| = \omega_1)$$

2. Let $\kappa$ be an uncountable measurable cardinal, and let $A_\alpha \subset \alpha$ for $\alpha < \kappa$. Show that for some $\alpha < \beta < \kappa$, $A_\alpha = A_\beta \cap \alpha$.

3. Assume $MA + \neg CH$. For each ordinal $\gamma < \omega_1$, let $A_\gamma \subset \gamma$, and assume $\gamma \neq \delta \rightarrow |A_\gamma \cap A_\delta| < \omega$. Show that there is an uncountable set $X \subset \omega_1$ such that $\gamma, \delta \in X \rightarrow \gamma \not\in A_\delta$.

4. Assume $V = L$. Show that $\{\alpha < \omega_1 : L(\alpha)$ is point-definable$\}$ is unbounded in $\omega_1$ and not stationary. A set $A$ is called point-definable iff every element of $A$ is first-order definable in $(A, \in)$.

5. Let $\mathfrak{a}$ be a family of countable sets such that $|\mathfrak{a}| = \omega_2$. Show that there is a $\mathfrak{B} \subset \mathfrak{a}$ and a countable $r$ such that $|\mathfrak{B}| = \omega_2$ and

$$\forall x, y \in \mathfrak{B}\, (x \neq y \rightarrow x \cap y \subset r).$$

Don't assume CH.