A. Elementary problems

1. State the Completeness Theorem for first-order logic and give a sketch of its proof.

2. Let T be a theory in a countable language with at least one function symbol, and assume that T has an infinite model. Show that T has a countable model which is not finitely generated.

3. Show $|\pi_n \omega_n| = (\omega_\omega)^\omega$ (cardinal exponentiation).

4. Assume ZFC is consistent. Show that there is a recursive extension T of ZFC such that T is consistent and

$$T \vdash \neg \mathrm{Con}(T).$$

5. For ordinals α, β, define

$$\alpha \# \beta = \sup \{\gamma : \exists A \subset \gamma \ (\mathrm{type}(A) = \alpha \ \& \ \mathrm{type}(\gamma - A) = \beta)\}.$$

Compute $(\omega^3 + \omega) \# (\omega^2 + 1)$, and prove your answer is correct.

B. Model Theory

1. Let $\mathfrak{A}$ be an infinite model and let $\kappa \geq |\mathfrak{A}|$ be such that $\kappa^\omega = \kappa$. Show that there is a $\mathfrak{B} \succ \mathfrak{A}$ such that $|\mathfrak{B}| = \kappa$.
<u>Note</u>. There is no restriction on the cardinality of the language.

2. Let T be the complete theory of the model

$$\langle \mathbb{Q} ; <, 1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots, -1, -\tfrac{1}{2}, -\tfrac{1}{3}, \ldots \rangle,$$

where $\mathbb{Q}$ is the set of rationals. Determine how many non-isomorphic countable models T has, and identify the prime and countable saturated models. <u>Note</u>. The language has one binary relation plus a constant symbol for each of $1, \frac{1}{2}, \frac{1}{3}, \ldots, -1, -\frac{1}{2}, -\frac{1}{3}, \ldots$.

3. Show that there is no set of sentences T, in the language of group theory, such that the models of T are precisely the free groups with ≥ 2 generators.

C. Recursion Theory

1. Define $a \mathrel{E} b$ iff $a \in W_b^1$. Show that there is a recursive $S \subset \omega$ and a recursive total order $\triangleleft$ of S isomorphic to the rationals such that for $a, b \in S$, $a \mathrel{E} b$ iff $a \triangleleft b$.

2. Show that there is a recursive total order whose well-founded initial segment has type ω_1^{CK}.

3. Show that there is no r.e. set $A \subset \omega$ such that for all $e \in \omega$, if φ_e^1 is total then φ_e^1 is 1-1 iff $e \in A$.

D. Set Theory

1. For $x, y \in \mathcal{P}(\omega)$, define $x \leq_L y$ iff $x \in L[y]$. Show that it is consistent with ZFC + GCH that there is an $\mathcal{F} \subset \mathcal{P}(\omega)$ such that $|\mathcal{F}| = \omega_1$ and the elements of $\mathcal{F}$ are pairwise incomparable under $\leq_L$.

2. Assume MA + $\neg$CH. Let $X_\alpha \subset [0, 1]$ for $\alpha < \omega_1$, and assume each X_α is Lebesgue measurable and has positive measure. Show that for some $A \subset \omega_1$, $|A| = \omega_1$ and $\bigcap_{\alpha \in A} X_\alpha$ has positive measure.

3. Assume $\exists A \subset \omega_1 (V = L[A])$. Prove CH.