LOGIC QUALIFYING EXAM

August 1979

Do 5 problems, at most 3 from Part A.

A. Elementary Problems.

A1. Let κ and λ be infinite cardinals such that $\lambda \leq \kappa$. Prove in ZFC that κ has exactly κ^λ subsets of power λ.

A2. For each set a let

$$G(a) = \text{the least ordinal not in } \{G(b) : b \in a\}.$$

Show that there is a formula $\varphi(x, y)$ of ZF such that for all a and b, $G(a) = b$ iff $\varphi(a, b)$.

A3. Let T and U be consistent theories such that every universal consequence of T is consistent with U. Show that there exist models M of T and P of U such that P is a submodel of M.

A4. Assume the twin prime conjecture is true, i.e. there are infinitely many p such that p and p+2 are prime. Prove that in every non-standard model M of Peano arithmetic there is an infinite x such that $M \models$ (x and x+2 are prime).

B. Model Theory.

Let N be the standard model of number theory.

B1. Let D be a free ultrafilter over ω. Prove that the ultrapower $\Pi_D N$ is isomorphic to a proper initial segment of $\Pi_D(\Pi_D N)$.

B2. Show that for each infinite cardinal κ, the theory of N has models of arbitrarily large cardinality which are κ^+-saturated but not κ^{++}-saturated.

B3. Let T be the theory with unary relations $U_0, U_1, U_2, \ldots$ and the axioms $\forall x (U_{n+1}(x) \rightarrow U_n(x))$, $n = 0, 1, 2, \ldots$. Show that T is ω-stable and find its Morley rank.

B4. Let $M = \langle A, <, R_n \rangle_{n<\omega}$ be an ω-homogeneous model such that $<$ well-orders A. Show that A has at most 2^ω elements.

C. Set Theory.

C1. Show that in ZFC one can prove the consistency of ZFC-P (ZFC without the power set axiom).

C2. Assume $V = L$. Prove that the class of α such that $L_\alpha = V_\alpha$ is c.u.b. in the class of all ordinals.

C3. Assume Martin's Axiom plus $2^\omega > \omega_1$. Show that there is no Borel measure on ω^{ω_1} such that every point has an open neighborhood of measure zero but ω^{ω_1} has measure one.

C4. A set $X \subseteq \omega_2$ is called fat iff for every c.u.b. set $C \subseteq \omega_2$, there is a closed set $D \subseteq C \cap X$ of order type $\omega_1 + 1$. Assuming $V = L$, prove that there exists $X \subseteq \omega_2$ such that both X and $\omega_2 - X$ are fat.

D. Recursion Theory.

D1. Show that the set $\{m \in \omega : \sin m > 0\}$ is recursive.

D2. Let φ_e be the e^{th} partial recursive function. Show that there are distinct $a, b, c \in \omega$ such that $\varphi_a(b) = c$, $\varphi_b(c) = a$, and $\varphi_c(a) = b$.

D3. Show that the set of Gödel numbers of true sentences of arithmetic is not arithmetical.

D4. A set $A \subseteq \omega$ is simple if A is r.e., $\omega - A$ is infinite, and for all $n \in \omega$, if W_n is infinite then $A \cap W_n$ is nonempty. Prove that if C is r.e. but not recursive then there is a simple set A such that C is not Turing reducible to A. (W_n is the n^{th} r.e. set).