Qualifying Examination in Logic

January 1980

<u>Instructions:</u> Do 5 problems, not more than 3 from one part.

.. Model Theory

1. Let $L(R_0, R_1, \ldots)$ be the language formed by adding countably many relation symbols $R_0, R_1, R_2, \ldots$ to the countable language L. Let T be a complete theory in $L(R_0, R_1, \ldots)$ and T_n the set of all consequences of T in $L(R_0, \ldots, R_n)$. Let $\Sigma(x)$ be a set of formulas of L. Suppose each T_n has a model which omits $\Sigma(x)$. Prove that T has a model which omits $\Sigma(x)$.

2. Let $\langle X, < \rangle$ be an infinite set of indiscernibles in a model A with built-in Skolem functions. Show that for each $Y \subseteq X$, A has an elementary submodel B such that $B \cap x = y$.

3. Give an example of a model A for a countable language such that A has ɔwer ω_1 but every proper elementary submodel of A is countable.

4. Let D be an ultrafilter over I. Suppose that for each $i \in I$, the model A is elementarily embeddable in the model B_i. Prove that A is elementarily embeddable in the ultraproduct $\pi_D B_i$.

B. Set Theory

1. Prove that if α and β are limit ordinals, $\alpha < \beta$, and $\langle R(\alpha), \varepsilon \rangle$ is an elementary submodel of $\langle R(\beta), \varepsilon \rangle$, then $\langle R(\alpha), \varepsilon \rangle$ is a model of ZFC.

2. (a) Show $ZF \vdash \forall x (P(x) \notin x)$

(b) Show that if ZF is consistent then so is $ZF^- + \exists x (P(x) \in x)$, where ZF^- is ZF without the axiom of regularity.

ınt: Try to find a model with an x, y such that $x = \{y\}$, $y = \{x, 0\}$ (so that $y = P(x)$).

3. Assume that ZF is consistent. Show that there is a finite subtheory T of ZF such that in ZF it cannot be proved that T $\cup$ "there is an uncountable inaccessible cardinal" is consistent.

4. Let M be a transitive model of ZF + "every uncountable cardinal is singular". Show that no transitive set N with $M \subseteq N$, $M \cap \mathrm{Ord} = N \cap \mathrm{Ord}$, satisfies ZFC.

C. Recursion Theory

1. Let T be a recursively axiomatized theory in a countable language with finite and infinite models such that T is ω_1-categorical. Prove that T has a decidable model.

2. Show that there is a sequence f_α, $\alpha < \omega_1$, of functions mapping ω into ω such that whenever $\alpha < \beta < \omega_1$, f_α is recursive in f_β but f_β is not recursive in f_α.

3. Show that there is an e such that d_e is the characteristic function of the set $\{0, 1, \ldots, e\}$, where $\{d_i \mid i < \omega\}$ is an effective enumeration of all partial recursive functions

4. Call a formula $\varphi(x)$ strongly finite if in every model M of Peano arithmetic, only a finite number of $m \in M$ satisfy φ. Prove that the set of Gödel numbers of strongly finite formulas is r.e. but not recursive.