Qualifying Examination in Logic

August 1981

Do 5 of the following 12 problems.

Glossary

$\omega = \mathbb{N} = \{0, 1, 2, \cdots\}$

$\mathbb{R}$ = the set of real numbers.

$\{\varphi_e : e \in \omega\}$ is a standard enumeration of all partial recursive functions.

$W_e = \mathrm{dom}(\varphi_e)$.

$A <_T B$ means that A is recursive in B but B is not recursive in A.

r.e. = recursively enumerable.

0' is the maximal r.e. degree.

CH = the Continuum Hypothesis

An ultrafilter, D, over a cardinal κ is <u>free</u> iff

$$\forall x \in D(|x| \geq \omega),$$

and <u>uniform</u> iff

$$\forall x \in D(|x| = \kappa).$$

A. Elementary Problems

A1. Assume CH is true. Show that there are ω_4 countable subsets of ω_4.

A2. Prove that there are models, M and N, of arithmetic such that:

1) M and N satisfy the same first-order sentences as the standard model, $(\omega, +, \cdot)$.

2) M and N are countable and non-standard.

3) M and N are not isomorphic.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.

B. Model Theory

B1. Let T be a complete theory with built-in Skolem functions. Let $\Gamma(x)$, $\Sigma(x)$ be complete 1-types T such that all models of T which realize $\Gamma(x)$ also realize $\Sigma(x)$. Show that the Cantor-Bendixon rank of $\Sigma(x)$ is $\leq$ than the Cantor-Bendixon rank of $\Gamma(x)$.

B2. Let D be a free ultrafilter over ω and E a uniform ultrafilter over ω_1. Show that $(\omega_1, <)^{\omega}/D$ is not isomorphic to $(\omega_1, <)^{\omega_1}/E$.

B3. Let $\mathfrak{A}$ be the structure $(\mathbb{R}, +, \cdot, \exp, \mathbb{N})$, where $\exp(x) = e^x$. Let $S \subset \mathbb{N}$ be any set of primes and let ${}^*\mathfrak{A} = ({}^*\mathbb{R}, \cdots)$ be any proper elementary extension of $\mathfrak{A}$. Prove that there is an $n \in {}^*\mathbb{N}$ such that for all primes $k \in \mathbb{N}$,

$$k \in S \longleftrightarrow k \mid n .$$

Here, $k \mid n$ means $\exists x \in {}^*N \ (k \cdot x = n)$.

C. Recursion Theory

C1. Prove that there is an infinite recursive set A for which $e \in A \rightarrow W_e = \omega \setminus \{e\}$.

C2. Use a priority argument to construct r.e. sets A, B, C with

$$0 <_T A <_T B <_T C <_T 0' .$$

C3. Show that there is an $f : \omega \rightarrow \omega$ such that:

1) For each recursive $g : \omega \rightarrow \omega$, $\{n : f(n) = g(n)\}$ is finite &

2) $\{\langle n, m \rangle \in \omega \times \omega : f(n) \neq m\}$ is r.e.

C4. Construct a countable decidable structure $\mathfrak{A}$ such that $Th(\mathfrak{A})$ is ω-categorical and the only recursive automorphism of $\mathfrak{A}$ is the identity map.

D. Set Theory

D1. Let x_α $(\alpha < \omega_1)$ be subsets of ω_1, where each x_α has order type ω. Show that there is an uncountable $A \subset \omega_1$ such that $\{x_\alpha : \alpha \in A\}$ forms a Δ-system —i.e., for some fixed r,

$$\forall \alpha, \beta \in A (\alpha \neq \beta \rightarrow x_\alpha \cap x_\beta = r) .$$

D2. Let M be a countable transitive model for ZFC plus $2^\omega = \omega_2$. Show that there is a c.c.c. forcing extension, $M[G]$, of M in which $2^\omega = \omega_2$ and Martin's Axiom is false.

D3. Let M be a countable transitive model for ZFC. In M, let $\mathbb{P} =$

$$\{p : p \subset \omega_1 \times \omega_1 \ \& \ |p| \leq \omega \ \& \ p \text{ is a 1-1 function}\} ;$$

define $p \leq q$ iff $q \subset p$. Let G be $\mathbb{P}$-generic over M. Show that $M[G]$ satisfies CH.