Qualifying Exam

LOGIC

August 26, 1982

Do <u>FOUR</u> of the following problems, at most two elementary.

I. Elementary Questions

1. Classify each of the following classes (with regard to first-order logic) as

   a) finitely axiomatizable (give the axioms)

   b) axiomatizable but not finitely axiomatizable (give the axioms and a proof that no finite set of axioms suffices).

   c) not axiomatizable (give proof).

   _____ i) The class of finite linearly ordered sets, $\langle A,<\rangle$

   _____ ii) The class of infinite linearly ordered sets, $\langle A,<\rangle$

   _____ iii) The class of densely linearly ordered sets, $\langle A,<\rangle$

   _____ iv) The class of well-ordered sets, $\langle A,<\rangle$.

2. Prove the following or give a counter-example. Let S be a decidable (i.e., recursive) set of sentences in propositional logic, using proposition letters P_n $(n \in \omega)$. Then $\{\varphi: S \vdash \varphi\}$ is decidable.

3. Give a counterexample to the following statement:

   If α,γ are ordinals and $A \subset \gamma$, $A \cong \alpha$ and $(\gamma - A) \cong \alpha$, then $\gamma = \alpha + \alpha$ or $\gamma = \alpha$.

II. Model Theory.

1. Which of the following properties of complete theories and/or models are preserved under the operation of taking reducts (explain):

   i) $\aleph_0$-categoricity;

   ii) $\aleph_1$-categoricity;

   iii) ω-stability;

   iv) being prime;

   v) being saturated;

   vi) having only finitely many countable models.

2. Assume T is a complete theory, $\{\Gamma_\eta | \eta \in 2^{<\omega}\}$ and $\{A_\eta | \eta \in 2^{<\omega}\}$ are types and models of T respectively, satisfying:

   i) if $\eta \subset \xi$, then A_ξ realizes Γ_η and $\Gamma_\eta \subset \Gamma_\xi$; and

   ii) if η and ξ are incompatible, then A_ξ omits Γ_η.

   Prove then that T has $2^{\aleph_0}$ pairwise non-isomorphic countable models.

3. Suppose $M \prec M'$, $a,b \in |M|$, $c \in |M'|$, p,q types over M , $\varphi(x,a)$ a formula with parameters from $|M| \cup \{a\}$ satisfy:

   i) a realizes p and b realizes q in M' ;

   ii) $p(\bar{x}) \cup q(\bar{y})$ is a complete type; and

   iii) $\varphi(x,a)$ is a complete formula for the type over $|M| \cup \{a\}$ realized by c .

   Prove then that $\varphi(x,a)$ is also a complete formula for the type that c realizes over $|M| \cup \{a,b\}$. [Hint: Otherwise there is $c' \in M'' \succ M'$ realizing the same type as c over $|M| \cup \{a\}$, but not over $|M| \cup \{a,b\}$. Proceed from there!].

III. Recursion Theory.

1. Prove that there is no recursive f satisfying:

   i) $W_{f(n)}$ codes a well order, $n < \omega$;

   ii) if W_n is a well order, then $W_n = W_{f(n)}$, $n < \omega$.

   [$A \subset \omega$ codes a well order if $\langle \text{ran } A \cup \text{dom } A, \{(x,y) \mid \langle x,y \rangle \in A\}\rangle$ is a well order, where $\text{ran } A = \{y \mid \exists x(\langle x,y \rangle \in A)\}$ and $\text{dom } A = \{x \mid \exists y(\langle x,y \rangle \in A)\}$] .

   [Hint: One way to do this uses the recursion theorem].

2. Suppose C is a non-recursive Δ^0_2 set. Prove that there is a simple set A such that $C \not\leq_T A$. Hint: Use the limit lemma to represent C as $\lim_s C_s$, where $\{C_s\}_{s<\omega}$ is a recursive sequence.

3. Suppose $\mathcal{A}$ is a countable, saturated structure realizing recursive types Γ_1, Γ_2 . Prove that there is a decidable structure realizing Γ_1 and Γ_2 .

4. Suppose $\{A_s\}_{s<\omega}$ is an effective enumeration of A , where each A_s is recursive, and $\forall e\{[\exists^\infty s\ \{e\}_s^{A_s}(e)\downarrow] \rightarrow \{e\}^A(e)\downarrow\}$.

   Prove A is low, i.e. $A' = 0'$.

IV. Set Theory.

1. Let $\langle A,<\rangle$ be a linear order such that $\forall X \subset A$ $(X \cong A$ or $A-X \cong A)$.

   Show A is a well order or an inverse well order.

2. Call $\mathcal{F}$ a mad f (maximal almost disjoint family of subsets of ω_1) iff

   i) $\mathcal{F} \subset \mathcal{P}(\omega_1)$.

   ii) $\forall X \in \mathcal{F}$ $(|X| = \omega_1)$.

   iii) $\forall X,Y \in \mathcal{F}$ $(X \neq y \rightarrow |X \cap Y| < \omega_1)$.

   iv) $\mathcal{F}$ is maximal with respect to (i) - (iii).

   Assume: M is a countable transitive model of ZFC, $\mathcal{F} \in M$, and $(\mathcal{F}$ is a mad f $)^M$. Show that $\mathcal{F}$ remains a mad f in any ccc forcing extension of M.

3. If $f,g \in \omega^\omega$, we say $f <^* g$ iff $\{n : g(n) \leqslant f(n)\}$ is finite. Let α be the least ordinal which is not the type of a chain in the partial order, $(\omega^\omega,<^*)$. Prove that α is a regular cardinal.

4. Assume that there is an ordinal α such that $R(\alpha) \models$ ZFC, and let α be the smallest such. Prove that α is a strong limit cardinal and that $cf(\alpha) = \omega$.