Qualifying Exam in LOGIC

January 20, 1983

INSTRUCTIONS: Do four questions; at most two elementary.

NOTATIONS:

1. $\{\varphi_e | e < \omega\}$ is a standard enumeration of all partial recursive functions.

2. $W_e =_{df} \text{dom } \varphi_e$

3. $W_{e,s} =_{df} \{x \leq s | \varphi_e(x)$ converges in $\leq s$ steps$\}$

4. $D_y = \{x_1 < x_2 < \cdots < x_n\}$ where $y = 2^{x_1} + 2^{x_2} + \cdots + 2^{x_n}$.

POLICY ON MISPRINTS

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.

I. Elementary Questions

1. Find a set S of sentences in an uncountable language such that

   a) for all $n \in \omega$, S has a finite model of size $\geq n$.

   b) S has no countably infinite models.

2. Let L be a fixed finite language. If S, T are consistent sets of sentences of L, say $S \Rightarrow T$ iff $S \vdash \varphi$ for all $\varphi \in T$.

   Show:

   a) If T is r.e. and consistent, there is a complete Π^0_1 (co-r.e.) S with $S \Rightarrow T$.

   b) There is an r.e. consistent T with no complete r.e. S such that $S \Rightarrow T$.

3. Prove the following or give a counterexample: Let S_n, T_n $(n \in \omega)$ be sets of sentences. Assume that for all n, $S_n \subset S_{n+1}$, $T_n \subset T_{n+1}$, and there is a model, $\mathfrak{A}_n$ such that $\mathfrak{A}_n \models T_n$ and $\mathfrak{A}_n \not\models S_n$. Then there is a L such that $L \models \cup_{n \in \omega} T_n$ and $L \not\models \cup_{n \in \omega} S_n$.

II. Recursion Theory

1. Prove or disprove:

   There exists a complete Δ_2^0 set; i.e. $\exists A \in \Delta_2^0 \; \forall B \in \Delta_2^0 \; \exists f$ recursive $\forall n$

   $n \in B$ iff $f(n) \in A$.

2. Let $\{U_{e,s} \mid e,s < \omega\}$ satisfy:

   i) $U_{e,s} \subset U_{e,s+1}$; and

   ii) $\exists F$ recursive $\forall e,s \, [U_{e,s} = D_{f(e,s)}]$.

   Let $U_e =_{df} \bigcup_{s<\omega} U_{e,s}$. Prove there is a recursive g such that for all e,s :

   1) $W_{g(e)} = U_e$; and

   2) $W_{g(e),s+1} \subset U_{e,s}$.

   Hint: Recursion Theorem.

3. Suppose a recursive f satisfies

   $$\forall n,m \quad [W_{f(n)} \neq \{0,1,2,\cdots,m-1\}] \quad .$$

   Prove that there is a recursive g satisfying:

   1) $\forall i \, \exists j \, [W_i = W_{f(j)}$ or $W_i = W_{g(j)}]$;

   2) $\forall i,j \, [W_{f(i)} \neq W_{g(j)}]$; and

   3) $\forall i \neq j \, [W_{g(i)} \neq W_{g(j)}]$.

   Hint: Construct $\{A_i\}_{i<\omega}$ such that there is a recursive g , $A_i = W_{g(i)}$ for all i .

III. Model Theory

1. A model A is <u>almost-ω-homogeneous</u> iff_{df} there is an $\bar{a} \in |A|^{<\omega}$ such that $\langle A,\bar{a}\rangle$ is ω-homogeneous. Prove that if every countable model of a theory T is almost-ω-homogeneous, then every model of T is almost-ω-homogeneous.

2. If Γ,Σ are complete types of a theory T, say that <u>Γ forces Σ</u> iff_{df} whenever Γ is realized in a model, then Σ is realized too. Suppose T is a complete theory with the property that if Γ forces Σ then either

   i) Σ is principal; or

   ii) $\Sigma(x_{i_1},\cdots,x_{i_m}) \subseteq \Gamma(x_1,\ldots,x_n)$ for some $1 \le i_1 \le i_2 \le \cdots \le i_m \le n$.

   Suppose $\Gamma(x)$ is a complete 1-type. Prove that there is a homogeneous model of T omitting $\Gamma(x)$.

3. Assume that $R,S \subset \omega\times\omega$ are recursive, $\langle\omega,R\rangle \cong \langle\omega,S\rangle$, and $\langle\omega,R\rangle$ is homogeneous. Prove that there is an $f \in \Delta^0_3$ such that $f : \langle\omega,R\rangle \cong \langle\omega,S\rangle$.

4. Let U be a uniform ultrafilter on ω_1. If K is a family of structures for L, let K^* be the set of all structures of the form $\prod_{\alpha\in\omega_1} \mathfrak{A}_\alpha/U$, where each $\mathfrak{A}_\alpha \in K$. Let S be a set of sentences of L with $|S| = \omega_1$. Assume that for each finite $F \subset S$, there is an $\mathfrak{A} \in K$ with $\mathfrak{A} \models F$. Show that there is an $\mathfrak{A} \in (K^*)^*$ with $\mathfrak{A} \models S$.

IV. Set Theory

1. Let M be a countable transitive model for ZFC + GCH. Assume that $\mathbb{P} \in M$, $\mathbb{P}$ is a partial order, and $(\mathbb{P}$ is c.c.c.$)^M$. Let G be $\mathbb{P}$-generic over M. Show that the following holds in $M[G]$:

$$\forall F \subset {}^{\omega_1}\omega_1 (|F| \geq \omega_3 \rightarrow \exists f,g \in F(f \neq g \wedge |\{\alpha<\omega_1 : f(\alpha) = g(\alpha)\}| = \omega_1)) \quad .$$

2. If X is a set of real numbers, call X a <u>Bernstein set</u> iff $|X| \geq \omega_1$ and X has no perfect subsets. Suppose that x_α, for $\alpha < \omega_1$, are distinct real numbers. Show that there is a closed unbounded $C \subset \omega_1$ such that $\{x_\alpha : \alpha \in C\}$ is a Bernstein set.

3. Assume $MA + \neg CH$. For each limit $\gamma \in \omega_1$, let A_γ be a cofinal ω-sequence. Show that there are $B_n \subset \omega_1$ for $n \in \omega$ such that each $B_n \cap A_\gamma$ is finite and $\bigcup_{n\in\omega} B_n = \omega_1$.