Qualifying Exam

in

LOGIC

August 25, 1983

INSTRUCTIONS: Do four questions; at most two elementary.

NOTATIONS AND DEFINITIONS:

1. $A \triangle B =_{df} (A-B) \cup (B-A)$ .

2. $M$ is maximal iff$_{df}$ $M$ is r.e. and $\omega-M$ is cohesive, i.e. $\omega-M$ is infinite and $\forall e[W_e \cap (\omega-M)$ is finite or $(\omega-W_e) \cap (\omega-M)$ is finite].

3. $\{W_e \mid e < \omega\}$ is a standard enumeration of all r.e. sets.

4. $\{\mu_e \mid e < \omega\}$ is a standard enumeration of all partial recursive functions.

## I. Elementary Questions

1. Let $L$ be a first order language. Suppose $T_i$ , $i < \alpha$ are theories in $L$ such that every $L$-structure is a model for exactly one of the $T_i$'s . For what $\alpha \leq \omega$ does it then follow that each of the $T_i$'s is finitely axiomatizable?

2. Let $T_1$ and $T_2$ be theories in the language $L$ . Assume that for every <u>finite</u> set $S$ of models for $T_1$ , there is a model $B$ for $T_2$ such that every model in $S$ is isomorphically embeddable in $B$ . Prove that the same is true for <u>all</u> sets $S$ of models for $T_1$ .

3. Let $A$ and $B$ be models of Peano arithmetic, with $A$ a submodel of $B$ . Let

$$C = \{a \in A : \forall c \in B\ (c < a \rightarrow c \in A)\}$$

Prove that if $a, b \in C$ , then $a + b \in C$ and $a \cdot b \in C$ .

## II. Recursion Theory

1. Let $M$ be a maximal set. Define

$$E =_{df} \{2n \mid n < \omega\} \quad ;$$

$$A =_{df} \{2n \mid n \in M\} \quad ; \quad \text{and}$$

$$B =_{df} \{2n+1 \mid n \in M\} \quad .$$

Prove that if $R \subset \omega$ satisfies

i) $R$ recursive;

ii) $A \subset R$ ; and

iii) $B \cap R = \phi$ ,

then $R \,\Delta\, E$ is finite.

2. Suppose $f : \omega \times \omega \to \omega$ satisfies

i) $f(0,y) = y + 1$ ;

ii) $f(x+1,0) = f(x,1)$ ; and

iii) $f(x+1,y+1) = f(x,f(x+1,y))$ .

Use the recursion theorem to prove that $f$ is recursive.

3. Given an r.e. non-recursive $C$, construct a low simple set $A \leq_T C$ .

[Hint: A negative requirement might be (for lowness)

$$N_e : \exists^{\infty} s\ \mu_{e,s}^{A_s}(e)\downarrow \to \mu_e^{A}(e)\downarrow\ ]\ .$$

## III. Model Theory

1. Let $L$ be the first order language whose only non-logical symbols are unary predicate symbols $P_i$, $i < \omega$. Find complete theories $T_1, T_2$ that are not $\aleph_0$-categorical, do have infinite models, and such that $i = 1$ satisfies and $i = 2$ fails to satisfy:

$$\text{For all countable } \mathcal{A}, \mathcal{B} \models T_i [\mathcal{A} \leq \mathcal{B} \text{ or } \mathcal{B} \leq \mathcal{A}] .$$

$\mathcal{A} \leq \mathcal{B}$ means: there exists an elementary embedding from $\mathcal{A}$ into $\mathcal{B}$.

2. Assume that $\{\Gamma_i \mid i < \omega\}$ is a set of complete types of a complete first order theory $T$ satisfying:

i) $\forall i,j\ \exists k\ [\Gamma_i(\bar{x}) \cup \Gamma_j(\bar{y}) \subset \Gamma_k(\bar{x},\bar{y})]$ ; and

ii) $\forall i \forall \varphi \in L(T) \exists j\ [(\exists \bar{y} \varphi(\bar{x},\bar{y})) \in \Gamma_i(\bar{x}) \rightarrow \Gamma_i(\bar{x}) \cup \{\varphi(\bar{x},\bar{y})\} \subset \Gamma_j(\bar{x},\bar{y})]$ .

Prove that there is a model $\mathcal{A} \models T$ realizing exactly the set of types $\{\Gamma_i \mid i < \omega\}$.

[Hint: Henkin construction].

3. Let $\mathfrak{A}$ be a structure for a language $L$. Let $\varphi_n(x)$ for $n \in \omega$ be formulas in $L_A$ (where $L_A$ is $L$ plus a new constant symbol for each element of $A$). Assume $\mathfrak{A} = \mathfrak{L}^{\omega}/U$, where $U$ is a non-principal ultrafilter on $\omega$. Show that

$$|\{a \in A : \mathfrak{A} \models \bigwedge_n \varphi_n(a)\}|$$

is either finite or $\geq 2^{\omega}$.

## IV. Set Theory

1. Let $F$ be a family of finite sets with $|F| = \lambda > \omega$. Show that there is an $r \subset \cup F$ and a $G \subset F$ with $|G| = \lambda$, $|r| < \lambda$, and

$$\forall x,y \in G \ (x \neq y \rightarrow x \cap y \subset r) .$$

<u>Caution</u>: $\lambda$ may be singular.

2. Assume there exists a strongly inaccessible cardinal. Show that there is an <u>un</u>countable transitive model for $ZFC + V \neq L$.

3. Let $M$ be a countable transitive model for $ZFC$ and let $\mathbb{P}$ be a partial order in $M$. Show

$$M = \cap \{M[G] : G \text{ is } \mathbb{P}\text{-generic over } M\} .$$