Qualifying Exam

LOGIC

January 19, 1984

INSTRUCTIONS: Do four questions, at most two elementary.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.

ELEMENTARY

1. Assume that first order predicate logic with equality and a binary predicate symbol, R, is undecidable. Prove that this remains true if you add the axiom:

$$\forall xy(R(x,y) \leftrightarrow R(y,x)) \quad .$$

2. Suppose that theories T_1, T_2 both have infinitely many countable models. Is it then true that if $T_1 \cup T_2$ is consistent then $T_1 \cup T_2$ has infinitely many countable models? (Give a proof or a counter-example).

3. Suppose $\forall n \in \omega \; (2^{\omega_n} = \omega_{\omega+17})$. Prove that $2^{\omega_\omega} = \omega_{\omega+17}$.

Model Theory

1. Suppose a complete consistent theory T does not have a countable saturated model. Prove that there exists $\mathfrak{B}_i$ for $i < 2^{\aleph_0}$ such that:

   i) each $\mathfrak{B}_i$ is a countable model of T, and

   ii) if $i \neq j$ then $\mathfrak{B}_i$ is not elementarily embeddable in $\mathfrak{B}_j$.

2. Suppose T is a complete theory with complete types Γ_1, Γ_2 that are Turing incomparable. Prove that T has at least 4 countable models.

3. Let T be a theory in a countable language which has infinite models. Prove that there are $\mathfrak{A}_\alpha \models T$ for $\alpha < \omega_1$ such that whenever $\alpha < \beta$,

$$\mathfrak{A}_\alpha \prec \mathfrak{A}_\beta, \quad \mathfrak{A}_\alpha \neq \mathfrak{A}_\beta, \quad \text{and} \quad \mathfrak{A}_\alpha \cong \mathfrak{A}_\beta$$

Set Theory

1. Let $X = \{\alpha<\omega_1 : L_\alpha \models ZFC\}$. Assume $|X| \geq 2$.
Show that there are $\alpha, \beta \in X$ and $a \subset \omega$ with $\alpha < \beta$, $L_\alpha[a] \models ZFC$ and $L_\beta[a] \not\models ZFC$.

2. Show that there are partial orders $\lhd_n$ on ω_1 for $n < \omega$ such that

   i) $\forall\alpha\beta\ (\alpha < \beta \leftrightarrow \exists n\ (\alpha \lhd_n \beta))$ and

   ii) $\forall\beta\forall n\ (|\{\alpha : \alpha \lhd_n \beta\}| < \omega)$.

3. Let M be a countable transitive model for ZFC . In M , let $\mathbb{P}$ be countable partial functions from ω_1 to 2 . Let G be $\mathbb{P}$-generic over M . Show that CH holds in $M[G]$.

Recursion Theory

1. True or False: If the intersection of two simple sets is infinite, then the intersection is simple. Prove that your answer is correct.

2. Suppose $\{\underset{\sim}{A}_i \mid i < \omega\}$ satisfies $A_i <_T A_{i+1}$ for all $i < \omega$. Prove that $\{\underset{\sim}{A}_i \mid i < \omega\}$ has $2^{\aleph_0}$ minimal upper bounds.

3. Prove that there exist low r.e. degrees $\underset{\sim}{a}, \underset{\sim}{b}$ such that for every r.e. degree $\underset{\sim}{c}$ there are r.e. degrees $\underset{\sim}{a}_0 \leq \underset{\sim}{a}$, $\underset{\sim}{b}_0 \leq \underset{\sim}{b}$ satisfying $\underset{\sim}{a}_0 \cup \underset{\sim}{b}_0 = \underset{\sim}{c}_0$.

   [Hint: Sack's splitting theorem says that for every r.e. C there are low disjoint r.e. A,B such that $A \cup B = C$. apply this to $K_0 =_{df} \{\langle n,e \rangle \mid n \in We\}$].

4. Prove that there is a Δ_2^0 degree that is not r.e.