Qualifying Exam

LOGIC

August 30, 1984

INSTRUCTIONS: Do four questions, at most two elementary.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.

DEFINITIONS

1. If $\eta, \xi \in \omega^{<\omega}$ then

$$\eta \lhd \xi \quad \text{iff}_{df} \quad \forall i < lh(\eta)[\eta(i) = \xi(i)]$$

2. $\eta \cap \xi =_{df}$ the maximal α such that $\alpha \lhd \eta, \xi$.

3. $\{\mu_i | i<\omega\}$ is the standard enumeration of all partial recursive functions $\mu : \omega \to \omega$.

4. $\{W_i | i<\omega\}$ standard enumeration of all r.e. sets, i.e. $W_i = dom(\mu_i)$.

5. $K =_{df} \{i | \mu_i(i)\downarrow\}$

6. $A|_T B$ just if A and B are Turing incomparable.

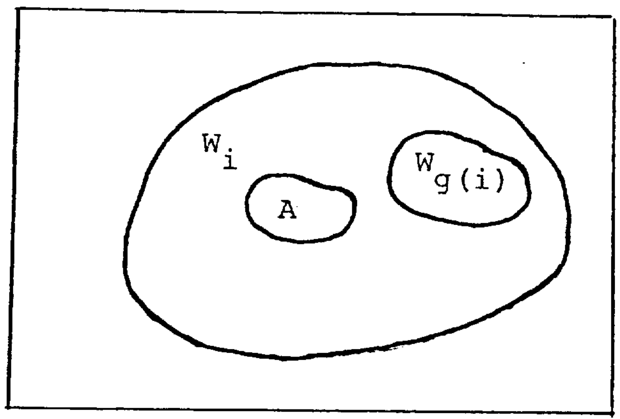
7. $T \subset \omega^{<\omega}$ is a tree iff$_{df}$ $\forall \eta, \xi \in \omega^{<\omega}[\eta \lhd \xi \in T \to \eta \in T]$.

Definitions--continued

8. A is effectively nowhere simple iff_{df}

$\exists g$ recursive $\forall i\ [W_{g(i)} \subset W_i - A$ and

$(W_i - A$ infinite $\rightarrow W_{g(i)}$ infinite$)]$



9. $A \leq_{\omega} B$ iff_{df} $\exists i[\mu_i^B = A$ and $\exists$ rec f $\forall x(\mu_i^{B \restriction f(x)}(x) = \mu_i^B(x))]$,

in other words, the use function is recursively bounded.

10. TySp(A) = set of complete types realized in A .

Elementary

1. Prove that if A is a countable structure and

$$\forall \bar{a} \in |A|^{<\omega} \; [\langle A, \bar{a} \rangle \text{ has more than one automorphism}] ,$$

then A has $2^{\aleph_0}$ automorphisms.

2. Find a structure $\mathfrak{A}$ for a countable language such that $\mathfrak{A}$ has exactly ω_1 elementary substructures. Don't assume the continuum hypothesis.

3. Which groups have the property that every subgroup is an elementary subgroup?

Model Theory

1. Assume that T is a complete theory and A_η , $\eta \in 2^{<\omega}$ are models of T satisfying:

   a) $TySp(A_\eta) \cap TySp(A_\xi) \subseteq TySp(A_{\eta \cap \xi})$ $\eta, \xi \in 2^{<\omega}$;

   b) $TySp(A_\eta) \subsetneq TySp(A_\xi)$ $\eta \ntrianglelefteq \xi$, $\eta, \xi \in 2^{<\omega}$.

   Prove that T has $2^{\aleph_0}$ pairwise non-isomorphic countable models.

2. Prove or disprove: If A is a structure realizing a non-principal type $\Gamma(x)$, then A has an elementary substructure omitting $\Gamma(x)$.

3. Prove or disprove:
   If T_i , $i < 3$ are complete theories in L_i , $i < 3$ respectively, and $T_0 \cup T_1 \cup T_2$ is inconsistent, then $\exists\, i,j < 3$ such that $T_i \cup T_j$ is inconsistent.

Model Theory--continued

4. Let $\mathfrak{A}$ be an infinite structure with a total order. Let $\mathfrak{B} = \mathfrak{A}^{\omega}/D$, where D is a non-principal ultrafilter on ω. Show that $\mathfrak{B}$ has an ordered set of indiscernibles of size 2^{ω}. Don't assume the Continuum Hypothesis.

Recursion Theory

1. Prove or disprove each of

   a) $\exists S$ r.e. $\exists \psi$ partial recursive $\forall i$

   $[W_i \text{ infinite} \rightarrow W_i \cap S \cap \{0,1,\ldots,\psi(i)\} \neq \phi]$;

   b) $\exists S$ r.e. $\exists g$ recursive $\forall i$

   $[W_i \text{ infinite} \rightarrow W_i \cap S \cap \{0,1,\ldots,g(i)\} \neq \phi]$.

2. Prove B is effectively nowhere simple iff $\exists C$ recursive $\forall i$ $[C \cap B = \phi$ and $(W_i - B \text{ infinite} \rightarrow C \cap W_i \text{ infinite})]$

   [Hint: Various subsets of $\bigcup_{i<\omega} W_{g(i)}$ are worth considering.]

3. Assume that $T \subset 2^{<\omega}$ is an infinite recursive tree. Prove that there is an $f \in {}^{\omega}2$ satisfying:

   a) $\forall n\, [f \restriction_n \in T]$;

   b) $f' \equiv_T 0'$ (i.e., f has low degree).

   [Hint: construct a recursive sequence $\{\eta_s\}_{s<\omega}$ such that $\lim_s \mu_s$ exists and is the desired f. To meet b), consider

   $$R_e : (\exists^\infty s\ \mu_{e,s}^{\eta_s}(e)\downarrow) \rightarrow \mu_e^{f}(e)\downarrow] .$$

4. Construct A, B r.e. such that $A \leq_T B$ but $A \not\leq_{T\omega} B$.

   [Hint: Diagonalize to ensure $A \not\leq_{\omega} B$. For $A \leq_T B$, for each n introduce a marker $x_{n,s}$ such that $\lim_s x_{n,s} = x_n$ exists and $x_n \in B$ iff $n \in A$. You must decide how to resolve conflicts.]

Set Theory

1. Prove that the Continuum Hypothesis is equivalent to the following statement:

$$\exists \mathcal{F} \subset {}^{\mathbb{R}}\omega \, [\, |\mathcal{F}| = c \quad \&$$

$$\forall f,g \in \mathcal{F} \; (f \neq g \rightarrow |\{x : f(x) \neq g(x)\}| \leq \omega)] \quad .$$

$\mathbb{R}$ is the set of real numbers, and $c = |\mathbb{R}|$.

2. Assume $L(\alpha) \prec L(\beta)$ and $\alpha < \omega_1 < \beta$. Show that $L(\alpha) \models ZFC$. $\prec$ means elementary submodel (with the ϵ relation understood). ω_1 means the <u>real</u> omega one (not $(\omega_1)^L$) .

3. Assume Martin's Axiom. Let k_n $(n \in \omega)$ be such that $0 < k_0 < k_1 < \cdots < \omega$. Let $\mathcal{F} \subset \prod_n k_n$ with $|\mathcal{F}| < 2^{\omega}$. Show that there is a $g \in \prod_n k_n$ such that $\{n : f(n) = g(n)\}$ is finite for all $f \in \mathcal{F}$.

4. Let M be a countable transitive model for ZFC. Within M, define the partial order $\mathbb{P}$ for adding a generic total order on ω_1; thus each $p \in \mathbb{P}$ is a pair, $\langle a_p, \lhd_p \rangle$, where a_p is a finite subset of ω_1 and $\lhd_p$ totally orders a_p. $\leq_{\mathbb{P}}$ is defined in the obvious way. Show that $\mathbb{P}$ is ccc in M, and that $\mathbb{1}$ forces that $\lhd$ is a separable dense total order without endpoints. Here, $\lhd$ is the name for

$$\bigcup_{p \in G} \lhd_p \quad .$$

Separable means that there is a countable subset of ω_1 which is dense in the order.