Qualifying Exam

LOGIC

January 17, 1985

INSTRUCTIONS: Do four questions, at most two elementary.

Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.

DEFINITIONS

1. If $\eta, \xi \in \omega^{<\omega}$ then

$$\eta \lhd \xi \quad \text{iff}_{df} \quad \forall i < lh(\eta)[\eta(i) = \xi(i)]$$

2. $\eta \cap \xi =_{df}$ the maximal α such that $\alpha \lhd \eta, \xi$.

3. $\{\mu_i | i<\omega\}$ is the standard enumeration of all partial recursive functions $\mu : \omega \to \omega$.

4. $\{W_i | i<\omega\}$ standard enumeration of all r.e. sets, i.e. $W_i = dom(\mu_i)$.

5. $K =_{df} \{i | \mu_i(i)\downarrow\}$

6. $A|_T B$ just if A and B are Turing incomparable.

7. $T \subset \omega^{<\omega}$ is a tree iff_{df} $\forall \eta, \xi \in \omega^{<\omega}[\eta \lhd \xi \in T \to \eta \in T]$.

8. If $T \subset \omega^{<\omega}$ is a tree, then $f \in {}^{\omega}\omega$ is a branch of T iff_{df} $\forall n<\omega[f\restriction_n \in T]$.

ELEMENTARY

1. Sort the following set of ordinals.

$$\{ (\omega^{\omega}) \cdot (\omega+\omega) , \quad (\omega+\omega) \cdot (\omega^{\omega}) ,$$

$$\omega^{\omega}\cdot\omega + \omega^{\omega}\cdot\omega , \quad \omega\cdot\omega^{\omega} + \omega\cdot\omega^{\omega} ,$$

$$\omega\cdot\omega^{\omega} + \omega^{\omega}\cdot\omega , \quad \omega^{\omega}\cdot\omega + \omega\cdot\omega^{\omega}$$

$$\}$$

Exponentiation is ordinal exp. Some values may be repeated.

2. Prove: There is a function $f : \mathbb{R} \to \mathbb{R}$ such that whenever P,Q are perfect subsets of $\mathbb{R}$, $|Q \cap f''P| = 2^{\aleph_0}$. $f''P = \mathrm{ran}(f \restriction P)$.

3. Let L be a finite language, T_i L-theories, $T_i \subset T_{i+1}$, and assume $\forall i<\omega \; \exists A[A \models T_i$ and not $A \models T_{i+1}]$.
Prove or disprove: $\bigcup_{i<\omega} T_i$ has an infinite model.

SET THEORY

1. Let $\mathbb{P}$ be the partial order $2^{<\kappa} = \bigcup_{\alpha<\kappa} 2^{\alpha}$, ordered by reverse inclusion. Under what conditions (cofinalities and cardinal arithmetic) will forcing with $\mathbb{P}$ preserve all cardinals. Why? Caution. Do not assume GCH and do not assume κ is regular.

2. Prove. If $cf(\kappa) > \omega$ and F is a family of κ finite sets, then there is a $G \subseteq F$ such that $|G| = \kappa$ and G forms a Δ-system.

3. Let I be the ideal of non-stationary subsets of ω_1, and let B be the Boolean algebra, $P(\omega_1)/I$. Prove that B is ω_2-complete (every set of size ω_1 has a supremum).

MODEL THEORY

1. Assume T is a first order consistent theory, not necessarily complete. Suppose further that $\{\varphi_i(x) \mid i<\omega\}$ is a set of formulas consistent with T in one free variable such that for any $\theta(x) \in L(T)$, if $T \cup \{\exists x\theta\}$ is consistent, then there is an $i<\omega$ such that $T \cup \{\exists x(\theta(x) \wedge \neg\varphi_i(x))\}$ is consistent.

   Prove or disprove: T has a model A such that the following set is infinite:

   $$\{a \in |A| \mid \forall i<\omega[\langle A,a\rangle \models \varphi_i(\underline{a})]\} .$$

2. Let T be a complete theory in a countable language with no finite models such that for some $A \models T$ the following is true:

   $$\forall B, C \text{ countable} \models T[A \not\cong B \ \& \ A \not\cong C \rightarrow B \prec C] .$$

   Prove or disprove: T is $\aleph_0$-categorical.

3. Suppose T is a first order theory and $\{A_i \mid i<\omega\}$ is a set of models of T such that

   $$\forall B \models T \ \exists i<\omega \ [A_i \prec B] .$$

   How many complete extensions (closed under deduction) can T have?

RECURSION THEORY

1. $\{A_i, i<3\}$ is a recursively enumerable, recursively inseparable triple iff_{df}

   i) A_i r.e. $i<3$;

   ii) $i \neq j \to A_i \cap A_j = 0$;

   iii) If R is recursive, $\{i,j,k\} = \{0,1,2\}$, and $A_i \cup A_j \subset R$, then $A_k \cap R \neq 0$.

   Prove there is a recursively enumerable, recursively inseparable triple.

2. Let $f_i(x_0,x_1,x_2)$, $i<3$ be total recursive functions. Prove there are ℓ_i, $i<3$ satisfying

$$\mu_{f_i(\ell_0,\ell_1,\ell_2)} = \mu_{\ell_i} \quad i<3 .$$

3. Assume Tr is a recursive tree with no terminal nodes. A set $\{f_i | i<\omega\}$ of recursive branches of Tr is Σ_n iff_{df} there is a Σ_n set $A \subset \omega$ such that:

   1) $\forall i \exists j [f_i = \mu_j$ and $j \in A]$;

   2) $\forall j \exists i [j \in A \to f_i = \mu_j]$.

   Prove that every Σ_2 set of recursive branches of Tr is contained in a Σ_1 set of recursive branches of Tr.