Qualifying Exam

LOGIC

January 16, 1986

INSTRUCTIONS: Do <u>four</u> questions, at most <u>two</u> elementary.

Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person.

Policy on Misprints

The Doctoral Exam Committee tries to proofread the exams as carefully as possible. Nevertheless, the exam may contain misprints. If you are convinced a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.

Elementary

1. Prove that for any ordinal α the following are equivalent:

   a) $\exists\beta\ [\alpha = \omega^{\beta}]$ (ordinal exponentiation).

   b) $\forall A \subseteq \alpha[(A,<) \cong (\alpha,<)$ or $(\alpha-A,<) \cong (\alpha,<)]$.

   ($\cong$ means isomorphic).

2. A transversal is a 1-1 choice function. Suppose F is a family of finite sets. Show that F has a transversal iff every finite subset of F has a transversal. Is this still true if F contains an infinite set?

3. Prove or disprove: If $\langle A,R\rangle \equiv \langle B,S\rangle$ then

   $\langle A,A\times A-R\rangle \equiv \langle B,B\times B-S\rangle$ ($\equiv$ means elementarily equivalent).

4. Let S be a countable set of ordinals. Show that

   $\{\beta \mid \exists\langle \alpha_n \mid n\in\omega\rangle \in S^{\omega}[\sum_{n<\omega} \alpha_n=\beta]\}$ is countable.

Recursion Theory

1. Let $A \subsetneq \omega$. Prove that A is creative iff

$$\forall \text{ r.e. } B[A \cap B = \emptyset \rightarrow A \equiv_1 A \cup B] \quad .$$

[Hint: For $\leftarrow$, prove first that A is not simple. Then be creative finding a set in the complement of A].

2. $A \leq_T K$ is n-r.e. iff$_{df}$ $A = \text{Lim}_s A_s$ for a recursive sequence $\{A_s\}_{s<\omega}$ such that

   1) $A_0 = \emptyset$; and

   2) $\{s \mid A_s(x) \neq A_{s+1}(x)\}$ has fewer than $n+1$ elements.

   Prove that if A is n=r.e. , then either A or $\overline{A}$ contains an infinite r.e. set.

3. $A \subset \omega$ is autoreducible iff$_{df}$ there is an e such that

$$\forall x \; [\; A(x) = \mu_e^{A-\{x\}}(x)] \quad .$$

   Prove there exists an r.e. set which is not autoreducible. [Hint: Priority style construction].

Recursion Theory--continued

4. A is <u>self-dual</u> iff_{df} $A \leq_m \overline{A}$. Let $B \subseteq \omega$ be an index set, i.e.

$$\forall e \forall i\ [\mu_e = \mu_i \rightarrow (e \in B \leftrightarrow i \in B)] \ .$$

Prove B is not self-dual.

Model Theory

All languages are countable.

1. Assume

   i) $L_0 \subset L_1 \subset L_2$ first order language;

   ii) T_i is a complete L_i theory $i < 3$;

   iii) $\mathcal{A}_i \models T_i$, $\mathcal{B}_j \models T_j$ $0 \leq i \leq 1$, $1 \leq j \leq 2$; and

   iv) $\mathcal{A}_0 \subsetneq_{\rightarrow} \mathcal{A}_1 \restriction_{L_0}$, $\mathcal{B}_1 \subsetneq_{\rightarrow} \mathcal{B}_2 \restriction_{L_1}$ [i.e. embeddable in a reduct] .

   Prove or disprove:

$$\forall \mathcal{A} \models T_0\ \exists \mathcal{B} \models T_2\ [\mathcal{A} \subseteq \mathcal{B} \restriction_{L_0}] \ .$$

Model Theory--continued

2. For an L-structure $\mathcal{A}$,

$$TySp(\mathcal{A}) =_{df} \{\Gamma \text{ complete n-type in } L \mid \mathcal{A} \text{ realizes } \Gamma , \; n < \omega\} .$$

Assume that $\mathcal{A} \models T$, T is complete, and T has only finitely many countable models. Prove

$$\exists \Gamma \in TySp(\mathcal{A}) \forall \Sigma \in TySp(\mathcal{A}) \forall \mathcal{B} \models T[\Gamma \in TySp(\mathcal{B}) \rightarrow \Sigma \in TySp(\mathcal{B})] .$$

3. Prove or disprove: If T is complete and has at least two non-isomorphic models of size ω_1 omitting the type Γ , then it must have infinitely many pairwise non-isomorphic models of size ω_1 omitting Γ .

4. Assume L has just a binary function symbol as its only non-logical symbol, and T has as axioms the universal closures of:

$$f(x,y) = f(z,t) \rightarrow x = z \wedge y = t \quad \text{and}$$

$$x \neq \tau$$

whenever τ is a term containing x . Prove that T is complete.

Set Theory

1. Assume $\mathbb{P}$ is c.c.c. and $S \subset \omega_1$ is stationary. Prove

$$1\!\!1 \Vdash_{\mathbb{P}} (\check{S} \text{ is stationary}) .$$

2. Show there exists a metric space (X, ρ) such that

$$\forall x \in X \, \forall r \in \mathbb{R}^+ \, \exists! y \in X \, [\rho(x,y) = r] .$$

($\mathbb{R}^+$ is the set of positive reals).

3. Suppose $n \in \omega$ and $\langle B_\alpha : \alpha < \omega_1 \rangle$ is an ω_1 sequence of countable sets such that for every $\alpha, \beta < \omega_1$, $\alpha \neq \beta \rightarrow |B_\alpha \cap B_\beta| \leq n$. Show there exists $\langle A_\alpha : \alpha < \omega_1 \rangle$ each A_α finite and $\{B_\alpha \setminus A_\alpha : \alpha < \omega_1\}$ are all pairwise disjoint. [Hint: Löwenheim Skolem theorem]

4. Let M be a countable transitive model for ZFC. Prove that for some $x \subset \omega$, $M[x] \models \text{ZFC}$, $\omega_1^M = \omega_1^{M[x]}$, and $\omega_2^M < \omega_2^{M[x]}$.