Qualifying Exam
Logic
Jan 15, 1987

Instructions: Do any **four** problems, but at most **two** elementary. Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person. If you think a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.

Elementary

1. A set of sentences Σ is independent iff

$$\forall \sigma \in \Sigma \; [\Sigma - \{\sigma\}] \nvdash \sigma$$

   Prove that if Γ, Σ are sets of formula satisfying:

   (a) $\Sigma \cap \Gamma = \emptyset$;

   (b) $|\Gamma| \leq |\Sigma|$;

   (c) $\forall \sigma \in \Sigma \; [(\Sigma \cup \Gamma) - \{\sigma\} \nvdash \sigma]$

   Then $\Sigma \cup \Gamma$ can be axiomatized by a set of independent sentences.

2. Find $A \subset B \subset C \subset D$ countable structures in the same language such that: A is isomorphic to C, B is elementarily equivalent to D but not isomorphic to it, and the theory of A is $\aleph_0$-categorical.

3. Show that the set of validities in the first order theory of pure equality is recursive.

4. Let $(P, <)$ be an infinite partial order. Show that P contains an infinite subset of order type ω or ω^* (i.e. converse ω), or an infinite set of pairwise incomparable elements.

Recursion Theory

5. Prove that if f,g are total recursive functions and A is a simple set, then $\exists n \in A$ satisfying
$$W_{f(n)} \cup W_{g(n)} = W_n$$

6. Assume A,B r.e. satisfying
   (a) $A \subset B$;
   (b) $\forall n \; [B^{[n]} - A^{[n]} \text{ infinite}]$.

   Prove $\exists C, D$ r.e. satisfying
   (a) $A \subset C, D \subset B$;
   (b) $C|_T D$.

7. Prove that there are no A,B r.e. recursively inseparable sets and simple set S such that $B \leq_m S$.

8. Prove there exists recursively incomparable maximal sets.

Model Theory

9. Find $T_i, \Gamma_i, L_i \;\; i < 2$ such that:

   (a) T_i is complete theory in L_i, $i < 2$;

   (b) Γ_i complete non-principal type of T_i, $i < 2$;

   (c) $T_0 \cup T_1$ is a consistent theory in $L_0 \cup L_1$; and

   (d) there exists a $L_0 \cup L_1$ formula $\theta(\bar{x})$ which is consistent with $T_0 \cup T_1$ and for every formula $\psi(\bar{x}) \in (\Gamma_1(\bar{x}) \cup \Gamma_2(\bar{x}))$

$$(T_0 \cup T_1) \vdash \theta(\bar{x}) \rightarrow \psi(\bar{x})$$

10. Assume T is a complete consistent theory such that no complete consistent expansion of T by finitely many constants has a complete principal type. Prove that every model of T has a proper elementary substructure.

11. $L(T) = \{<, c_i \;\; : \;\; i < \omega\}$. T is a complete consistent theory which says that $<$ is a dense linear order without endpoints and the c_i's are distinct constants. What are the possible cardinalities of the class of countable isomorphism types of models of T?

12. Let T be the theory with countably many unary relation symbols $\{P_n : n \in \omega\}$ and all axioms of the form:

$$\exists x (\wedge_{n \in A} P_n \wedge \wedge_{n \in B} \neg P_n)$$

    where A and B are disjoint finite subsets of ω. Show that T is a complete theory.

Set Theory

13. Suppose A_α for $\alpha < \omega_1$ are countable and for all $\alpha < \omega_1$:

$$A_\alpha \cap (\cup_{\beta<\alpha} A_\beta) \text{ is finite}$$

Show there exists $X \in [\omega_1]^{\omega_1}$ and a set Z such that for every distinct $\alpha, \beta \in X$, $A_\alpha \cap A_\beta = Z$, i.e. an uncountable Δ-system.

14. Assume $MA+\neg CH$ and suppose $a_\alpha \subset \omega$ for $\alpha < \omega_1$. Show there exists $X \in [\omega]^\omega$ such that for every $\alpha < \omega_1$

$$\text{either } X \subset^* a_\alpha \text{ or } X \cap a_\alpha =^* \emptyset$$

where * means modulo finite.

15. Show there exists an almost disjoint family F of countably infinite subsets of the real line R such that for every uncountable $X \subset R$ there exists a $Y \in F$ such that $Y \subset X$.

16. Assume $MA + \neg CH$ and suppose that P is a poset with the ccc. and τ is a term in the forcing language of P such that

$$1 \Vdash \tau \subset \omega_1 \text{ is stationary}$$

Show that for some P-filter G the set

$$\{\alpha < \omega_1 \ : \ \exists p \in G \ p \Vdash \alpha \in \tau\}$$

is stationary in ω_1.