Qualifying Exam

Logic

Aug 27 1987

Instructions: Do any **four** problems, but at most **two** elementary. Please use a separate packet of paper for each problem since not all of your answers will be graded by the same person. If you think a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.

DEFINITIONS

1. $\omega = N$ = the set of natural numbers.

2. $\mathcal{A} \precsim \mathcal{B}$ means $\mathcal{A}$ is elementarily embeddable in $\mathcal{B}$.

3. $C^{<\omega}$ are all finite sequences from C .

4. $\{W_e^{(n)} \mid e<\omega\}$ is the standard enumeration of all r.e. subsets of ω^n . $W_e = W_e^{(1)}$.

5. $K = \{e \mid e \in W_e\}$.

6. $Tr \subset \omega^{<\omega}$ is a tree iff $\forall \alpha, \beta \in \omega^{<\omega}$: if $\alpha \subset \beta$ and $\beta \in Tr$, then $\alpha \in Tr$.

7. $\mathcal{A} \prec \mathcal{B}$ means $\mathcal{A}$ is an elementary submodel of $\mathcal{B}$.

8. $\mathcal{A} \leq_T \mathcal{B}$ means $\mathcal{A}$ is Turing reducible to $\mathcal{B}$.

Elementary

1. Given a countable set of students and a countable set of classes. Suppose each student wants one of a finite set of classes, and each class has a finite enrollment limit. Use the compactness theorem to prove that if each finite set of students can be accommodated, then the whole set can.

2. Let T and U be first order theories in a language L. Suppose that for each finite subset $T_o \subseteq T$ and $U_o \subseteq U$ there are models $\mathbb{A}_o \models T_o$ and $\mathbb{B}_o \models U_o$ such that $\mathbb{B}_o$ is a submodel of $\mathbb{A}_o$. Prove that there are models $\mathbb{A} \models T$ and $\mathbb{B} \models U$ such that $\mathbb{B}$ is a submodel of $\mathbb{A}$.

3. Given a partial order $< A, <^* >$ with no infinite decreasing sequences. Prove that there is a well order $< A, < >$ such that $<^* \subseteq <$.

4. Let κ be an uncountable cardinal of countable cofinality. Show there exists $< f_\alpha : \omega \to \kappa \mid \alpha < \kappa^+ >$ such that for all $\alpha \neq \beta$ and for all but finitely many n, $f_\alpha(n) \neq f_\beta(n)$.

Recursion Theory

1. Prove that there is no recursive g such that for all $e<\omega$:

   1) $W_{g(e)}$ is finite; and

   2) if W_e is finite, then $W_e=W_{g(e)}$.

2. Given an infinite r.e. set A, construct a low simple set S containing the complement of A.

3. Prove

$$\forall A \subseteq N \ \exists B,C \subseteq N \ \exists e \ [\ A = \phi_e^B = \phi_e^C \ \text{ and } \ B \mid_T C\ ].$$

4. Prove that there exist a minimal triple of Turing degrees such that no two of the degrees form a minimal pair.

[$\underset{\sim}{A}$, $\underset{\sim}{B}$, $\underset{\sim}{C}$ are a <u>minimal triple</u> iff_{df}

they are non-recursive and $\forall D[\ D \leq_T A,B,C \rightarrow D$ recursive]]

Model Theory

1. Let $\mathfrak{A}$ be an infinite model of a countable language. Prove that for each $b \in A$, $Th(\mathfrak{A})$ is ω-categorical iff $Th(\mathfrak{A}, b)$ is ω-categorical.

2. Let $\mathfrak{A}$ be a model with the property that each subset U of A is a relation of $\mathfrak{A}$ and each function $f: A \to A$ is a function of $\mathfrak{A}$. Suppose $\mathfrak{A} \prec \mathbb{B}$ and there is an element $b \in B$ such that $\mathbb{B}$ has no proper submodels containing b. Prove that there is an ultrafilter D over A such that
$\mathbb{B} \cong \Pi_D \mathfrak{A}$.

3. Let T be a complete theory with infinite models in a countable language and let κ be an infinite cardinal. Prove that T has models $\mathfrak{A}$ and $\mathbb{B}$ of power κ where $\mathbb{B}$ is a proper submodel of $\mathfrak{A}$ and there is an automorphism f of $\mathfrak{A}$ such that

$$\mathbb{B} \prec f(\mathbb{B}) \prec f(f(\mathbb{B})) \prec f(f(f(\mathbb{B}))) \prec \ldots$$

and

$$\mathfrak{A} = \mathbb{B} \cup f(\mathbb{B}) \cup f(f(\mathbb{B})) \cup f(f(f(\mathbb{B}))) \cup \ldots .$$

Hint: use indiscernibles.

4. Prove that for any consistent complete theory T there is a model $\mathfrak{A} \models T$ such that

$\forall a, b \in A$ [a,b realize the same 1-type in $\mathfrak{A}$ iff

$$\exists \theta(x,y)[\ \mathfrak{A} \models \theta(a,b) \text{ and } \forall \sigma(x)[\ T \vdash \forall x,y[\theta(x,y) \to [\sigma(x) \leftrightarrow \sigma(y)]]]]],$$

where 'θ' and 'σ' range over formulas of L(T).

Set Theory

1. Assume CH and let Lim be the set of limit ordinals less than ω_1. Show that there exists $< A_\alpha \mid \alpha \in \text{Lim} >$ such that for every $\alpha \in$ Lim , $A_\alpha \subseteq \alpha$ and for $\alpha \neq \beta$, $A_\alpha \cap A_\beta$ is finite, but there does not exist $X \in [\text{Lim}]^{\omega_1}$ and $\alpha < \omega_1$ such that for every $\gamma \neq \beta \in X$,

$A_\gamma \cap A_\beta \subseteq \alpha$.

2. Assume there exists an uncountable transitive model of ZFC. Show there exists an uncountable transitive model of ZFC+V$\neq$L.

Hint: Consider forcing with $P = (2^{<\kappa})^{L_\alpha}$ for appropriate α, κ .

3. Let P=FIN(ω_2) be the partial order of functions with finite domaincontained in ω_2 and range $\{0,1\}$. Show that in the generic extension obtained by forcing with P that there does not exist a linear orderof cardinality ω_1 such that every other linear order of cardinality ω_1 can be embedded.