QUALIFYING EXAM
LOGIC
August, 1989

INSTRUCTIONS: Do any **four** problems, including at most **two** elementary problems. Please use a separate packet of paper for each problem, since not all of your answers will be graded by the same person. If you think a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

NOTATION: ω is the set of natural numbers. $\mathbb{R}$ is the set of real numbers. $\mathbb{Q}$ is the set of rational numbers. $\Pi_D M_i$ is the ultraproduct of the models M_i modulo D. ϕ_n is the partial recursive function with index n. W_n is the domain of ϕ_n.

ELEMENTARY PROBLEMS

E1. Let $f(n)$ be the n^{th} digit after the "." in the decimal expansion of e. (So $f(1)=7$, $f(2)=1$, $f(3)=8$). Prove that the function f is computable.

E2. Let S be the set of all finite sequences of elements of ω. If $s,t \in S$, we say that $s \to t$ iff t is obtained by replacing one term of s by a finite (possibly empty) sequence of smaller natural numbers. So, for example,

$$(5,5) \to (4,1,3,4,5) \to (4,1,3,2,1,3,5).$$

Prove that there is no infinite subset $\{s_n : n \in \omega\} \subseteq S$ such that $s_0 \to s_1 \to s_2 \to s_3 \to \cdots$.

E3. Let $g(n)$ be the number of distinct prime divisors of n, with $g(0)=g(1)=0$. Prove that in every nonstandard model M of Peano Arithmetic, there is an element b such that in M,

$$b > g(b) > g(g(b)) > g(g(g(b))) > \cdots.$$

SET THEORY

S1. Assume that $2^{\omega} = \omega_2$ and $2^{\omega_1} = \omega_4$. Prove that there is an almost disjoint family of ω_4 uncountable subsets of ω_1. Here, "almost disjoint" means that the intersection of any two distinct elements of the family is countable.

S2. Define a relation $\leq$ on the set of all ω_1-Suslin trees by saying that $X \leq Y$ iff the 1 condition of Y forces that there is an uncountable chain in X. Prove that $\leq$ is transitive.

MODEL THEORY

M1. Let T be the complete first order theory of the model $M = (\mathbb{R}, \mathbb{Q}, \leq, q)_{q \in \mathbb{Q}}$. Prove that every model of T has a proper elementary submodel.

M2. Let κ be the first cardinal such that $cf(\kappa) > \omega$, and $\lambda < \kappa$ implies $\lambda^{\omega} < \kappa$. Let $\mathcal{L}$ be the language which is formed by adding to first order logic the extra quantifier (Qx) where $(Qx)\theta(x, \cdots)$ holds if and only there exist at least κ x's such that $\theta(x, \cdots)$. Let D be an ultrafilter over a countable set I. Prove that for any family of models M_i, $i \in I$ and any sentence θ of $\mathcal{L}$,

$$\Pi_D M_i \models \theta \text{ if and only if } \{i \in I : M_i \models \theta\} \in D.$$

RECURSION THEORY

R1. Prove that for every partial recursive function $f(x,y)$ there exists $n \in \omega$ such that

$$W_n = \{f(n,y) : y \in \omega \text{ and } f(n,y) \text{ is defined}\}.$$

R2. Prove that the set

$$\{x \in \omega : \phi_x \text{ is total and } \lim_{n \to \infty} \frac{\phi_x(n)}{n} = \infty\}$$

is complete Π^0_3.