QUALIFYING EXAM IN LOGIC

January, 1991

INSTRUCTIONS: Do any four problems. Use a separate packet of paper for each problem, since not all of your answers will be graded by the same person. You should not hand in more than four problems; if you do more, only the first four will be graded.

If you think a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

NOTATION: ω is the set of natural numbers. All languages are understood to be languages in first order predicate logic. The universe of a model $\mathfrak{A}$ is denoted by A. A set $X \subseteq Y$ is <u>cofinite</u> in Y if $Y - X$ is finite. The <u>cofinality</u> of a linearly ordered set $\langle S, \leq \rangle$ is the least cardinal κ such that for some $T \subseteq S$ of size κ, $(\forall s \in S)(\exists t \in T) s \leq t$. The <u>coinitiality</u> of $\langle S, \leq \rangle$ is defined similarly but with $s \geq t$. If $A, B \subseteq \omega$, $A \equiv_T B$ means that A is Turing equivalent to B, and $A <_T B$ means that A is Turing reducible to B and not $A \equiv_T B$. W_x is the domain of the partial recursive function with Godel number x. $A^{(n)}$ is the n^{th} jump of A. $A \oplus B$ is the disjoint union $\{2^x : x \in A\} \cup \{3^y : y \in B\}$. ZFC is Zermelo-Fraenkel set theory with choice. MA is Martin's Axiom, CH is the continuum hypothesis, and GCH is the generalized continuum hypothesis.

ELEMENTARY PROBLEMS

E1. Let T be a theory in a finite language which has no infinite models. Show that T is decidable.

E2. Let $\mathfrak{A}$ be elementarily equivalent to the model $\langle\omega,0,s\rangle$ where s is the successor function. Show that for every formula $\phi(x)$ in the language of $\mathfrak{A}$, the set $\{a \in A : \mathfrak{A} \models \phi[a]\}$ is either finite or cofinite in A.

MODEL THEORY

M1. Let T be a theory with infinite models in a countable language. Prove that T has a countable model $\mathfrak{A}$ which has 2^{ω} distinct elementary submodels.

M2. Let κ and λ be infinite regular cardinals. Prove that the standard model of arithmetic has an elementary extension $\mathfrak{A} = \langle A, +^{\mathfrak{A}}, *^{\mathfrak{A}}, \leq^{\mathfrak{A}}\rangle$ such that $\langle A - \omega, \leq^{\mathfrak{A}}\rangle$ has coinitiality κ and cofinality λ.

RECURSION THEORY

R1. Show that there is no partial recursive function $\phi(x)$ on ω such that for all x, $W_x \neq \emptyset$ implies that $\phi(x)$ is defined and equals $\min\{y : y \in W_x\}$.

R2. Show that there are r.e. sets A and B such that for all n,
$A^{(n)} <_T \emptyset^{(n+1)}$, $B^{(n)} <_T \emptyset^{(n+1)}$, and $A^{(n)} \oplus B^{(n)} \equiv_T \emptyset^{(n+1)}$.
Hint: Use the Sacks Splitting Theorem, the Robinson Jump Interpolation Theorem, and the Recursion Theorem.

SET THEORY

S1. Let M be a countable transitive model of ZFC + GCH. Show that there is a forcing extension of M satisfying ZFC + GCH together with the statement that not every subset of ω_1 is constructible from a subset of ω.

S2. Assume MA and $\neg$CH. Let X be a set of real numbers of size $\aleph_1$. For each $x \in X$, let S_x be an ω-sequence of elements of X which converges to x. Prove that there is an uncountable $Y \subseteq X$ such that $Y \cap S_x$ is finite for all $x \in X$.

SET THEORY

S1. Let M be a countable transitive model of $ZFC + GCH$. Show that there is a forcing extension of M satisfying $ZFC + GCH$ together with the statement that not every subset of ω_1 is constructible from a subset of ω.

Solution: In M, let $\mathcal{P}$ be finite partial functions from ω_1 to 2. Since (in M) $\mathcal{P}$ is *ccc* and has size ω_1, $M[G]$ still satisfies GCH. Every element of $M[G \cap \mathcal{P}\ ]$ is of the form τ_G for some $\mathcal{P}$ name τ. In particular, $G \notin M[G \cap \mathcal{P}\ ]$ for any $\alpha < \omega_1$. In $M[G]$, if $x \subset \omega$, then $x \in M[G \cap \mathcal{P}\ ]$ for some $\alpha < \omega_1$, so $G \notin M[x]$, so $G \notin L[x]$. Thus, if $A = \{\alpha \in \omega_1 : \ G(\alpha) = 1\}$, then A is a subset of ω_1 not constructible from any subset of ω.

S2. Assume MA and ¬CH. Let X be a set of real numbers of size $\aleph_1$. For each $x \in X$, let S_x be a simple sequence in X which converges to x. Prove that there is an uncountable $Y \subset X$ such that $Y \cap S_x$ is finite for all $x \in X$.

Solution: Let $\mathcal{S} = \{S_x : x \in S\}$. Let $\mathcal{P}$ be the set of all pairs, $p = \langle a_p, F_p \rangle$ such that a_p is a finite subset of X and F_p is a finite subset of $\mathcal{S}$. Say $q \leq p$ iff $a_q \supseteq a_p$, $F_q \supseteq F_q$, and

$$\forall S \in F_p \forall x \in (a_q \backslash a_p)(x \notin S) \quad .$$

Dense sets: Say $X = \bigcup \{X_\alpha : \alpha \in \omega_1\}$, where each X_α is countable. If G meets $\{p : a_p \backslash X_\alpha \neq \emptyset\}$ for each α and $\{p : S \in F_p\}$ for each $S \in \mathcal{S}$, then $Y = \bigcup_{p \in G} a_p$ satisfies the requirements of the problem.

ccc: Suppose A is an uncountable antichain in $\mathcal{P}$. By the standard Δ-system and thinning arguments, we may assume that the a_p for $p \in A$ form a Δ-system, and then that the root is empty. We may then assume that $A = \{p_\xi : \xi < \omega_1\}$, where $a_{p_\xi} = \{x^1 \ldots x^n_\xi\}$. Let $T_\xi = \bigcup F_{p_\xi}$. Thinning again, we may assume $\alpha > \xi$ implies $x^i_\alpha \notin T_\xi$ Now, fix $\alpha < \omega_1$ such that whenever $I_1 \ldots I_n$ are rational intervals, if there exists a ξ such that each $x^i_\xi \in I_i$ $(i = 1 \ldots n)$, then there is such a ξ less than α. Since T_α has countable closure, we may find a ξ such that each $x^i_\xi \notin T_\alpha$, and then fix rational neighborhoods I_i of each x^i_ξ missing T_α. Now, by our assumption on α, we can choose $\xi < \alpha$ – but then p_ξ and p_α are compatible.