Qualifying Exam
Logic
January 21, 1994

Instructions: If you signed up for recursion theory, do two E and two R problems. If you signed up for set theory, do two E and two S problems. Nobody signed up for model theory. If you think a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.

E1. Suppose L is a finite language, T is an L-theory, L' a language extending L, and θ is an L'-sentence such that the models of T are exactly the reducts to L of the models of θ. Prove that T is recursively axiomatizable.

E2. For any theory T let F be the set of all sentences in the language of T which are true in some finite model of T. Assume the language of T is recursive.

a. If T is finitely axiomatizable show F is recursively enumerable.

b. If T is decidable show F is recursively enumerable. Warning: The language of T might be infinite and even if it is finite, T might not be finitely axiomatizable.

c. Give an example of a recursively axiomatizable T in the language of pure equality such that F is not recursively enumerable.

E3. A linear order has real type iff it is order isomorphic to a subset of the real line. Let c be the cardinality of the the real line. Prove that there are 2^c non-isomorphic real types. Hint: There are only c isomorphisms of the reals to the reals.

R1. Define $V_e = \{x : (e, x) \in V\}$. Prove there exists a recursively enumerable set V such that V_e is not simple for every e, and for every nonsimple recursively enumerable set X there exists e such that $X = V_e$.

R2. Prove or disprove: For every total recursive function f, there exists $n \in \omega$ such that $W_{f(f(n))} = W_{f(n)}$.

R3. Suppose $T \subseteq 2^{<\omega}$ is an infinite recursive tree. Show that there exists $f \in 2^\omega$ such that f is an infinite branch through T and there exists a recursively enumerable set A such that $A \equiv_T f$.

S1. Suppose that κ is strongly inaccessible. Let V_α be the set of all sets of rank less than α. Prove that there is an ω-sequence $\alpha_0 < \alpha_1 < \ldots < \kappa$ such that each V_{α_n} is a model of ZFC, but V_λ where $\lambda = \sup_{n\in\omega} \alpha_n$ is not a model of ZFC.

S2. A tree is special iff it is a countable union of antichains. Let

$$T = \{s : \exists \alpha < \omega_1 \;\; s : \alpha \to \omega \text{ is 1-1 and range of } s \text{ is coinfinite } \}.$$

Order T by inclusion. Prove T is not special. Hint: Build an increasing sequence of nodes whose limit is in T (the hard part is to make sure the range is coinfinite). Make sure the limit cannot be in any of the antichains.

S3. Assume MA+notCH. A set $X \subseteq 2^\omega$ is ω_1 dense iff every nonempty clopen set contains ω_1 elements of X. Show that any two ω_1 dense sets X and Y are homeomorphic.

Hint: Consider the poset of finite 1-1 partial maps from X to Y. Add side conditions to force the generic map to be a homeomorphism. Note if any $x \in X$ can be mapped to uncountably many $y \in Y$, then the poset will fail to have ccc. Break X and Y in ω_1 countable pieces and demand that the homeomorphism map the α^{th} piece of X to the α^{th} piece of Y for each $\alpha < \omega_1$.

Answers to Logic Qual Jan 94

E1. If S is the set of all L sentences which θ proves, then S axiomatizes T and is recursively enumerable, hence T is recursively axiomatizable. If A is a model of T then it is the reduct of a model of θ and hence A is a model of S. If A is a model of S, then $Th(A) \cup \{\theta\}$ is consistent so A is elementarily equivalent to a model of T and so A is a model of T.

E2. (a) Since the language is recursive we can uniformly list all finite structures in all finite sublanguages of the given language so that every every such structure is isomorphic to one on the list. Say $(A_n : n < \omega)$. By dovetailing and using the fact that T is finitely axiomatizable, we can recursively enumerate every sentence true in some A_n which is also a model of T.

(b) Since T is decidable we can effectively decide given any sentence ψ whether or not $T \cup \{\psi\}$ is consistent. Let ρ_n be the sentence of pure equality that asserts "there are less than n elements in the universe". Then $\theta \in F$ iff there exist $n < \omega$ such that $T \cup \{\theta, \rho_n\}$ is consistent.

(c) Let K be a recursively enumerable set which is not recursive. Let ρ_n be the sentence which says "there are not exactly n elements in the universe". Let T be the theory axiomatized by $\{\rho_n : n \in K\}$.

E3. Call two sets of reals equivalent iff they are order isomorphic. Each equivalence class has size c, but there are 2^c subsets of the reals, hence there are 2^c equivalence classes.

R1. Think of e as coding two 1-1 functions f and g such that range f is V_e and range of g is disjoint from V_e. Do this by waiting for convergence:

$$f(0), g(0), f(1), g(1), \ldots$$

If you see that f or g are not 1-1 or their ranges not disjoint or they fail to converge on some input, then V_e ends up being finite.

R2. This is false. Let f be any recursive function such that for every n $W_{f(n)} = \{n\}$ and $f(n) \neq n$.

R3. Let f be the leftmost infinite branch of T, i.e. for every n if $f(n) = 1$ then there are only finitely many nodes of T which extend

$$(f(0), f(1), \ldots, f(n-1), 0).$$

Let A be the set of all nodes of T such that s and f are the same up to to i but $s(i) < f(i)$. Then f and A are Turing equivalent. To see that A is recursively enumerable consider f_n for each n the leftmost branch through $T \cap 2^{\leq n}$ of length n. A is the set of all nodes of T which are to the left of some f_n.

S1. Let α_n be the n^{th} cardinal such that V_{α_n} is a model of ZFC.

S2. Suppose for contradiction that T is the union of countably many antichains A_n. Let $s_0 \in T$ be arbitrary. Build and a chain $s_n \subseteq s_{n+1}$ in T and an increasing sequence of finite sets $F_n \subseteq \omega$ so that F_n is disjoint from the range of s_n and such that there exists no extension of s_n in A_n whose range is disjoint from F_n.

S3. Write X and Y as the disjoint union of countable dense sets X_α, Y_α for $\alpha < \omega_1$. Consider the poset P of conditions of the form $p = (f, C, D)$ where

(a) f is a finite 1-1 partial map from X to Y,
(b) if $f(x)$ is defined and $x \in X_\alpha$, then $f(x) \in Y_\alpha$,
(c) C and D are (finite) clopen partitions of 2^ω, and
(d) each element of the domain of f is in a unique element of C and each element of the range of f is in a unique element of D.

Define $p \leq q$ iff $f_p \supseteq f_q$, C_p refines C_q, D_p refines D_q, and $f_p(U) \subseteq V$ for any $U \in C_q$ and $V \in D_q$ such that $f_q(U) \subseteq V$. To see that P has ccc, let p_α for $\alpha < \omega_1$ be an uncountable set of conditions. Because there are only countably many clopen subsets of 2^ω. We may assume that C_α and D_α are all the same. By applying a Delta-system argument and using (b) we can find $\alpha \neq \beta$ such that $f_\alpha \cup f_\beta$ is a 1-1 function. D_n is dense where D_n is the set of all $p \in P$ such that the partition elements have diameter less than $1/n$. Let D_x for $x \in X$ be the elements of $p \in P$ with some x in domain of f_p. Similarly D_y. These are dense and any generic filter meeting all D_n for $n < \omega$, D_x for $x \in X$, and D_y for $y \in Y$ determines a homeomorphism taking X to Y.