Qualifying Exam
Logic
August 30, 1994

Instructions: If you signed up for recursion theory, do two E and two R problems. If you signed up for model theory, do two E and two M problems. (Nobody signed up for set theory.) If you think a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases do not interpret the problem in such a way that it becomes trivial.

E1. State, and sketch the proof of, the Gödel Completeness Theorem. (You may assume the language is countable if you wish; *some* proofs are easier in that case.)

E2. Let $\mathcal{L}$ be the language of one unary function f, and T the theory stating that f is a bijection such that $f^n(x) \neq x$ for all x and all $n \in \omega$ (where $f^n(x)$ is obtained by applying f to x n many times). Show that (the set of consequences of) T is decidable.

E3. Prove that there is a subset of the plane which meets each line in exactly two points. Here, the plane is $\mathcal{R} \times \mathcal{R}$, where $\mathcal{R}$ is the set of real numbers. Then, "line" has the usual geometric meaning; a line does not have to be parallel to one of the axes.

R1. A set $A \subseteq \omega$ is d.r.e. (a difference of recursively enumerable sets) if A can be written as the difference of two r.e. sets $A_0 - A_1$. Show that a set A is many-one reducible to $K \times \overline{K}$ iff A is d.r.e.

R2. An infinite set $A = \{a_0 < a_1 < \ldots\}$ is called retraceable if there is a partial recursive function ϕ such that $\mathrm{dom}\phi \subseteq A$, $\phi(a_0) = a_0$, and $\phi(a_{n+1}) = a_n$ for all $n \in \omega$. Show that if A and its complement are retraceable then A is recursive.

R3. A set $A \subseteq \omega$ is 1-generic if for any r.e. set S of binary strings, there is a string $\sigma \subset A$ such that $\sigma \in S$ or no extension of σ is in S. Show a 1-generic set A cannot be (a) recursive, or even (b) r.e.

M1. Let $\mathcal{L} = \{+, \cdot, 0, 1, F\}$, where F is a 1-place predicate. Let T be the theory in the language $\mathcal{L}$ which says that the model is an algebraically closed field of characteristic 0 (using $+, \cdot, 0, 1$), and that F is a proper algebraically closed subfield. Prove that T is complete.

M2. Let $\mathcal{L}$ be a first order language that has exactly one non-logical symbol – a binary predicate symbol R. Let T' be the set of sentences:

1. $\exists x \exists y R(x, y)$
2. $\neg \exists x \exists y [R(x, y) \wedge R(y, x)]$
3. $\neg \exists x \exists y \exists z [R(x, y) \wedge R(y, z) \wedge R(z, x)]$
4. $\neg \exists x \exists y \exists z \exists w [R(x, y) \wedge R(y, z) \wedge R(z, w) \wedge R(w, x)]$

Prove

1. If $T \supset T'$ is a consistent $\mathcal{L}$-theory that admits elimination of quantifiers, then T proves
$$\forall x \exists y R(x, y)$$
and
$$\forall x \forall y \forall z [[R(x, y) \wedge R(y, z)] \rightarrow R(x, z)];$$
2. Such a T must be unstable.

M3. Let T be a theory in a countable language, and assume that T has infinite models. Let κ be any infinite cardinal. Prove that there is a model for T of size κ in which every definable subset is either finite or has size κ. Here, "definable" allows the use of a finite set of elements of the model as parameters.