Qualifying Exam
Logic
January 19, 1996

Instructions: Do two E and two R problems. If you think that a problem has been stated incorrectly, mention this to the proctor and indicate your interpretation in your solution. In such cases, do not interpret the problem in such a way that it becomes trivial.

E1. Let S be the set of all finite sequences of natural numbers. For $s, t \in S$, define sRt iff s is obtained by replacing some element of t by a sequence of zero or more smaller numbers. For example,

$$() \ R \ (0) \ R \ (0,3) \ R \ (0,3,6) \ R \ (6,6) \ R \ (7)$$

Note that R is not transitive, but it is well-founded, so it has a rank function ρ such that $\rho(s) = sup\{\rho(t)+1 : tRs\}$. Prove that the range of ρ is the ordinal ω^ω.

E2. Let $\mathcal{L}$ be the language consisting of $=$ plus one unary relation symbol R. Let ϕ be a sentence of $\mathcal{L}$ which is true in all finite structures for $\mathcal{L}$. Prove that ϕ is also true in all infinite structures for $\mathcal{L}$.

E3. Let $<$ be a strict partial order (transitive and irreflexive) on some infinite set A of size κ. Suppose there are *fewer than* 2^κ different partial orders on A which are isomorphic to $<$. Prove that there is a subset B of A of size less than κ such that $x \not< y$ for all $x, y \in A \backslash B$.

In the Recursion Theory problems, φ_e is the e^{th} partial recursive function of one variable, using some standard enumeration. If $\mathbf{b}$ and $\mathbf{d}$ are Turing degrees, then $\mathbf{b} \leq \mathbf{d}$ means that $\mathbf{b}$ is recursive in $\mathbf{d}$, and $\mathbf{b} < \mathbf{d}$ means that $\mathbf{b} \leq \mathbf{d}$ and $\mathbf{d} \not\leq \mathbf{b}$.

R1. Prove that there is a primitive recursive function $f : \omega \to \omega$ such that:

1. For each x, i: $\varphi_{f(x)}(i) = x \cdot f(i)$.
2. For each x, y: If $x < y$ then $f(x) < f(y)$.

R2. Let $\mathbf{d}_n$ and $\mathbf{b}_n$, for $n \in \omega$, be Turing degrees. Assume that $\mathbf{d}_n < \mathbf{d}_{n+1}$ for each n, and $\mathbf{d}_m < \mathbf{b}_n$ for each m, n. Prove that there is a Turing degree $\mathbf{c}$ such that $\mathbf{d}_n < \mathbf{c}$ and $\mathbf{b}_n \not\leq \mathbf{c}$ for each n.

R3. Say sets $A, B \subset \omega$ are *r.e. inseparable* iff they are disjoint and there is no r.e. set that contains one and is disjoint from the other. Prove that there are Δ^0_2 sets that are r.e. inseparable.

Answers to Logic Qual January 1996

E1. The empty sequence has rank 0. If s is non-empty, and $n_1, \ldots, n_k$ are the numbers occurring in s, where $n_1 > \cdots > n_k$ and each n_i occurs exactly r_i times, then $\rho(s) = \omega^{n_1} \cdot r_1 + \cdots + \omega^{n_k} \cdot r_k$.

E2. Use the Compactness and Löwenheim-Skolem theorems. If ϕ fails in some infinite model, consider three cases: $R, \neg R$ are both infinite, R is finite and $\neg R$ is infinite, or $\neg R$ is finite and R is infinite.

E3. If there is no such B, find distinct a_α, b_α in A for $\alpha < \kappa$ such that $a_\alpha < b_\alpha$. Then, for each subset S of κ, one can construct a different isomorphic copy of $<$ by exchanging $\{a_\alpha, b_\alpha\}$ for $\alpha \in S$.

R1. It is easy to find primitive recursive functions, s, t satisfying:

$$\varphi^{(1)}_{s(a,e,x)}(i) = \varphi^{(3)}_a(e, x, i)$$

(by the s^m_n theorem) and

$$\varphi_{t(w,j)} = \varphi_w \quad \text{and} \quad t(w, j) > j$$

Now, fix a such that $\varphi^{(3)}_a(z, x, i) = x \cdot \varphi_z(i)$. By the Recursion Theorem, fix e such that

$$\varphi_e(0) = s(a, e, 0) \quad \text{and} \quad \varphi_e(x + 1) = t(s(a, e, x), \varphi_e(x))$$

Let $f(x) = \varphi_e(x)$. Then,

$$\varphi_{f(x)}(i) = \varphi_{s(a,e,x)}(i) = \varphi^{(3)}_a(e, x, i) = x \cdot \varphi_e(i) = x \cdot f(i)$$

R2. Fix sets $D_n \subseteq \omega$ of degree $\mathbf{d}_n$. $\mathbf{c}$ can be the degree of some $C \subseteq \omega \times \omega$. Choose C so that for each n, $\{i : (n, i) \in C\}$ is equal to D_n modulo some finite set; the finite sets are chosen so that there is no way to compute any $\mathbf{b}_n$ from C.

R3. Let $A_0 = \{2n : n \in \omega\}$ and $B_0 = \{2n + 1 : n \in \omega\}$. Define $C = \{4n : 4n, 4n + 2 \in W_n\}$ and $D = \{4n + 1 : 4n + 1, 4n + 3 \in W_n\}$. Then $A = (A_0 \cup D) - C$ and $B = (B_0 \cup C) - D$ are Δ^0_2 and *r.e.* inseparable.